CHAPTER 5

A TYPE OF EXTENSIONS OF

TOPOLOGICAL SPACES
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5.1. INTRODUCTION

In Chapter 1 we discussed to some extent about a few types of extensions of topological spaces. It was pointed out there as to how principal extensions and \( \theta \)-principal extensions can be applied to many topological investigations of pivotal importance like the study of compactness, \( H \)-closedness and above all proximities. The intent of this chapter is to build up a basic theory for a kind of extensions in a way similar to the existing theories done in [128, 25], and we believe that such a theory has close bearing to the study of a well known covering property viz. near compactness, as may be revealed when this line of approach is continued to further extent in future investigations.

In Section 1 we first introduce the concept of \( \delta \)-extension of a topological space in a natural way. Then follows a number of grill-related concepts like \( \delta \)-adherence grill, \( \delta \)-grill, \( \delta \)-trace system of a \( \delta \)-extension etc. After defining \( \delta \)-equivalence of \( \delta \)-extensions, we prove a few results which ultimately lead us to establish that the \( \delta \)-equivalence of \( \delta \)-extensions of a certain class of topological spaces can be
A typical $\delta$-extension, termed $\delta$-principal extension, is introduced in Section 3, and it is observed that $\delta$-principal extensions are uniquely determined by their $\delta$-trace systems. Then we give the construction of a $\delta$-principal extension $X^*$ of a topological space $X$, $\delta$-homeomorphic to any given $\delta$-principal extension of $X$, where $X^*$ is a set of some typical grills on $X$. Finally, we show that this constructed $\delta$-extension $X^*$ satisfies a kind of covering property iff the collection $X^*$ of grills satisfies a specific condition.

### 5.2. $\delta$-EXTENSIONS, $\delta$-TRACE AND $\delta$-EQUIVALENCE

First we clarify certain well known concepts before we introduce the intended notion of $\delta$-extensions.

**Definition 5.2.1.** [199] Let $(X, \tau)$ be a topological space. A point $x$ of $X$ is said to be a $\delta$-adherent point of $A$ if $\text{int}clU \cap A \neq \emptyset$, for each open nbd $U$ of $x$. The set of all $\delta$-adherent points of $A$ is called the $\delta$-closure of $A$, to be denoted by $\delta-clA$.

$A$ set $A(\subseteq X)$ is called $\delta$-closed if $A = \delta-clA$, and the complement of a $\delta$-closed set is said to be $\delta$-open.

The notion of semiregularization topology plays a predominant role in this chapter. Although its definition was given in Chapter 2, we prefer to recall it here once again.

**Definition 5.2.2.** [121] The set of all $\delta$-open sets of a topological space $(X, \tau)$ forms a topology on $X$, called the semiregularization topology on $X$ and denoted
by \( \tau_s \), such that \( \tau_s \subseteq \tau \) and for which the set of all regular open sets forms an open base. Thus a set \( A \) in \( X \) is \( \delta \)-open iff it is a union of some regular open sets.

**Definition 5.2.3.** [143] A function \( f : X \to Y \) is said to be \( \delta \)-continuous if for each \( x \in X \) and each open nbd \( V \) of \( f(x) \), there exists an open nbd \( U \) of \( x \) in \( X \) such that \( f(\text{int}U) \subseteq \text{int}V \).

The function \( f \) is called a \( \delta \)-homeomorphism if \( f \) is a bijection such that \( f \) and \( f^{-1} \) are \( \delta \)-continuous; in such a case \( X \) and \( Y \) are said to be \( \delta \)-homeomorphic.

**Theorem 5.2.4.** (a) [123] A function \( f : X \to Y \) is a \( \delta \)-homeomorphism iff \( f(\delta\text{-cl}A) = \delta\text{-cl}(f(A)) \), for each \( A \subseteq X \).

(b) [123] Two spaces \((X, \tau)\) and \((Y, \sigma)\) are \( \delta \)-homeomorphic iff \((Y, \sigma)\) is homeomorphic to some space \((X, \Sigma)\) such that \( \Sigma_s = \tau_s \).

We now introduce the following definition.

**Definition 5.2.5.** A \( \delta \)-extension of a topological space \( X \) is defined to be a pair \((\Psi, Y)\) if \( Y \) is a topological space and \( \Psi : X \to Y \) is an injective map such that \( \Psi(\delta\text{-cl}A) = \delta\text{-cl}(\Psi(A)) \cap \Psi(X) \) for each \( A \subseteq X \), and \( \delta\text{-cl}\Psi(X) = Y \).

**Remark 5.2.6.** It is easy to observe that \((\Psi, Y)\) is a \( \delta \)-extension of a space \((X, \tau)\) iff \((\Psi, Y_s)\) is an extension of \((X, \tau_s)\).

We next define two typical grills which will turn out to play crucial roles throughout the rest of the chapter. To that end, we first see that a routine check establishes the following result.

**Theorem 5.2.7.** For each point \( x \) of a topological space \( X \), \( G(\delta, x) = \{ A \subseteq X : x \in \delta\text{-cl}A \} \) is a grill on \( X \).
Definition 5.2.8. For each $x \in X$, the grill $G(\delta, x)$, as defined above, is called the $\delta$-adherence grill at $x$ on $X$.

Definition 5.2.9. A grill $G$ on a space $X$ is called a $\delta$-grill if for any $A \subseteq X$, $\delta-cl A \in G \Rightarrow A \in G$.

Observation 5.2.10. Clearly, every $\delta$-adherence grill $G(\delta, x)$ on a space $X$ (where $x \in X$) is a $\delta$-grill on $X$. Indeed, for any $A \subseteq X$, $\delta-cl A \in G(\delta, x) \Rightarrow x \in \delta-cl(\delta-clA) = \delta-cl A \Rightarrow A \in G(\delta, x)$.

Definition 5.2.11. A space $(X, \tau)$ is said to be $\delta$-$T_0$ if for any two distinct points $x, y$ of $X$, there exists a regular open set $U$ in $X$ containing one of $x$ and $y$, and not the other.

Remark 5.2.12. A space $(X, \tau)$ is $\delta$-$T_q$ iff $(X, t_s)$ is $T_q$.

Theorem 5.2.13. A topological space $(X, \tau)$ is $\delta$-$T_0$ iff $\forall x, y \in X, G(\delta, x) = G(\delta, y) \Rightarrow x = y$.

Proof. Let $(X, \tau)$ be $\delta$-$T_0$ and $x, y \in X$ such that $x \neq y$. Then $x \notin \delta-cl\{y\}$ (say).

But as $y \in \delta-cl\{y\}$, $G(\delta, x) \neq G(\delta, y)$.

Conversely, let $x, y \in X$ with $x \neq y$. By hypothesis, there exists $A \subseteq X$ such that $A \in G(\delta, x)$ but $A \notin G(\delta, y)$ (say), i.e., $x \in \delta-cl A$ but $y \notin \delta-cl A$. Then there is a regular open set $U$ in $X$ such that $y \in U$ and $U \cap A = \emptyset$. But since $x \in \delta-cl A$, $x \notin U$ (as $x \in \delta-cl A$ and $x \in U \Rightarrow U \cap A \neq \emptyset$). Thus $X$ is $\delta$-$T_0$.

We now define a specific type of grills associated with each point of a $\delta$-extension.

Definition 5.2.14. Let $E = (\Psi, Y)$ be a $\delta$-extension of a topological space $X$. Then, for any $y \in Y$, the $\delta$-trace of $y$ on $Y$ is defined to be a collection $\tau(y, E)$ given by

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The collection \( \{r(y, E) : y \in Y\} \) is called the \( \delta \)-trace system of the extension \( E \) on \( X \).

**Theorem 5.2.15.** Let \( E = (\Psi, Y) \) be a \( \delta \)-extension of a topological space \( X \). Then for each \( y \in Y \), \( r(y, E) \) is a \( \delta \)-grill on \( X \).

**Proof.** That \( r(y, E) \) is a grill can be established by a routine check. To show that \( r(y, E) \) is a \( \delta \)-grill, let \( A \subseteq X \). Then \( \delta-clA \in r(y, E) \Rightarrow y \in \delta-cl\Psi(\delta-clA) = \delta-cl[\delta-cl\Psi(A) \cap \Psi(X)] \subseteq \delta-cl\Psi(A) \cap Y = \delta-cl(\Psi(A)) \Rightarrow A \in r(y, E) \). Hence \( r(y, E) \) is a \( \delta \)-grill on \( X \), for each \( y \in Y \).

**Theorem 5.2.16.** Let \( E = (\Psi, Y) \) be a \( \delta \)-extension of a topological space \( X \). Then for each \( x \in X \), \( r(\Psi(x), E) = G(\delta, x) \).

**Proof.** We have, \( r(\Psi(x), E) = \{A \subseteq X : \Psi(x) \in \delta-cl(\Psi(A))\} = \{A \subseteq X : \Psi(x) \in \delta-cl(\Psi(A)) \cap \Psi(X)\} = \{A \subseteq X : x \in \delta-clA\} \) (as \( \Psi \) is injective) = \( G(\delta, x) \).

**Theorem 5.2.17.** Let \( E = (\Psi, Y) \) be a \( T_2 \) \( \delta \)-extension of a \( T_2 \) topological space \( (X, \tau) \). Then \( \delta \)-traces of different points of \( Y \) are different.

**Proof.** Consider two distinct points \( y_1, y_2 \) of \( Y \). By Hausdorffness of \( Y \), there exist open sets \( A \) and \( B \) in \( Y \) such that \( y_1 \in A \), \( y_2 \in B \) and \( A \cap B = \emptyset \). Clearly \( y_2 \notin \delta-cl(\Psi(X)) \). We put \( U = \Psi^{-1}(A) \) which gives \( \Psi(U) = A \cap \Psi(X) \). Thus \( y_2 \notin \delta-cl\Psi(U) \) and hence \( U \notin r(y_2, E) \) \( \cdots (1) \).

If possible, let \( y_1 \notin \delta-cl\Psi(U) = \delta-cl(\Psi(X)) \). Since \( \delta-cl(\Psi(X)) \) is a closed set, \( A \setminus \delta-cl(\Psi(X)) \) is an open nbhd of \( y_1 \). Then \( cl(A \setminus \delta-cl(A \cap \Psi(X))) = \delta-cl(A \setminus \delta-cl(A \cap \Psi(X)) \subseteq A \setminus \Psi(X) \). So, intcl(\( A \setminus \delta-cl(\Psi(X)) \)) \( \subseteq Y \setminus \Psi(X) \), which gives \( intcl(A \setminus \delta-cl(\Psi(X))) \cap \Psi(X) = \emptyset \), contradicting the fact that \( \delta-cl\Psi(X) = Y \).
Thus \( y_1 \in \delta\text{-cl}(\Psi(X) \cap A) = \delta\text{-cl}(\Psi(U)) \), i.e., \( U \in \tau(y_1, E) \cdots (2) \).

From (1) and (2) it follows that \( \tau(y_1, E) \neq \tau(y_2, E) \).

**Definition 5.2.18.** Two \( \delta \)-extensions \((\Psi_1, Y_1)\) and \((\Psi_2, Y_2)\) of a topological space \( X \) are called \( \delta \)-equivalent if there is some \( \delta \)-homeomorphism \( f : Y_1 \to Y_2 \) such that \( f \circ \Psi_1 = \Psi_2 \).

Our aim now is to show that for a certain class of topological space, the \( \delta \)-equivalence of \( \delta \)-extensions of such a space can be characterized in terms of the concept of \( \delta \)-trace systems. The following theorem is the first step towards that direction.

**Theorem 5.2.19.** If two \( \delta \)-extensions of a topological space are \( \delta \)-equivalent, then the extensions have the same \( \delta \)-trace systems.

**Proof.** Let \( E_1 = (\Psi_1, Y_1) \) and \( E_2 = (\Psi_2, Y_2) \) be two \( \delta \)-equivalent \( \delta \)-extensions of a given topological space \( X \). Then there exists a \( \delta \)-homeomorphism \( f : Y_1 \to Y_2 \) such that \( f \circ \Psi_1 = \Psi_2 \cdots (1) \).

Then for any \( y \in Y_1 \), \( A \in \tau(y, E_1) \iff y \in \delta\text{-cl}(\Psi_1(A)) \iff f(y) \in f(\delta\text{-cl}(\Psi_1(A))) = \delta\text{-cl} f(\Psi_1(A)) \) (as \( f \) is a \( \delta \)-homeomorphism) \( \iff f(y) \in \delta\text{-cl} \Psi_2(A) \) (by (1)) \( \iff A \in \tau(f(y), E_2) \). Thus \( \tau(y, E_1) = \tau(f(y), E_2) \), for each \( y \in Y_1 \). Since \( f \) is a bijection, it then follows that \( \{\tau(y, E_1) : y \in Y_1\} = \{\tau(z, E_2) : z \in Y_2\} \).

In order to achieve the converse of the theorem above, we require to introduce certain new concepts, which we explain in the sequel.

**Definition 5.2.20.** A collection \( B \) of \( \delta \)-closed sets in a space \( X \) is called a \( \delta \)-base for \( X \) if each \( \delta \)-closed set is expressible as an intersection of some members of \( B \).

Clearly the above definition could equivalently be given as follows.

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Theorem 5.2.21. A collection $B$ of $\delta$-closed sets in a topological space $X$ is a $\delta$-base of $X$ iff for each $\delta$-closed set $F$ and each point $x \notin F$, there exists $B \in B$ such that $x \notin B$ and $F \subseteq B$.

Theorem 5.2.22. Let $f : X \rightarrow Y$ be a bijection. If $f$ maps a $\delta$-base of $X$ onto a $\delta$-base of $Y$, then $f$ is a $\delta$-homeomorphism.

Proof. Let $B$ be a base of $X$ and $f(B) = \{f(B) : B \in B\}$ be a $\delta$-base of $Y$. In order to show that $f$ is a $\delta$-homeomorphism, it suffices to show that $f(\delta-cl(A)) = \delta-clf(A), \forall A \subseteq X$.

Consider any $A \subseteq X$ and $y \notin \delta-clf(A)$. Since $f(B)$ is a $\delta$-base of $Y$, $\delta-clf(A) = \cap\{f(B_\alpha) : B_\alpha \in B_0\}$, where $B_0 \subseteq B$. Then $y \notin f(B_\alpha)$ for some $B_\alpha \in B_0$. Now, $f(A) \subseteq \delta-clf(A) \subseteq f(B_\alpha) \Rightarrow A \subseteq B_\alpha$ (as $f$ is a bijection) $\Rightarrow \delta-clA \subseteq \delta-clB_\alpha = B_\alpha$ (since $B_\alpha$ is $\delta$-closed). But $y \notin f(B_\alpha)$, so that $y \notin f(\delta-clA)$.

Hence $f(\delta-clA) \subseteq \delta-clf(A)$ \cdots (1).

Now let $y \notin f(\delta-clA)$. Since $\delta-clA = \cap\{B_\alpha : B_\alpha \in B_0\}$ where $B_0 \subseteq B$, $y \notin f(\delta-clA) = f(\cap\{B_\alpha : B_\alpha \in B_0\}) = \cap\{f(B_\alpha) : B_\alpha \in B_0\}$ (since $f$ is a bijection) $\Rightarrow y \notin f(B_\beta)$, for some $B_\beta \in B_0 \subseteq B$.

Again, $A \subseteq \delta-clA \subseteq B_\beta \Rightarrow \delta-clf(A) \subseteq \delta-clf(B_\beta) = f(B_\beta)$ ($f(B)$ being a $\delta$-base in $Y$) $\Rightarrow y \notin \delta-clf(A)$. Thus $\delta-clf(A) \subseteq f(\delta-clA)$ \cdots (2).

From (1) and (2) the result follows.

Analogous to the concept of complete regularity, we now define as follows.

Definition 5.2.23. A topological space $X$ is said to be $\delta$-completely regular if for each $\delta$-closed set $F$ in $X$ and each $x \in X \setminus F$, there exists a $\delta$-continuous function $f : X \rightarrow \mathbb{R}$ such that $f(x) = 1$ and $f(F) = \{0\}$.

Theorem 5.2.24. Let $(\Psi, Y)$ be a $\delta$-completely regular $\delta$-extension of a $\delta$-completely regular topological space $X$. Then the family $\{\delta-cl\Psi(A) : A \subseteq X\}$ constitutes a
Proof. Consider any \( \delta \)-closed set \( F \) in \( Y \) and \( y \in Y \setminus F \). Then by \( \delta \)-complete regularity of \( Y \), there exists a \( \delta \)-continuous function \( f : Y \to \mathbb{R} \) such that \( f(y) = 1 \) and \( f(F) = \{0\} \). Define \( B = \{ z \in \Psi(X) : f(z) \leq 1/2 \} \). We shall show that \( F \subseteq \delta \text{-cl} B \) and \( y \notin \delta \text{-cl} B \).

It is clear that \( y \notin \delta \text{-cl} B \) (since \( y \in f^{-1}(2/3, \infty) \) such that \( \text{int}(f^{-1}(2/3, \infty)) \cap B = \emptyset \)). If possible, let \( F \subseteq \delta \text{-cl} B \). Then there exists some \( z \in F \) such that \( z \notin \delta \text{-cl} B \). Thus there is an open set \( U \) containing \( z \) such that \( \text{int}(U \cap B) = \emptyset \).

Now, \( U \cap f^{-1}(-\infty, 1/2) \) is an open set containing \( z \) (as \( f(z) = 0 \)) such that \( \text{cl}(U \cap f^{-1}(-\infty, 1/2)) \subseteq \text{cl}(U \cap f^{-1}(-\infty, 1/2]) \subseteq \text{int}(U \cap f^{-1}(-\infty, 1/2]) \) and hence \( \text{int}(U \cap f^{-1}(-\infty, 1/2]) \cap \Psi(X) \subseteq \text{int}(U \cap f^{-1}(-\infty, 1/2]) \cap \Psi(X) \subseteq \text{int}(U \cap f^{-1}(-\infty, 1/2]) \cap \Psi(X) = \emptyset \), which contradicts that \( \delta \text{-cl} \Psi(X) = Y \).

So, \( F \subseteq \delta \text{-cl} B \). Let us choose \( A = \Psi^{-1}(B) \). Then \( \Psi(A) = B \) and \( F \subseteq \delta \text{-cl} \Psi(A) \) with \( y \notin \delta \text{-cl} \Psi(A) \). By Theorem 5.2.21, it follows that \( \{ \delta \text{-cl} \Psi(A) : A \subseteq X \} \) is a \( \delta \)-base for \( Y \).

We are now in a position to prove the following result towards the converse of Theorem 5.2.19.

**Theorem 5.2.25.** Let \( X \) be a \( T_2 \), \( \delta \)-completely regular space and \( E_1 = (\Psi_1, Y_1) \) and \( E_2 = (\Psi_2, Y_2) \) be two \( T_2 \), \( \delta \)-completely regular \( \delta \)-extensions of \( X \) such that \( E_1 \) and \( E_2 \) have identical \( \delta \)-trace systems. Then \( E_1 \) and \( E_2 \) are \( \delta \)-equivalent.

**Proof.** Since \( \{ \tau(y, E_1) : y \in Y_1 \} = \{ \tau(z, E_2) : z \in Y_2 \} \), for each \( y \in Y_1 \) there exists a unique (by virtue of Theorem 5.2.17) \( z \in Y_2 \) such that

\[
\tau(y, E_1) = \tau(z, E_2) \quad \cdots (1).
\]

Define a map \( f : Y_1 \to Y_2 \) given by \( f(y) = z \), where \( \tau(y, E_1) = \tau(z, E_2) \). From (1) it follows that \( f \) is a bijection of \( Y_1 \) onto \( Y_2 \), and

\[
\tau(y, E_1) = \tau(f(y), E_2), \forall y \in Y_1 \quad \cdots (2).
\]
In particular, for each \( x \in X \), \( \tau(\Psi_1(x), E_1) = \tau(f(\Psi_1(x)), E_2) \). But by Theorem 5.2.16, \( \tau(\Psi_1(x), E_1) = G(\delta, x) = \tau(f(\Psi_1(x)), E_2) \), and \( \tau(\Psi_2(x), E_2) = G(\delta, x) \). Hence \( \tau(f(\Psi_1(x), E_2)) = \tau(\Psi_2(x), E_2) \). Then by Theorem 5.2.17, \( f(\Psi_1(x)) = \Psi_2(x) \), for each \( x \in X \). Thus \( f(\Psi) = \Psi_2 \).

Now for any \( A \subseteq X \), \( y \in \delta-cl \Psi_1(A) \Leftrightarrow f(y) \in \delta-cl \Psi_2(A) \) (by (2)). So, \( f(\delta-cl \Psi_1(A)) = \delta-cl \Psi_2(A) \), \( \forall A \subseteq X \). Since by Theorem 5.2.24, \( \{\delta-cl \Psi_1(A) : A \subseteq X \} \) and \( \{\delta-cl \Psi_2(A) : A \subseteq X \} \) are \( \delta \)-bases of \( E_1 \) and \( E_2 \) respectively, it follows by Theorem 5.2.22 that \( f \) is a \( \delta \)-homeomorphism.

From Theorems 5.2.19 and 5.2.25 it follows that

**Corollary 5.2.26.** Two \( T_2 \), \( \delta \)-completely regular \( \delta \)-extensions \( E_1 = (\Psi_1, Y_1) \) and \( E_2 = (\Psi_2, Y_2) \) of a \( T_2 \), \( \delta \)-completely regular space \( X \) are \( \delta \)-equivalent iff \( E_1 \) and \( E_2 \) have identical \( \delta \)-trace systems.

### 5.3. A TYPICAL \( \delta \)-EXTENSION AND LINKAGE NEAR COMPACTNESS

In this section we shall introduce a special type of \( \delta \)-extension which will be called \( \delta \)-principal extension. We shall give an explicit construction of such an extension and show that this extension satisfies a kind of covering property if the grills, under consideration, are chosen suitably.

**Definition 5.3.1.** A \( \delta \)-extension \( E = (\Psi, Y) \) of a topological space \( X \) is called a \( \delta \)-principal extension of \( X \) if the following two conditions hold:

(i) For all \( y_1, y_2 \in Y \), \( \tau(y_1, E) = \tau(y_2, E) \Rightarrow y_1 = y_2 \)

(ii) \( \{\delta-cl \Psi(A) : A \subseteq X \} \) is a \( \delta \)-base for \( Y \).
Remark 5.3.2. By virtue of Theorems 5.2.17 and 5.2.24 it follows that any Hausdorff $\delta$-completely regular $\delta$-extension of any $\delta$-completely regular Hausdorff space is an example of a $\delta$-principal extension.

Looking at the proof of Theorem 5.2.25 it can be easily seen that

Theorem 5.3.3. Let $E_1$ and $E_2$ be two $\delta$-principal extensions of a topological space $X$. If $E_1$ and $E_2$ have the identical $\delta$-trace systems then $E_1$ and $E_2$ are $\delta$-equivalent; i.e., $\delta$-principal extensions are uniquely determined by their $\delta$-trace systems.

From Theorem 5.2.13 it follows at once that any topological space admitting a $\delta$-principal extension is necessarily $\delta$-$T_0$. In fact, we have even more:

Theorem 5.3.4. Every $\delta$-principal extension $E = (\Psi, Y)$ of a space $X$ is a $\delta$-$T_0$ space.

Proof. For any $y_1, y_2 \in Y$ we have, $G(\delta, y_1) = G(\delta, y_2) \Rightarrow \tau(y_1, E) = \{A \subseteq X : y_1 \in \delta-cl\Psi(A)\} = \{A \subseteq X : \Psi(A) \in G(\delta, y_1)\} = \{A \subseteq X : \Psi(A) \in G(\delta, y_2)\} = \tau(y_2, E) \Rightarrow y_1 = y_2$. The rest follows from Theorem 5.2.13.

We now give a method for construction of $\delta$-principal extension of a $\delta$-$T_0$ topological space.

Theorem 5.3.5. Let $E = (\Psi, Y)$ be a $\delta$-principal extension of a $\delta$-$T_0$ topological space $X$. Then $E$ is $\delta$-homeomorphic to a $\delta$-extension $E^* = (g, X^*)$ of $X$ where $X^*$ is a collection of $\delta$-grills on $X$ containing all the $\delta$-adherence grills.

Proof. Let $X^* = \{\tau(y, E) : y \in Y\} \equiv \delta$-trace system of $E$ on $X$. Then by Theorems 5.3.15 and 5.2.16, $X^*$ is a collection of $\delta$-grills, containing all the $\delta$-adherence grills in $X$. We define a mapping $g : X \rightarrow X^*$ given by $g(x) = G(\delta, x)$. Now for any two distinct points $x_1, x_2$ of $X$, we have $G(\delta, x_1) \neq G(\delta, x_2)$ as $X$ is
\(\delta-T_0\) (by Theorem 5.2.13) and hence \(g\) is injective.

Let us put \(A^C = \{G \in X^*: A \in G\}\) for any \(A \subseteq X\), and \(B = \{A^C: A \subseteq X\}\). It is clear that \(\Phi^C = \Phi\) and \((A \cup B)^C = A^C \cup B^C\) for any \(A, B \subseteq X\). Thus \(B\) forms a base for closed sets for some topology on \(X^*\). The Kuratowski closure operator, inducing this topology, is given by \(cl(\alpha) = \cap\{A^C: \alpha \subseteq A^C, A \subseteq X\}, \forall \alpha \subseteq X^*\).

We have the following properties:

(i) \(g(\delta-clA) = A^C \cap g(X)\):

Indeed, \(g(x) \in g(\delta-clA) \iff x \in \delta-clA\) (since \(g\) is injective) \(\iff A \in G(\delta, x) = g(x) \iff g(x) \in A^C \cap g(X)\).

(ii) \(g(A) \subseteq A^C\): This is a consequence of (i).

(iii) For any \(A \subseteq X\), \(cl(g(A)) = A^C\): In view of (ii), it is enough to show that whenever \(g(A) \subseteq B^C\) for some subset \(B\) of \(X\), then \(A^C \subseteq B^C\). So, we assume that \(g(A) \subseteq B^C\) for some \(B \subseteq X\). Then, \(x \in A \Rightarrow G(\delta, x) = g(x) \in B^C \Rightarrow B \in G(\delta, x) \Rightarrow x \in \delta-clB\). Thus \(A \subseteq \delta-clB\). This implies that \(A^C \subseteq (\delta-clB)^C = B^C\).

For, \(G \in B^C \iff B \in G \iff \delta-clB \in G\) (as in Observation 5.2.10 it can be shown that every element of \(X^*\) is a \(\delta\)-grill) \(\iff G \in (\delta-clB)^C\).

From (i) and (iii) we obtain, \(g(\delta-clA) = cl(g(A)) \cap g(X) \cdots (1)\).

If we denote the \(\delta\)-topology of \(X\) by \(\tau\), then \((X, \tau)\) is homeomorphic to \(g(X)\) (in view of (1)). We know that \((X, \tau)\) is \(\delta\)-equivalent to \((X, \tau)\) (as \((\tau_\delta) = \tau)\) [121]).

So, \((X, \tau)\) is \(\delta\)-homeomorphic to \(g(X)\) (by Theorem 5.2.4 (b)). In view of (iii) we have, \(\delta-clg(X) \supseteq clg(X) = X^C = X^*\). Hence \(E^* = (g, X^*)\) is a \(\delta\)-extension of \((X, \tau)\).

Now, we define a map \(f: Y \rightarrow X^*\) by \(f(y) = \tau(y, E)\). As \(Y\) is a \(\delta\)-principal extension of \(X\), \(f\) is a bijection. Again, for any \(A \subseteq X\), \(f(\delta-cl\Psi(A)) = A^C\); indeed, \(y \in \delta-cl\Psi(A) \iff A \in \tau(y, E) \iff \tau(y, E) \in A^C\) so that \(f(\delta-cl\Psi(A)) = A^C\).

Since \(\{\delta-cl\Psi(A): A \subseteq X\}\) is a \(\delta\)-base for \(Y\), the semiregularization space of \(Y\) is
homeomorphic to \( X^* \), and hence \( Y \) is \( \delta \)-homeomorphic to \( X^* \).

We now like to point out that as a particularization of our unified theory of \( (P, G) \)-compactness in Chapter 2, one can characterize near compactness of a topological space in terms of grills. To do this, we need to frame the following definition analogous to the corresponding definition of the convergence of grills.

**Definition 5.3.6.** A grill \( G \) on a topological space \( X \) is said to \( \delta \)-converge to a point \( x \) of \( X \) if for each regular open set \( U \) containing \( x \), there is some \( G \in \mathcal{G} \) such that \( G \subseteq U \).

It follows by Observation 2.2.8 and Theorem 2.2.20 that

**Theorem 5.3.7.** A topological space \( (X, \tau) \) is nearly compact iff every grill \( G \) on \( X \) \( \delta \)-converges.

Our intention here is to define a type of covering property, allied to near compactness, and to ultimately show that the \( \delta \)-principal extension, constructed in Theorem 5.3.5, satisfies the proposed covering property when the grills in \( X^* \) satisfies a suitable restriction.

**Definition 5.3.8.** Let \( (\mathcal{G}, X^*) \) be the \( \delta \)-extension of a topological space \( X \), as obtained in Theorem 5.3.5. A collection of grills \( \{G_\alpha : \alpha \in \Lambda\} \) on \( X \) is said to be a \( \delta \)-binary collection of grills on \( X \), if for any grill \( G \) on \( X \), whenever each two membered subfamily \( \mathcal{A} \) of \( G \) is contained in some \( G_\alpha \), then \( \exists \beta \in \Lambda \) such that \( G_\beta \subseteq \bigcap_{G \in \mathcal{G}} \delta-clG \).

**Definition 5.3.9.** [25] A grill \( G \) on a topological space \( X \) is called a linked grill if for any two members \( A, B \) of \( G \), \( clA \cap clB \neq \Phi \).

**Definition 5.3.10.** A topological space \( X \) is called linkage nearly compact if for every linked grill \( G \) on \( X \), there exists a point \( x \) in \( X \) such that \( x \in \bigcap_{G \in \mathcal{G}} \delta-clG \).
Theorem 5.3.11. The $\delta$-principal extension $(g, X^*)$ of a $\delta$-$T_0$ topological space $X$ (as observed in Theorem 5.3.5) is linkage nearly compact iff $X^*$ is a $\delta$-binary collection of grills on $X$.

Proof. Let us first assume that $X^*$ is linkage nearly compact. Let $G$ be a grill on $X$ such that each two membered subfamily of $G$ is contained in some member of $X^*$. Then for any two given $A, B \in G$, there exists some $H \in X^*$ such that $A, B \in H$, so that $H \in A^C \cap B^C$. Thus for all $A, B \in G$, $A^C \cap B^C \neq \Phi$.

Let us define $A = \{ \alpha \subseteq X^* : g(A) \subseteq \alpha$ for some $A \in G \}$. It is clear that $\Phi \notin A$ and $A$ is closed under the formation of supersets. To show that $A$ is a grill on $X^*$, let $\alpha, \beta$ be any two subsets of $X^*$ such that $\alpha \cup \beta \in A$. Then by construction of $A$, there exists some $A \in G$ with $g(A) \subseteq \alpha \cup \beta$.

Let us put $A_1 = \{ x \in A : g(x) \in \alpha \}$ and $A_2 = \{ x \in A : g(x) \in \beta \}$. Then $A_1 \cup A_2 = A \in G$ so that $A_1 \in G$ or $A_2 \in G$, which in turn implies that $g(A_1) \subseteq \alpha \in A$ or $g(A_2) \subseteq \beta \in A$. It now follows that $A$ is a grill on $X^*$.

We next observe that $A$ is a linked grill on $X^*$. In fact, $A^C = \text{cl}(g(A))$ (see (iii) of the proof of theorem 5.3.5) $\subseteq \text{cl} \alpha$, and $B^C = \text{cl}(g(B)) \subseteq \text{cl} \beta$, so that $\Phi \neq A^C \cap B^C \subseteq \text{cl}(\alpha) \cap \text{cl}(\beta)$, proving that $\text{cl}(\alpha) \cap \text{cl}(\beta) \neq \Phi$. Thus $A$ is a linked grill on $X^*$. As $X^*$ is linkage nearly compact, $\exists G_0 \in X^*$ such that $G_0 \in \bigcap_{A \in A} \delta-\text{cl} A$, i.e., $A \in G(\delta, G_0)$, $\forall A \in A$, and hence $A \subseteq G(\delta, G_0)$.

Now, $A \in G \Rightarrow g(A) \in A \Rightarrow g(A) \in G(\delta, G_0) \Rightarrow G_0 \in \delta-\text{cl}(g(A))$. Thus $G_0 \in \bigcap_{A \in G} \delta-\text{cl}(A)$, so that $X^*$ becomes a $\delta$-binary collection of grills.

Conversely, let $X^*$ be a $\delta$-binary collection of grills on $X$. We shall show that $X^*$ is linkage nearly compact.
Let \( A \) be any linked grill on \( X^* \). We are to find some \( G_0 \in X^* \) such that \( A \subseteq G(\delta, G_0) \). Let us put \( A_1 = \{ A \subseteq X : g(A) \in A \} \) and \( A_2 = \{ A \subseteq X : A^C \setminus g(X) \in A \} \).

It can be checked that \( A_1 \) and \( A_2 \) are grills on \( X \), so that \( A_1 \cup A_2 \) (\( = A^* \), say) is also a grill on \( X \). Now, let \( \{ A, B \} \subseteq A^* \) be any two membered family in \( A^* \).

We consider the following cases to show that there exists some \( \Omega \in X^* \) such that \( A, B \in \Omega \).

Case 1: \( \{ A, B \} \subseteq A_1 \). Then \( g(A), g(B) \in A \Rightarrow \text{cl}(g(A)) \cap \text{cl}(g(B)) \neq \Phi \) (as \( A \) is a linked grill) \( \Rightarrow A^C \cap B^C \neq \Phi \Rightarrow \exists \Omega \in A^C \cap B^C \Rightarrow A \in \Omega \) and \( B \in \Omega \).

Case 2: \( \{ A, B \} \subseteq A_2 \). Then \( A^C \setminus g(X), B^C \setminus g(X) \in A \Rightarrow \text{cl}(A^C \setminus (g(X))) \cap \text{cl}(B^C \setminus (g(X))) \neq \Phi \Rightarrow \text{cl}(A^C) \cap \text{cl}(B^C) \neq \Phi \Rightarrow A^C \cap B^C \neq \Phi \) (see (iii) of the proof of Theorem 5.3.5) \( \Rightarrow \exists \Omega \in A^C \cap B^C \Rightarrow A, B \in \Omega \).

Case 3: \( A \in A_1 \) and \( B \in A_2 \). Then \( g(A) \in A \) and \( B^C \setminus g(X) \in A \Rightarrow \text{cl}(g(A)) \cap \text{cl}(B^C \setminus (g(X))) \neq \Phi \Rightarrow \text{cl}(g(A)) \cap \text{cl}(B^C) \neq \Phi \Rightarrow A^C \cap B^C \neq \Phi \Rightarrow \exists \Omega \in A^C \cap B^C \Rightarrow A, B \in \Omega \).

Since \( X^* \) is a \( \delta \)-binary collection of grills, there exists some \( G_0 \in X^* \) such that \( G_0 \in \delta - \text{cl}(K), \forall K \in A^* \cdots (1) \)

It now suffices to show that \( A \subseteq G(\delta, G_0) \). For this let \( \alpha \in A \) so that \( \alpha \in X^* \).

Now, \( \alpha = (\alpha \cap g(X)) \cup (\alpha \setminus g(X)) \) so that we have \( \alpha \cap g(X) \in A \) or \( \alpha \setminus g(X) \in A \). If \( \alpha \cap g(X) \in A \) then \( g^{-1}(\alpha) \in A_1 \). So by (1), \( G_0 \in \delta - \text{cl} \alpha \), and hence \( \alpha \in G(\delta, G_0) \), proving that \( A \subseteq G(\delta, G_0) \). The case when \( \alpha \setminus g(X) \in A \) can suitably be tackled.

This completes the proof.