CHAPTER 4

s-CLOSED SPACES VIA GRILLS
AND s-CLUSTER SETS
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VIA GRILLS AND
s-CLUSTER SETS

4.1. INTRODUCTION

It was Thompson [195] who introduced the concept of S-closed spaces by using the notion of semi-open sets of Levine [108] and closure of such sets. Crossley and Hildebrand [33] introduced the notion of semiclosure of a subset of a topological space; Mayo and Noiri [113] used it along with semi-open sets to initiate the concept of s-closed spaces and also derived certain characterizations. Subsequently s-closedness has been studied extensively by many researchers. Ganster and Reilly [61] have established that every infinite topological space can be embedded as a closed subspace of a connected S-closed space which is not s-closed. It has been observed from the literature that many researchers have worked on this concept by adopting a number of approaches and also from different angles. Different techniques, mainly covers, nets, filters and filterbase etc. are employed so far to study the said type of spaces. The purpose of the deliberation in this chapter is to continue the investigations of the concept of s-closedness through
two new approaches, viz. by means of grills and in terms of the notion of a new kind of cluster sets for functions and multifunctions between topological spaces.

As mentioned in Chapter 1, Choquet [28] was first to introduce the concept of grills. Afterwards Thron [197] investigated extensively the theory and structure of grills and many other researchers (see for instance [24, 25, 22, 23]) have worked on it in detail for the study of closure spaces, proximity spaces, compactifications and some other topological concepts. For the purpose we require to establish certain concepts concerning grills and then apply this general discussion to the intended study of s-closed spaces. Thus the theory of grills has been developed to certain extent in Section 2, in order to facilitate the study of s-closed spaces, in Section 3, via grills.

Weston [203] was the initiator of the concept of cluster sets of arbitrary functions between topological spaces. Since then a good number of mathematicians have been investigating similar theories for different classes of functions and multifunctions. It has already been observed in the recent time literature that such theories have an enormous applications specially to the characterizations of different covering properties and certain separation axioms. For instance, many well known covering properties like, compactness and H-closedness have been studied by Joseph [91] in terms of cluster sets and he also derived certain nice applications of the said theory. In [132] and [134], the notions of $\delta$-cluster sets and p-cluster sets respectively have been introduced and studied in detail, wherein the two notions have been nicely employed in investigating the near compactness and p-closedness respectively among other things.

Here our prime objective is to introduce the notions of s-cluster sets of functions and multifunctions by means of semi-open sets and some other allied concepts as supporting tools, and to ultimately formulate s-closed spaces just by
exploiting effectively this new concept of s-cluster sets. Moreover, we have derived explicit characterization of Hausdorffness through cluster sets for suitable functions.

In Section 4, the notion of a new kind of cluster sets for functions and multifunctions between topological spaces has been investigated by means of semi-open sets and allied concepts. Hausdorffness has been characterized in terms of the newly introduced cluster sets of a certain class of functions. A momentous application of the study has been achieved in characterizing s-closed spaces through the induced idea of cluster set of multifunctions in Section 5.

Throughout this chapter, by a multifunction \( F : X \rightarrow Y \) we mean, as usual, a function which maps the points of the space \( X \) to nonempty subsets of the space \( Y \). Similarly by the notation \( f : X \rightarrow Y \) we shall mean a function \( f \) from the space \( X \) to the space \( Y \).

As has already been stated in Chapter 2, a subset \( A \) of \( X \) is said to be semiopen [108] if \( A \subseteq clint A \), the complement of a semiopen set is called semiclosed [34]. The set of all semiopen (resp. open) sets in a space \( X \), each containing a given subset \( A \) of \( X \) will be denoted by \( SO(A) \) (resp. \( \tau(A) \)). In case \( A \) is singleton, say \( A = \{ x \} \) (for \( x \in X \)), we shall write \( SO(x) \) (resp. \( \tau(x) \)) instead of \( SO(\{ x \}) \) (resp. \( \tau(\{ x \}) \)). Here it is to be noted that \( \tau \subseteq SO(X) \).

The semiclosure and semiinterior [35] of a set \( A \) in a space \( X \), to be denoted respectively by \( scl A \) and \( sint A \), are defined by

\[
\begin{align*}
    scl A &= \{ x \in X : \forall U \in SO(x), U \cap A \neq \Phi \}, \\
    sint A &= \{ x \in X : \exists U \in SO(x), U \subseteq A \}.
\end{align*}
\]

It is well known [35] that \( A(\subseteq X) \) is semiclosed iff \( A = scl A \). For any subset \( A \) of a space \( X \), the \( \theta \)-closure [199] (resp. semi-\( \theta \)-closure [113]) of \( A \), to be denoted by \( \theta-cl A \) (resp. \( s(\theta)cl A \)) is the set of all those points \( x \) of \( X \) such that for every \( U \in \tau(x) \) (resp. \( U \in SO(x) \)), \( cl U \cap A \neq \Phi \) (resp. \( scl U \cap A \neq \Phi \)). The set \( A \) is called \( \theta \)-closed.
[199] (resp. semi-$\theta$-closed [113] or in short, $s(\theta)$-closed) if $A = \theta\text{-cl} A$ (resp. $A = s(\theta)\text{-cl} A$). The complement of such sets are known as $\theta$-open (resp. $s(\theta)$-open). If $A \in SO(X)$, then $\text{cl} A$, $\text{scl} A$ and $s(\theta)\text{-cl} A$ are all members of $SO(X)$. It is also true that $U \in \tau \Rightarrow \text{cl} U = \theta\text{-cl} U$ [180]. Now we are to consider the following results required for the purpose of studying the said $s$-closed spaces.

**Result 4.1.1.** [113, 122] (a) Semiclosure and semi-$\theta$-closure are both idempotent operators.

(b) For any semi-open set $U$ in $X$, $\text{scl} U = s(\theta)\text{-cl} U$.

**Result 4.1.2.** For any subset $A$ of $X$, $s(\theta)\text{-cl} A = \cap \{ \text{sd} U : A \subseteq U \in SO(X) \}$.

**Proof.** Indeed, for any $U \in SO(X)$ with $A \subseteq U$ we have, $s(\theta)\text{-cl} A \subseteq s(\theta)\text{-cl} U = \text{scU}$ (by Result 4.1.1(b)). So, L.H.S. $\subseteq$ R.H.S. Next let, $x \notin s(\theta)\text{-cl} A$. Then for some $V \in SO(x)$, $\text{sd} V \cap A = \Phi$, which implies that $A \subseteq X \setminus \text{sd} V = W$ (say) $\in SO(x)$ such that $\text{sd} W = \text{sc}(X \setminus \text{sd} V) = X \setminus \text{sd} V$ (since $\text{sd} V$ is semi-open) $\subseteq X \setminus V$. Hence $x \notin \text{sd} W$. Thus $x \notin$ R.H.S.

### 4.2. GRILLS AND ASSOCIATED CONCEPTS

In this section, we propose to discuss the theory of grills and a few allied concepts to some extent so that the notion of $s$-closedness can be studied in the subsequent sections with these concepts as prerequisite tools. Thus our prime objective is to develop the theory of grills and to observe the interplay between grills and filters as required for our purpose. In Chapter 2 we gave the definition of a grill. However, we append it here once again for the sake of ready reference.
Definition 4.2.1. [28] A nonempty collection \( \mathcal{G} \) of subsets of \( X \) is called a grill on \( X \) if (i) \( \emptyset \notin \mathcal{G} \) (ii) \( A \in \mathcal{G}, A \subseteq B(\subseteq X) \Rightarrow B \in \mathcal{G} \), and (iii) \( A \cup B \in \mathcal{G}(A, B \subseteq X) \Rightarrow A \in \mathcal{G} \) or \( B \in \mathcal{G} \).

Definition 4.2.2. A grill \( \mathcal{G} \) on a space \( X \) is said to

(a) s-adhere (resp. \( s(\theta) \)-adhere) at a point \( x \) of \( X \), if for each \( U \in \text{SO}(x) \) and each \( G \in \mathcal{G} \), \( U \cap G \neq \emptyset \) (resp. \( sclU \cap G \neq \emptyset \)),

(b) s-converge (resp. \( s(\theta) \)-converge ) to \( x \in X \) if to each \( U \in \text{SO}(x) \), there corresponds some \( G \in \mathcal{G} \) such that \( G \subseteq U \) (resp. \( G \subseteq sclU \)).

Remark 4.2.3. It is clear from the above definition that a grill \( \mathcal{G} \) on a space \( X \) is s-convergent (resp. \( s(\theta) \)-convergent) to a point \( x \in X \) iff \( \mathcal{G} \) contains the collection \( \text{SO}(x) \) (resp.\( \{sclU : U \in \text{SO}(x)\})

Definition 4.2.4. [113] A filter (or a filterbase) \( \mathcal{F} \) on a space \( X \) is said to \( s(\theta) \)-adhere at a point \( x \) of \( X \) (resp. \( s(\theta) \)-converge to \( x \) in \( X \)), if for each \( U \in \text{SO}(x) \) and for each \( F \in \mathcal{F} \), \( F \cap sclU \neq \emptyset \) (resp. to each \( U \in \text{SO}(x) \), there corresponds an \( F \in \mathcal{F} \) such that \( F \subseteq sclU \)).

Let us now recall a few terminologies and results from the literature which are required for the rest of the section.

Definition 4.2.5. [197] For any grill (or a filter) \( \mathcal{G} \) on a space \( X \), the section of \( \mathcal{G} \), denoted by \( \text{sec} \mathcal{G} \), is given by \( \text{sec} \mathcal{G} = \{A \subseteq X : A \cap G \neq \emptyset, \forall G \in \mathcal{G}\} \).

Theorem 4.2.6. [197] (a) For any grill (filter) \( \mathcal{G} \) on a space \( X \), \( \text{sec} \mathcal{G} \) is a filter (resp. grill) on \( X \).

(b) If \( \mathcal{F} \) and \( \mathcal{G} \) are respectively a filter and a grill on a space \( X \), with \( \mathcal{F} \subseteq \mathcal{G} \), then there is an ultrafilter \( U \) on \( X \) such that \( \mathcal{F} \subseteq U \subseteq \mathcal{G} \).
Theorem 4.2.7. If a grill $\mathcal{G}$ on a topological space $X$ $s(\theta)$-adheres at some point $x \in X$, then $\mathcal{G}$ is $s(\theta)$-convergent to $x$.

Proof. Let $\mathcal{G}$ be a grill on a topological space $X$, which $s(\theta)$-adheres at $x \in X$. Then for each $U \in SO(x)$ and each $G \in \mathcal{G}$, $sclU \cap G \neq \emptyset$ and so, $sclU \in sec \mathcal{G}$, for each $U \in SO(x)$ and hence $X \setminus sclU \notin \mathcal{G}$. Then $sclU \in \mathcal{G}$ (as $\mathcal{G}$ is a grill on $X$ and $X \in \mathcal{G}$) for each $U \in SO(x)$. Hence $\mathcal{G}$ must $s(\theta)$-converge to $x$.

The following example shows that the $s(\theta)$-convergence of a grill does not imply its $s(\theta)$-adherence.

Example 4.2.8. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ and $\mathcal{G} = \{\{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Then it is easy to verify that $\mathcal{G}$ is a grill on $X$ and $\tau$ is a topology on $X$ such that $SO(X) = \tau$. Now $scl\{a\} = \{a, c\}$, $scl\{b\} = \{b\}$, $scl\{a, b\} = X$ and $scl\{a, c\} = \{a, c\}$. So for each $x \in X$, $\{sclU : U \in SO(x)\} = \{\{a, c\}, \{b\}, X\} \subseteq \mathcal{G}$. Hence $\mathcal{G}$ $s(\theta)$-converges to each point $x$ of $X$ but does not $s(\theta)$-adhere at any point $x \in X$.

Let us consider the following notations to be used in the sequel.

Notation 4.2.9. Let $X$ be topological space. Then for any $x \in X$, we take the following notations:

(a) $G(s(\theta), x) = \{A \subseteq X : x \in s(\theta)\text{-}clA\}$.

(b) $sec G(s(\theta), x) = \{A \subseteq G : A \cap G \neq \emptyset, \forall G \in G(s(\theta), x)\}$.

Theorem 4.2.10. A grill $\mathcal{G}$ on a space $X$, $s(\theta)$-adheres at a point $x \in X$, iff $\mathcal{G} \subseteq G(s(\theta), x)$.

Proof. A grill $\mathcal{G}$ on a space $X$, $s(\theta)$-adheres at $x \in X$

$\Rightarrow sclU \cap G \neq \emptyset, \forall U \in SO(x)$ and $\forall G \in \mathcal{G}$

$\Rightarrow x \in s(\theta)\text{-}clG, \forall G \in \mathcal{G}$

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Conversely, let $G \subseteq G(s(\theta), x)$. Then for all $U \in SO(x)$ and for all $G \in G$, $x \in s(\theta) \cdot cl G$. So, for all $U \in SO(x)$ and for all $G \in G$, $sclU \cap G \neq \emptyset$. Hence $G$ $s(\theta)$-adheres at $x$.

Theorem 4.2.11. A grill $G$ on a topological space $X$ is $s(\theta)$-convergent to a point $x \in X$ iff $sec G(s(\theta), x) 

Proof. Let $G$ be a grill on $X$, $s(\theta)$-converging to $x \in X$. Then for each $U \in SO(x)$ there exists $G \in G$ such that $G \subseteq sclU$ and hence $sclU \in G$ for each $U \in SO(x) \cdots (i)$. 

Now, $B \in sec G(s(\theta), x) \Rightarrow X \setminus B \notin G(s(\theta), x) \Rightarrow x \notin s(\theta) \cdot cl(X \setminus B) \Rightarrow$ there exists $U \in SO(x)$ such that $sclU \cap (X \setminus B) = \emptyset \Rightarrow sclU \subseteq B$, where $U \in SO(x)$ 

$\Rightarrow B \in G \ (by \ (i))$. So, $sec G(s(\theta), x) \subseteq G$.

Conversely, let if possible, $G$ do not $s(\theta)$-converge to $x$. Then for some $U \in SO(x)$, $sclU \notin G$ and hence $sclU \notin sec G(s(\theta), x)$. Thus for some $A \in G(s(\theta), x)$, $A \cap sclU = \emptyset \cdots (ii)$. But $A \in G(s(\theta), x) \Rightarrow x \in s(\theta) \cdot cl A \Rightarrow sclU \cap A \neq \emptyset$, contradicting (ii).

4.3. $s$-CLOSEDNESS THROUGH GRILLS

Our purpose in this section is to do some investigations concerning $s$-closedness of a topological space, on the basis of the discussion in the last section involving grills. We begin by restating the definition of an $s$-closed space, given already in Chapter 2.

Definition 4.3.1. [113] A non-empty subset $A$ of a topological space $X$ is said to be $s$-closed relative to $X$ if for each cover $\{U_\alpha : \alpha \in \Lambda\}$ of $A$ by semi-open sets
of $X$, there exists a finite subset $A_0$ of $A$ such that $A \subseteq \bigcup_{\alpha \in A_0} \text{scl} U_\alpha$.

If in addition $A = X$, the space $X$ is called an $s$-closed space.

Some of the various known characterizations of an $s$-closed space are given as follows.

**Theorem 4.3.2.** [113, 122, 150] For a topological space $(X, \tau)$, the following are equivalent:

(a) $(X, \tau)$ is $s$-closed.

(b) Every maximal filterbase on $X$ $s(\theta)$-converges to some point of $X$.

(c) Every filterbase $s(\theta)$-adheres at some point of $X$.

(d) For every family $\{V_\alpha : \alpha \in A\}$ of semiclosed subsets of $X$ such that $\bigcap_{\alpha \in A} V_\alpha = \emptyset$, there exists a finite subset $A_0$ of $A$ such that $\bigcap_{\alpha \in A_0} \text{scl} V_\alpha = \emptyset$.

(e) For every semi-$\theta$-open cover $\{V_\alpha : \alpha \in A\}$ of $X$, there exists a finite subset $A_0$ of $A$ such that $X = \bigcup_{\alpha \in A_0} V_\alpha$.

From part (e) of the above theorem it follows at once that

**Theorem 4.3.3.** For a topological space $(X, \tau)$, the following are equivalent:

(a) $(X, \tau)$ is $s$-closed.

(b) For every family $\{V_\alpha : \alpha \in A\}$ of semi-$\theta$-closed subsets of $X$, such that $\bigcap_{\alpha \in A} V_\alpha = \emptyset$, there exists a finite subset $A_0$ of $A$ such that $\bigcap_{\alpha \in A_0} V_\alpha = \emptyset$.

**Theorem 4.3.4.** A subset $A$ of a topological space $X$ is $s$-closed relative to $X$ iff every grill $G$ on $X$ with $A \in G$, $s(\theta)$-converges to a point in $A$.

**Proof.** Let $A$ be $s$-closed relative to $X$ and $G$ be a grill on $X$ satisfying $A \in G$ such that $G$ does not $s(\theta)$-converge to any point of $A$. Then to each $a \in A$, there corresponds some $U_a \in SO(a)$ such that $\text{scl} U_a \notin G$. Now, $\{U_a : a \in A\}$ is a cover of $A$ by semi-open sets of $X$. Then $A \subseteq \bigcup_{i=1}^{n} \text{scl} U_{a_i} = K$ (say), for some positive integer $n$. Since $G$ is a grill, $K \notin G$ and hence $A \notin G$, which is a contradiction.
Conversely, let A be not s-closed relative to X. Then for some cover \( U = \{ U_\alpha : \alpha \in \Lambda \} \) of A by semi-open sets of X, \( F = \{ A \setminus \bigcup_{\alpha \in \Lambda_0} \text{scl}U_\alpha : \Lambda_0 \text{ is a finite subset of } \Lambda \} \) is a filterbase on X. Then the family \( F \) can be extended to an ultrafilter \( F^* \) on X. Then \( F^* \) is a grill on X with \( A \in F^* \) (as each \( F \in F \), is a subset of A).

Now for each \( x \in A \), there must exist \( \beta \in \Lambda \) such that \( x \in U_\beta \), as \( U \) is a cover of A. Then for any \( G \in F^* \), \( G \cap (A \setminus \text{scl}U_\beta) \neq \emptyset \), so that \( G \not\subseteq \text{scl}U_\beta \), for all \( G \in G \).

Hence \( F^* \) cannot \( s(\theta) \)-converge to any point of A. This contradiction proves the desired result.

For any space \( X \), \( X \) being a member of every grill \( G \) on \( X \), the following theorem is an immediate consequence of Theorem 4.3.4.

**Theorem 4.3.5.** A topological space \( X \) is s-closed iff every grill on \( X \) is \( s(\theta) \)-convergent in \( X \).

Our intention now is to characterize s-closedness in terms of the concept of \( s(\theta) \)-adherence of grills. We see in the next theorem that such a characterization is possible for a suitable class of grills satisfying a certain condition.

**Theorem 4.3.6.** A topological space \( X \) is s-closed iff every grill \( G \) on \( X \) satisfying the following condition \( C_1 \), \( s(\theta) \)-adheres in \( X \), where

\[
C_1 : \text{For every finite subfamily } \{ G_1, G_2, \ldots, G_n \} \text{ of } G, \bigcap_{i=1}^n s(\theta)\cdot \text{cl}G_i \neq \emptyset.
\]

**Proof.** Let \( U \) be any ultrafilter on \( X \). Then \( U \) is a grill on \( X \) and also for each finite subcollection \( \{ U_1, U_2, \ldots, U_n \} \) of \( U \), \( \bigcap_{i=1}^n s(\theta)\cdot \text{cl}U_i \supseteq \bigcap_{i=1}^n U_i \neq \emptyset \), so that \( U \) is a grill on \( X \) with the given condition. Hence by hypothesis, \( U \) \( s(\theta) \)-adheres.

Consequently, by Theorem 4.3.2, the space \( X \) is s-closed.

Conversely, consider any grill \( G \) satisfying the condition \( C_1 \) on an s-closed space \( X \). Now for any \( A \subseteq X \), \( s(\theta)\cdot \text{cl}A \) is semi-\( \theta \)-closed. Thus \( \{ s(\theta)\cdot \text{cl}A : A \in G \} \) is a collection of semi-\( \theta \)-closed sets in \( X \) such that \( \bigcap_{i=1}^n s(\theta)\cdot \text{cl}A_i \neq \emptyset \), for any finite
subcollection $A_1, A_2, ..., A_n$ of $G$. Then by Theorem 4.3.3, $igcap_{A \in G} s(\theta)\text{-cl} A \neq \emptyset$, i.e., there exists $x \in X$ such that $x \in s(\theta)\text{-cl} A$ for all $A \in G$. Hence, $G \subseteq G (s(\theta), x)$.

So by Theorem 4.2.10, $G$ $s(\theta)$-adheres at $x \in X$.

An improved version of the necessity part of the above theorem is given as follows.

**Theorem 4.3.7.** In an $s$-closed space $X$, every grill $G$ with the property that $s(\theta)\text{-cl} A \cap s(\theta)\text{-cl} B \neq \emptyset$ for any $A, B \in G$, $s(\theta)$-adheres in $X$.

**Proof.** Let $X$ be $s$-closed and let $G$ be any grill on $X$ with the given property such that $G$ does not $s(\theta)$-adhere in $X$. Then for each $x \in X$, there exists $G_x \in G$ such that $x \notin s(\theta)\text{-cl} G_x = s(\theta)\text{-cl} (s(\theta)\text{-cl} G_x)$ (by Result 4.1.1(a)). Thus there exists $U_x \in SO(x)$ such that $sclU_x \cap s(\theta)\text{-cl} G_x = \emptyset$ and hence $s(\theta)\text{-cl} U_x \cap s(\theta)\text{-cl} G_x = \emptyset$ (by Result 4.1.1(b)). Since $s(\theta)\text{-cl} G_x \in G$ and $G$ is a grill, $sclU_x = s(\theta)\text{-cl} U_x \notin G$.

Now $\{U_x : x \in X\}$ is a cover of $X$ by semi-open sets of $X$. So, by $s$-closedness of $X$, $X = \bigcup_{i=1}^{n} sclU_{x_i}$, for a finite subset $\{x_1, x_2, ..., x_n\}$ of $X$. It then follows that $X \notin G$ (since $sclU_{x_i} \notin G$, for $i = 1, 2, ..., n$) which is a contradiction. Hence $G$ must $s(\theta)$-adhere in $X$.

### 4.4. SEMICOMPACTNESS AND $s$-CLOSEDNESS

Charles Dorsett [46] introduced the notion of semicompact spaces. According to him a topological space $X$ is semicompact if every cover of $X$ by semi-open sets of $X$ has a finite subcover. It is well known that every semicompact space is an $s$-closed space but not conversely. In this section we try to investigate for the
condition under which the class of s-closed spaces may coincide with the class of semicompact spaces. We first derive a characterization of semicompactness in terms of grills.

**Theorem 4.4.1.** A topological space $X$ is semicompact iff every grill on $X$ s-converges to some point $x$ of $X$.

**Proof.** Let $G$ be a grill on a topological space $X$, such that $G$ does not s-converge to any point $x \in X$. Then for each $x \in X$, there exists $U_x \in SO(x)$ with $U_x \notin G \cdots (i)$.

As $\{U_x : x \in X\}$ is a cover of the semicompact space $X$ by semi-open sets in $X$, there exist finitely many points $x_1, x_2, \ldots, x_n$ in $X$ such that $X = \bigcup_{i=1}^{n} U_{x_i} \notin G$ (since $U_{x_i} \notin G$ for $i = 1, 2, \ldots, n$) which is a contradiction.

Conversely, let every grill on $X$ s-converge and if possible, let $X$ be not semicompact. Then there exists a cover $\mathcal{U}$ of $X$ by semi-open sets of $X$ having no finite subcover. Thus $\mathcal{F} = \{X \setminus \cup \mathcal{U}_0 : \mathcal{U}_0$ is a finite subset of $\mathcal{U}\}$ is a filterbase on $X$. Then $\mathcal{F}$ is contained in an ultrafilter $\mathcal{F}_0$ and then $\mathcal{F}_0$ is a grill (since every ultrafilter is a grill) on $X$. By hypothesis, $\mathcal{F}_0$ s-converges to some $x \in X$. Then for some $U \in \mathcal{U}$, $x \in U$ and hence $U \in \mathcal{F}_0$. Again $X \setminus U \in \mathcal{F} \subseteq \mathcal{F}_0$. Thus $U$ and $X \setminus U$ both belong to $\mathcal{F}_0$, which is a filter. So, we arrive at a contradiction. Hence the theorem.

It is known that for a space satisfying regularity, compactness becomes equivalent to certain weaker forms of it. We may thus consider a sort of regularity criterion, as given below, by the use of semi-open sets.

**Definition 4.4.2.** A topological space $X$ is s-regular, if for each $x \in X$ and each $U \in SO(x)$, there exists $V \in SO(x)$ such that $scl V \subseteq U$.

Now it can be easily seen in a straightforward manner that in an s-regular
space, there is no difference between $s$-closedness and semicompactness of a space, i.e., we have:

**Theorem 4.4.3.** An $s$-regular $s$-closed space is semicompact.

Now we like to infer semicompactness from $s$-closedness with the assumption of a weaker condition than $s$-regularity. For this, we now define a sort of strictly weaker form of $s$-regularity in terms of grills, termed $s(\theta)$-regularity and prove the equivalence of the concepts of $s$-closedness and semicompactness in the presence of $s(\theta)$-regularity rather than $s$-regularity.

**Definition 4.4.4.** A topological space $X$ is called $s(\theta)$-regular if every grill on $X$ which $s(\theta)$-converges must $s$-converge (not necessarily to the same point).

**Theorem 4.4.5.** Every $s$-regular space is $s(\theta)$-regular.

**Proof.** Let $G$ be a grill on an $s$-regular space $X$, such that it $s(\theta)$-converges to a point $x$ of $X$. For each $U \in SO(x)$, there exists by $s$-regularity of $X$, some $V \in SO(x)$ such that $sclV \subseteq U$. By hypothesis, $sclV \in G$. So, $U \in G$, i.e., $G$ $s$-converges to $x$, proving $X$ to be $s(\theta)$-regular.

The following example shows that a topological space may be $s(\theta)$-regular without being $s$-regular i.e., former one is strictly a weaker form of the latter one.

**Example 4.4.6.** Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$. It can be easily verified that $\tau$ is a topology on $X$ and $SO(X, \tau) = \tau$. Now $(X, \tau)$ is semicompact ($X$ being a finite set). Hence by Theorem 4.4.1, every grill on $X$ $s$-converges. Then $X$ is $s(\theta)$-regular. But the space is not $s$-regular. Indeed, $\{a\} \in SO(a)$, but $scl(\{a\}) = X \not\subseteq \{a\}$.

Now from the discussion so far, we are in a position to arrive at our desired conclusion as follows.
Theorem 4.4.7. A semicompact space $X$ is $s$-closed and the converse is also true if $X$ is $s(\theta)$-regular.

4.5. $s$-CLUSTER SET AND ITS DEGENERACY

Our proposed definition of the desired type of cluster sets for arbitrary functions or multifunctions goes as follows.

Definition 4.5.1. Let $F : X \to Y$ be a multifunction, where $X$ and $Y$ are any topological spaces and $x \in X$. Then the $s$-cluster set of $F$ at $x$, denoted by $s(F, x)$ is defined to be the set $\cap \{\theta-cl F(sclU) : U \in SO(x)\}$.

Similarly for any function $f : X \to Y$, the $s$-cluster set $s(f, x)$ of $f$ at $x$, is given by $s(f, x) = \cap \{\theta-cl f(sclU) : U \in SO(x)\}$.

The following theorem gives two characterizations of $s$-cluster sets of arbitrary functions in terms of the concepts, already defined in this chapter.

Theorem 4.5.2. For any function $f : (X, \tau) \to (Y, \sigma)$, the following are equivalent:
(a) $y \in s(f, x)$.
(b) The filterbase $f^{-1}(cl(\sigma(y)))$ $s(\theta)$-adheres at $x$, where $cl(\sigma(y)) = \{clV : V \in \sigma(y)\}$.
(c) There is a grill $G$ on $X$ such that $G$ $s(\theta)$-converges to $x$ (say) in $X$ and $y \in \cap \{\theta-cl f(G) : G \in G\}$.

Proof. (a) $\Rightarrow$ (b): $y \in s(f, x) \Rightarrow$ for each $T \in SO(x)$ and each $R \in \sigma(y)$,
\(clR \cap f(sclT) \neq \emptyset, \text{ and hence } f^{-1}(clR) \cap sclT \neq \emptyset.\) This implies that \(F = \{f^{-1}(clR) : R \in \sigma(y)\}\) is a filterbase on \(X\) and \(F s(\theta)\)-adheres at \(x\).

(b) \(\Rightarrow\) (c): Let \(F\) denote the filterbase \(f^{-1}(clr(y))\) and let \(G = \{K \subseteq X : K \cap F \neq \emptyset, \text{ for all } F \in F\}\). We verify that \(G\) is a grill on \(X\). Indeed, obviously \(\emptyset \notin G\) and \(G\) is nonvoid; and moreover \(A \in G, \text{ and } A \subseteq B \Rightarrow B \in G\). Now, let \(A \cup B \in G\) (where \(A, B \subseteq X\), i.e., \((A \cup B) \cap F \neq \emptyset, \forall F \in F\). If possible, let \(A \notin G\) and \(B \notin G\). Then for some \(F_1, F_2 \in F\), \(A \cap F_1 = B \cap F_2 = \emptyset\). Since \(F\) is a filterbase, there exists \(F_3 \in F\) such that \(F_3 \subseteq F_1 \cap F_2\). Then \(F_3 \cap (A \cup B) = \emptyset\), a contradiction. Thus \(G\) is a grill on \(X\). Now by (b), for each \(G \in SO(x)\) and each open neighbourhood \(H \in \sigma(y), f^{-1}(clH) \cap sclG \neq \emptyset\), i.e., \(F \cap sclG \neq \emptyset\) for each \(F \in F\) and each \(G \in SO(x)\) in \(X\). Hence for each \(G \in SO(x)\) in \(X\), \(sclG \in G\). So, \(G\) is \(s(\theta)\)-convergent to \(x\). From the definition of \(G\) it then follows that \(f(K) \cap clH \neq \emptyset, \text{ for each } K \in G\) and each \(H \in \sigma(y)\).

Thus \(y \in \cap \{\theta-cl f(K) : K \in G\}\).

(c) \(\Rightarrow\) (a): Let \(G\) be a grill on \(X\) such that \(G\) is \(s(\theta)\)-convergent to \(x\), and \(y \in \cap \{\theta-clf(K) : K \in G\}\). Then \(\{sclT : T \in SO(x)\} \subseteq G\) and \(y \in \theta-clf(K), \text{ for all } K \in G\), i.e., \(y \in \cap \{\theta-clf(sclT) : T \in SO(x)\}\). So, \(y \in s(f, x)\).

It is evident from the definition of \(s\)-cluster set of a function \(f : X \to Y\) (or a multifunction \(F : X \to Y\)) at any point \(x\) of \(X\) that \(f(x) \in s(f, x)\) (resp. \(F(x) \subseteq s(F, x)\)). Now we shall observe in the following theorems that under certain conditions, \(s(f, x)\) (resp. \(s(F, x)\)) becomes degenerate, i.e., \(s(f, x) = \{f(x)\}\) (resp. \(s(F, x) = F(x)\)).

**Theorem 4.5.3.** A topological space \(Y\) is Hausdorff if for some space \(X\) and some surjection \(f : X \to Y, s(f, x)\) is degenerate for each \(x \in X\).

**Proof.** Let \(y_1, y_2 \in Y\) with \(y_1 \neq y_2\). As \(f\) is onto, there exist \(x_1, x_2 \in X\) such that \(f(x_1) = y_1\) and \(f(x_2) = y_2\). Now the degeneracy of \(s(f, x)\) for each \(x \in X\) implies
that $y_2 = f(x_2) \notin s(f, x_1)$. So there exist $G \in \tau(y_2)$ in $Y$ and $H \in SO(x_1)$ such that $f(sclH) \cap clG = \emptyset$, whence we get $f(sclH) \subseteq Y \setminus clG \in \tau(y_1)$ and $G \in \tau(y_2)$ such that $G \cap (Y \setminus clG) = \emptyset$. This shows that $Y$ is Hausdorff.

The following example shows that the converse of the above theorem is false even if the codomain space is a $T_5$-space and the function is bijective.

Example 4.5.4. Consider the identity map $f : (\mathcal{R}, \tau_1) \rightarrow (\mathcal{R}, \tau_2)$, where $\tau_1, \tau_2$ respectively denote the cofinite topology and the usual topology on the set $\mathcal{R}$ of real numbers. We verify that $s(f, x) = \mathcal{R}$, for each $x \in \mathcal{R}$, although $(\mathcal{R}, \tau_2)$ is known to be a $T_5$-space and $f$ is a bijective map. Indeed, in $(\mathcal{R}, \tau_1)$, each nonempty semi-open set $U$ has finite complement and so its semiclosure in $(\mathcal{R}, \tau_1)$ is $\mathcal{R}$ itself. As $f$ is the identity map, we have $f(sclU) = f(\mathcal{R}) = \mathcal{R}$, consequently, $s(f, x) = \cap\{\theta-cl f(sclU) : U \in SO(x)\} = \mathcal{R}$.

Now we derive a weaker version of the converse of Theorem 4.5.3.

Theorem 4.5.5. If a topological space $(Y, \sigma)$ is Urysohn, then for some space $(X, \tau)$ and some function $f : X \rightarrow Y$, $s(f, x)$ is degenerate for each $x \in X$.

Proof. Let $(Y, \sigma)$ be a Urysohn space and if possible, let for each space $X$ and each function $f : X \rightarrow Y$, $s(f, x)$ is not degenerate for at least one $x \in X$. Now if we consider the identity map $i : (Y, \sigma) \rightarrow (Y, \sigma)$, then for some $x \in X$, there exists $y \in Y$ with $y \neq x = i(x)$ such that $y \in s(i, x)$. Then for all $G \in SO(x)$ and all $H \in \sigma(y)$, $sclG \cap clH \neq \emptyset$ and $clG \cap clH \neq \emptyset$, $\forall G \in \sigma(x)$ and $\forall H \in \sigma(y)$. Then $Y$ is not Urysohn.

The converse of Theorem 4.5.3 is however true for a suitable class of functions as defined below.

Definition 4.5.6. A function $f : X \rightarrow Y$ is said to $s(\theta)$-irresolute if for each
Let $x \in X$ and each open set $V$ containing $f(x)$ in $Y$, there exists $U \in SO(x)$ in $X$ such that $f(scl U) \subseteq V$.

**Theorem 4.5.7.** Let a function $f : (X, \tau) \rightarrow (Y, \sigma)$ be $s(\theta)$-irresolute with $Y$ a Hausdorff space. Then $s(f, x)$ is degenerate, for each $x \in X$.

**Proof.** Let $x \in X$. As $f$ is $s(\theta)$-irresolute, for each open set $G$ containing $f(x)$ in $Y$, there exists $H \in SO(x)$ in $X$ such that $f(scl H) \subseteq G$. Now, $s(f, x) = \bigcap \{\theta-clf(scl H) : H \in SO(x)\} \subseteq \bigcap \{\theta-cl G : G \in \sigma(f(x))\} \cdots (1)$.

Let $y \in Y$ with $y \neq f(x)$. As $Y$ is Hausdorff, there are disjoint open sets $S$ and $T$ in $Y$ such that $y \in S$ and $f(x) \in T$. Then obviously $S \cap cl T = \emptyset$. So $y \notin cl T = \theta-cl T$. Thus by (1), $y \notin s(f, x)$. Hence $s(f, x) = \{f(x)\}$.

From Theorems 4.5.3 and 4.5.7 we obtain the following characterization of the Hausdorffness of a topological space.

**Corollary 4.5.8.** A topological space $Y$ is Hausdorff iff for some space $X$ and some $s(\theta)$-irresolute surjection $f : X \rightarrow Y$, $s(f, x)$ is degenerate for each $x \in X$.

Let us now recall a few terminologies to be used in the sequel.

**Definition 4.5.9.** [37] A function $f : X \rightarrow Y$ is $\theta$-closed if the image of each $\theta$-closed set in $X$ is $\theta$-closed in $Y$.

**Definition 4.5.10.** [180] A topological space $X$ is said to be almost regular if for every regular closed set $A$ (i.e., $A = cl int A$) in $X$ and each point $x \in X \setminus A$, there exist disjoint open sets $U$ and $V$ such that $x \in U$ and $A \subseteq V$.

It is shown in [180] that in an almost regular space $X$, $\theta-cl A$ is $\theta$-closed for any $A \subseteq X$.

In the following theorems we arrive at certain other situations when the $s$-cluster set of a function or a multifunction may be degenerate.
Theorem 4.5.11. Let \( f : X \rightarrow Y \) be a \( \theta \)-closed map from an almost regular space \( X \) into a space \( Y \). If \( f^{-1}(y) \) is \( \theta \)-closed for each \( y \in Y \), then \( s(f,x) \) is degenerate for each \( x \in X \).

Proof. We have for any \( x \in X \),
\[
\mathcal{s}(f,x) = \bigcap \{ \theta-\text{cl} f(sclU) : U \in SO(x) \} \subseteq \\
\cap \{ \theta-\text{cl} f(\theta-\text{cl } U) : U \in SO(x) \} \quad \text{(since } sclU \subseteq clU \subseteq \theta-\text{cl } U) \quad \text{. Now, } \theta-\text{cl } U \text{ is} \\
\theta-\text{closed for each } U \in SO(x) \text{ by almost regularity of } X \text{. So, } \theta-\text{cl}(f(\theta-\text{cl } U)) = f(\theta-\text{cl } U), \text{ since } f \text{ is a } \theta-\text{closed map. Hence, } \\
s(f,x) \subseteq \cap \{ f(\theta-\text{cl } U) : U \in SO(x) \} \ldots \quad (1)
\]

Let \( y \in Y \) and \( y \neq f(x) \). Then by hypothesis, \( f^{-1}(y) \) is \( \theta \)-closed and \( x \notin f^{-1}(y) \).

Hence there exists an open set \( K \) containing \( x \) in \( X \) such that \( clK \cap f^{-1}(y) = \emptyset \), and then \( y \notin f(clK) = f(\theta-\text{cl } K) \) (since \( K \) is open). Thus by (1), \( y \notin s(f,x) \) and \( s(f,x) \) becomes degenerate.

Theorem 4.5.12. If \( f : X \rightarrow Y \) is a \( \theta \)-closed injective map and \( X \) is a Hausdorff as well as an almost regular space, then \( s(f,x) \) is degenerate for each \( x \in X \).

Proof. Since \( X \) is almost regular and \( f \) is a \( \theta \)-closed map, we have \( \theta-\text{cl}(f(\theta-\text{cl } U)) = f(\theta-\text{cl } U) \) for all \( U \subseteq X \cdot \cdot \cdot (1) \).

Let \( x, x_1 \in X \) such that \( x \neq x_1 \). As \( f \) is injective, \( f(x) \neq f(x_1) \). By the Hausdorffness of \( X \), there exist disjoint open sets \( S \) and \( T \) in \( X \) such that \( x \in S \) and \( x_1 \in T \). Then \( clT \cap S = \emptyset \) and hence \( x_1 \notin \theta-\text{cl } S \), so that \( f(x_1) \notin f(\theta-\text{cl } S) = \theta-\text{cl}(f(\theta-\text{cl } S)) \) (by (1)) \( \supseteq \theta-\text{cl}(sclS) \), as \( S \in \tau(x) \subseteq SO(x) \). It thus follows that \( f(x_1) \notin s(f,x) \). Hence \( s(f,x) \) is degenerate.

Our aim now is to derive a condition under which an arbitrary multifunction \( F : X \rightarrow Y \) becomes degenerate. In this connection we adopt the usual definition of the lower inverse \( F^- \) of any multifunction \( F : X \rightarrow Y \), given by \( F^-(B) = \{ x \in X : F(x) \cap B \neq \emptyset \} \) for any subset \( B \) of \( Y \).

Theorem 4.5.13. If a multifunction \( F : X \rightarrow Y \) has a \( \theta \)-closed graph \( G(F) \), then for each \( x \in X \), \( s(f,x) = F(x) \), where \( G(F) = \{ (x,y) \in X \times Y : y \in F(x) \} \).
Proof. Let \( x \in X \) and \( y \in s(f,x) \). Then for each \( U \in SO(x) \) and each open neighbourhood \( V \) of \( y \) in \( Y \), \( \overline{clV} \cap F(sclU) \neq \emptyset \), i.e., \( F^-(\overline{clV}) \cap sclU \neq \emptyset \). Then for any basic open set \( S \times T \) in \( X \times Y \) containing \( (x,y) \), \( F^-(\overline{clT}) \cap sclS \neq \emptyset \), so that \( (\overline{clS} \times \overline{clT}) \cap G(F) \neq \emptyset \). Hence \( \overline{cl(S \times T)} \cap G(F) \neq \emptyset \), where \( S \times T \) is a basic open set in \( X \times Y \) containing \( (x,y) \). Hence \( (x,y) \in \theta-cl G(F) = G(F) \) and thus \( (x,y) \in G(F) \cap (\{x\} \times Y) \). Then \( y \in p_2(G(F) \cap (\{x\} \times Y)) = F(x) \), where \( p_2 : X \times Y \to Y \) is the second projection map. Hence \( s(F,x) = F(x) \), \( \forall x \in X \).

In the next result we achieve a partial converse of the above theorem. For this purpose we require the following lemma, the proof of which is quite straightforward.

Lemma 4.5.14. Let \( X, Y \) be topological spaces, \( A \subseteq X, B \subseteq Y, x \in X \) and \( y \in Y \). Then

(a) \( U \in SO(x) \) and \( V \in SO(y) \Rightarrow U \times V \in SO((x,y)) \)

(b) \( scl(A \times B) \subseteq sclA \times sclB \).

Theorem 4.5.15. For a multifunction \( F : (X, \tau) \to (Y, \sigma) \), if \( s(F,x) = F(x) \) for each \( x \in X \), then the graph \( G(F) \) of \( F \) is \( s(\theta) \)-closed.

Proof. Let \( (x,y) \in X \times Y \setminus G(F) \). Now, \( y \notin F(x) = s(F,x) \Rightarrow \exists V \in \sigma(y) \) in \( Y \) and \( U \in SO(x) \) in \( X \) such that \( \overline{clV} \cap F(sclU) = \emptyset \Rightarrow (sclU \times sclV) \cap G(F) = \emptyset \) \( \) (since \( sclV \subseteq \overline{clV} \) \( \Rightarrow scl(U \times V) \cap G(F) = \emptyset \) \( \) (by Lemma 4.5.14(b)) \( \Rightarrow (x,y) \notin s(\theta)-clG(F) \) (using Lemma 4.5.14(a)). Hence \( G(F) = s(\theta)-clG(F) \) and so \( G(F) \) is \( s(\theta) \)-closed.

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4.6. $s$-CLOSEDNESS AND $s$-CLUSTER SETS

As has already been proposed, our only concern in this section is to find an application of the introduced idea of $s$-cluster sets towards a new characterization of the well known concept of $s$-closedness.

For any subset $A$ of a topological space $X$, let us now introduce the notation $S_A$ to denote the following collection of subsets of $X$ given by

$$S_A = \left\{ \bigcup_{i=1}^{n} \text{scl } U_i : U_i \in SO(X) \text{ and } \bigcup_{i=1}^{n} \text{scl } U_i \supseteq A \text{ for some } n \in \mathbb{N} \right\}.$$  

we now need to prove the following result which incidentally gives a characterization of the $s$-closedness of a subset $A$ relative to a space $X$.

**Theorem 4.6.1.** A subset $A$ of a topological space $(X, \tau)$ is $s$-closed relative to $X$ iff whenever for any filterbase $F$ on $X$, $F \cap D \neq \emptyset$ holds for each $F \in F$ and each $D \in S_A$, then $A \cap s(\theta)\text{-ad } F \neq \emptyset$.

**Proof.** Let $A$ be $s$-closed relative to $X$ and $F$ be a filterbase on $X$ satisfying the given property. If possible, let $A \cap s(\theta)\text{-ad } F = \emptyset$. Then for each $a \in A$, there are $U(a) \in SO(a)$ and $F(a) \in F$ such that $F(a) \cap \text{scl } U(a) = \emptyset$. Since $\{U(a) : a \in A\}$ is a semi-open cover of $A$, by $s$-closedness of $A$ relative to $X$, there exists a finite subset $A_0$ of $A$ such that $A \subseteq \bigcup_{a \in A_0} \text{scl } U(a)$. Let $F_0 \in F$ such that $F_0 \subseteq \bigcap_{a \in A_0} F(a)$. Then $F_0 \cap \left( \bigcup_{a \in A_0} \text{scl } U(a) \right) = \emptyset$. But $\bigcup_{a \in A_0} \text{scl } U(a) = W$ (say) $\in S_A$ and $F_0 \in F$ with $F_0 \cap W = \emptyset$, a contradiction.

Conversely, suppose $A$ is not $s$-closed relative to $X$. Then there exists a cover $\{U_\alpha : \alpha \in \Lambda\}$ of $A$ by semi-open subsets of $X$ such that $A \nsubseteq \bigcup_{\alpha \in \Lambda_0} \text{scl } U_\alpha$ for any finite subset $\Lambda_0$ of $\Lambda$. Then $F = \{A \setminus \bigcup_{\alpha \in \Lambda_0} \text{scl } U_\alpha : \Lambda_0 \text{ is a finite subset of } \Lambda\}$ is a filterbase on $X$, such that $D \cap F \neq \emptyset$ for each $F \in F$ and each $D \in S_A$. But $A \cap s(\theta)\text{-ad } F = \emptyset$. Indeed, for $a \in A$ there exists $U_\alpha (\alpha \in \Lambda)$ such that $a \in U_\alpha$.
and then $A \setminus \text{scl}U_a \in F$ such that $(\text{scl}U_a) \cap (A \setminus \text{scl}U_a) = \emptyset$.

To arrive at our desired result, we need the following lemma as our final prerequisite.

**Lemma 4.6.2.** [113] A semi-$\theta$-closed subset of an $s$-closed space $X$ is $s$-closed relative $X$.

**Theorem 4.6.3.** A topological space $X$ is $s$-closed iff for each space $Y$, each semi-$\theta$-closed set $A \subseteq X$ and each multifunction $F : X \to Y$, $s(F, A) \supseteq \theta$-$\text{ad}F(S_A)$.

**Proof.** Let $X$ be $s$-closed, and suppose that for a space $Y$ and a multifunction $F : X \to Y$, $y \in \theta$-$\text{ad}F(S_A)$ for some semi-$\theta$-closed subset $A$ of $X$. Then for each $D \in S_A$ and each open neighbourhood $G$ of $y$ in $Y$, $\text{cl}_G F(D) \neq \emptyset$, i.e., $F^-(\text{cl}_G) \cap D \neq \emptyset$. Now, $\{F^-(\text{cl}_G) : G \text{ is an open neighbourhood of } y \text{ in } Y\}$ ($= F$, say) forms a filterbase on $X$. As $A$ is a semi-$\theta$-closed subset of the $s$-closed space $X$, $A$ becomes $s$-closed relative to $X$ (by Lemma 4.6.2). Hence by Theorem 4.6.1, we have $(s(\theta)-\text{ad}F) \cap A \neq \emptyset$. Let $x \in (s(\theta)-\text{ad}F) \cap A$. Then for all $U \in SO(x)$ and all open neighbourhoods $G$ of $y$ in $Y$ we have, $F^-(\text{cl}_G) \cap \text{scl}U \neq \emptyset$ which implies that $\text{cl}_G F(\text{scl}U) \neq \emptyset \Rightarrow y \in \theta$-$\text{cl}F(\text{scl}U)$, $\forall U \in SO(x)$ $\Rightarrow y \in s(F, x) \subseteq s(F, A)$ (since $x \in A$).

Conversely, let $F$ be a filterbase on $X$ and $y_0$ be an object such that $y_0 \notin X$. Put $Y = X \cup \{y_0\}$ and $\tau_Y = \{U \subseteq Y : y_0 \notin U\} \cup \{U \subseteq Y : y_0 \in U \text{ and } \exists F \in F \text{ such that } F \subseteq U\}$. Then it is easy to verify that $\tau_Y$ is a topology on $Y$. We consider the function $g : (X, \tau) \to (Y, \tau_Y)$ given by $g(x) = x$. We first show that $\theta$-$\text{cl}_Y X \subseteq \theta$-$\text{ad} g(S_X)$, where $\theta$-$\text{cl}_Y X$ stands for the $\theta$-closure of the subset $X$ of $Y$ in $(Y, \tau_Y)$ (similarly $\text{cl}_Y A$ will stand for the closure of a set $A(\subseteq Y)$ in $(Y, \tau_Y)$). In fact, $y \in \theta$-$\text{cl}_Y X \Rightarrow$ for each $\tau_Y$-open neighbourhood $G(y)$ of $y$ in $Y$, $\text{cl}_Y G(y) \cap X \neq \emptyset$, i.e., $\text{cl}_Y G(y) \cap g(X) \neq \emptyset$. Let $S \in S_X$ (in $X$), then $S = \bigcup_{i=1}^{n} \text{scl}U_i$.
for some $n \in \mathbb{N}$, where $U_i \in SO(X)$ and $X \subseteq \bigcup_{i=1}^{n} \text{scl} U_i$. So it is clear that $D = X$. Thus $\text{cl}_Y G(y) \cap g(S) \neq \emptyset$, $\forall S \in S_X$ (in $X$) and for all open neighbourhood $G(y)$ of $y$ in $Y$, and $y \in \theta\text{-ad}(g(S_X))$. Thus $\theta\text{-cl}_Y X \subseteq \theta\text{-ad}(g(S_X)) \subseteq s(g,x)$ (by the given condition). Now, $y_0 \in \theta\text{-cl}_Y X \Rightarrow y_0 \in s(g,x)$ for some $x \in X \Rightarrow y_0 \in \theta\text{-cl}_Y (g(sclG)), \forall G \in SO(X) \Rightarrow \text{cl}_Y (F \cup \{y_0\}) \cap g(sclG) \neq \emptyset, \forall G \in SO(x)$ and $\forall F \in F$ (as $F \cup \{y_0\}$ is a $\tau_Y$-open neighbourhood of $y_0$) $\Rightarrow F \cap sclG \neq \emptyset$, $\forall G \in SO(x)$ and $\forall F \in F$ (since $g(sclU) = sclU \subseteq X$ and $F \cup \{y_0\}$ is closed in $Y$) $\Rightarrow x \in s(\theta)\text{-ad}F$. Hence $X$ is $s$-closed, by Theorem 4.3.2 ((c) $\Rightarrow$ (a)).