CHAPTER II

OPERATIONAL DERIVATION OF GENERATING FUNCTIONS FOR CLASSICAL POLYNOMIALS AND THEIR GENERALIZATIONS
In this chapter we shall derive generating functions of certain special functions with the use of operational technique. The operational technique is frequently very useful as well as elegant. In 1941, J.L. Burchnall introduced the following operational formula for the Hermite polynomials:

\[(D - 2x)^n \frac{n!}{r! (n-r)!} H_{n-r}(x) \frac{D^r}{Dx}, D = \frac{d}{dx}\]

Later in 1960, L. Carlitz introduced the operational formula for the Laguerre polynomials in the form:

\[\sum_{r=0}^{n} \frac{n!}{r! (n-r)!} L_{n-r}^{(\alpha+r)}(x) \frac{D^r}{Dx}\]

Burchnall and Carlitz derived many properties of Hermite and Laguerre polynomials respectively from this viewpoint. In 1969, S.K. Chatterjea observed that the Rodrigues' formula for certain classical polynomials was not so convenient for obtaining a class of bilateral generating functions, especially when applications were considered from operational point of view, because the expression under \(D^n\) in Rodrigues formula would depend upon 'n' in general. Carrying out the technique adopted by Chatterjea we have found unilateral and bilateral generating functions for classical polynomials and their generalizations.
A GENERAL THEOREMS FOR BILINEAR GENERATING FUNCTIONS

In this section we shall furnish a general bilinear generating relation for
\[ f_n(x) = \mu(n) G(x) D^n g(x) \]
where \(g(x)\) is of the form \(x^n w(x)\).

Following the method of S.K.Chatterjea \cite{1}, S.Saran \cite{3} has recently proved the following general theorem for bilinear generating functions:

If
\[ (2.1.1) \quad f_n(x) = \mu(n) G(x) D^n G(x) \quad D = \frac{d}{dx} \]
where \(g(x)\) and \(G(x)\) are independent of \(n\),

and
\[ (2.1.2) \quad F(x,t) = \sum_{m=0}^{\infty} a_m t^m f_m(x) \]

then
\[ (2.1.3) \quad \frac{G(x) F(x-t,ty)}{G(x-t)} = \sum_{r=0}^{\infty} \frac{(-t)^r}{r!} b_r(y) f_r(x), \]
where
\[ b_r(y) = \sum_{m=0}^{r} (-1)^m \mu(m) a_m y^m \]

The interest of such a theorem lies in the fact that in the theory of classical orthogonal polynomials, every polynomial system \(\{P_n(x)\}\) satisfies a generalized Rodrigues' formula.
\[ (2.1.4) \quad P_n(x) = \left\{ k_n w(x) \right\}^{-1} D^n\{w(x) X^n\} \]

where \( k_n \) is a constant, \( w(x) \) is independent of \( n \) and \( X \) is a polynomials of degree at most two in \( x \) whose coefficients are independent of \( n \), and therefore Saran's theorem is only applicable when the expression under \( D^n \) is independent of \( n \) and for this reason he has shown applications on \( H_n(x) \), the Hermite polynomials; \( P_n(\alpha-n, \beta-n) \) the Jacobi polynomials with parameters \( \alpha = \alpha-n \) and \( \beta = \beta-n \); \( L_n(\alpha-n) \) the Laguerre polynomials with parameters \( \alpha = \alpha-n \); without thinking of the applications on the proper Jacobi and Laguerre polynomials viz \( P_n(\alpha, \beta) \) and \( L_n(\alpha) \).

Here we prove the following theorem:

Theorem:

If
\[ (2.1.5) \quad P_n(x) = \left\{ k_n W(x) \right\}^{-1} D^n\{W(x) X^n\} \]

where \( k_n \) is a constant and \( W(x) \) is independent of \( n \), and

\[ (2.1.6) \quad G(x,t) = \sum_{m=0}^{\infty} a_m t^m P_m(x) \]

then

\[ (2.1.7) \quad G\left(\frac{x}{1-xt}, \frac{txy}{1-xt}\right) = \sum_{r=0}^{\infty} \frac{k_{r}(tx)^r}{r!} b_r(y) P_r(x) \]
It may be noted that our theorem can be directly applied to $L_n^{(\alpha)}(x)$, $P_n^{(\alpha-n, \beta)}$ and $Y_n(x, a-n, b)$, where $Y_n(x, a, b)$ is the generalized Bessel polynomial introduced by H.L. Krall and O. Frink [4].

(A). Proof of the theorem:
Replacing $\frac{1}{x}$ for $x$ and $t/x$ for $t$ in (2.1.6)
we obtain
$$G\left(\frac{1}{x}, \frac{t}{x}\right) = \sum_{m=0}^{\infty} a_m t^m x^{-m} P_m \left(\frac{1}{x}\right)$$
Again replacing $ty$ for $t$ we have
(2.1.8) $G(x, \frac{ty}{x}) = \sum_{m=0}^{\infty} a_m t^m y^m x^{-m} P_m \left(\frac{1}{x}\right)$

Now if $P_n(x) = \frac{1}{k_n w(x)} D^n (w(x) x^n)$
we have on replacing $x$ by $\frac{1}{z}$
$$P_n\left(\frac{1}{z}\right) = \frac{(-1)^n}{k_n w\left(\frac{1}{z}\right)} (z^2 \frac{d}{dz})^n \left[ w\left(\frac{1}{z}\right)\left(\frac{1}{z}\right)^n \right]$$
$$= \frac{(-1)^n}{k_n w\left(\frac{1}{z}\right)} \left(z \frac{d}{dz}\right)\left(z \frac{d}{dz}\right) ... \left(z \frac{d}{dz}\right) w\left(\frac{1}{z}\right)\left(\frac{1}{z}\right)^n$$
$$= \frac{(-1)^n}{k_n w\left(\frac{1}{z}\right)} z \partial (\partial - 1) (\partial - 2) ... (\partial - n+1) z^{n-1} w\left(\frac{1}{z}\right)\left(\frac{1}{z}\right)^n$$
where \( \mathcal{D} = z \frac{d}{dz} \)
\[
= \frac{(-1)^n}{k_n w(\frac{1}{z})} z^{n+1} \left( \frac{d}{dz} \right)^n \left( w(\frac{1}{z}) \frac{1}{z} \right)
\]

So that we have
\[
(2.1.9) \quad x^{-n} p_n \left( \frac{1}{x} \right) = \frac{(-1)^n}{k_n w(\frac{1}{x})} x^D^n \left( w(\frac{1}{x}) \frac{1}{x} \right)
\]

Now (2.1.8) becomes
\[
G \left( \frac{1}{x}, \frac{t^y}{x} \right) = \sum_{m=0}^{\infty} \frac{a_m t^m y^m (-1)^m x^D^m \left( w(\frac{1}{x}) \frac{1}{x} \right)}{k_m w(\frac{1}{x})}
\]

Thus we obtain
\[
W \left( \frac{1}{x-t} \right) \frac{1}{x-t} G \left( \frac{1}{x-t}, \frac{ty}{x-t} \right) = \sum_{m=0}^{\infty} \frac{a_m t^m y^m (-1)^m D^m \left( w(\frac{1}{x}) \frac{1}{x} \right)}{k_m}
\]

Applying the operator \( e^{-tD} \) and noticing that \( e^{-tD} f(x) = f(x-t) \)

we obtain,
\[
W \left( \frac{1}{x-t} \right) \frac{1}{x-t} G \left( \frac{1}{x-t}, \frac{ty}{x-t} \right)
\]
\[
= \sum_{r=0}^{\infty} \frac{(-tD)^r}{r!} \sum_{m=0}^{\infty} \frac{a_m t^m y^m (-1)^m D^m \left[ W \left( \frac{1}{x} \right) \frac{1}{x} \right]}{k_m}
\]
Now using (2.1.9) and replacing $x$ by $\frac{1}{x}$ we obtain (2.1.7)

(B). Some applications of the theorem.

(a) First we consider the well-known generating function for Laguerre polynomial.

\[
\sum_{n=0}^{\infty} \frac{(c)_n}{(1+\alpha)_n} t^n L_n^{(\alpha)}(x) = (1-t)^{-c} {}_1F_1 \left( c; 1 + \alpha; \frac{-xt}{1-t} \right)
\]

If we take $P_n(x) = L_n^{(\alpha)}(x)$, $k_n = n!$

\[
w(x) = xe^{-x} ; \quad a_n = \frac{(c)_n}{(1+\alpha)_n}
\]

and $G(x,t) = (1-t)^{-c} {}_1F_1 \left( c; 1 + \alpha; \frac{-xt}{1-t} \right)$

then with the help of (2.1.7) we obtain,
\((-xt)^{-\alpha+c-1} (1-xt-xy)^{-c} \exp \left( \frac{x^2y}{1-xt} \right) \).

\[
\begin{align*}
\left. _1F_1 \left( c; 1+\alpha; \frac{-x^2y}{(1-xt)(1-xt-xy)} \right) \right|_{(1-xt-txy)}^\infty \\
= \sum_{r=0}^{\infty} 2F_1 \left[ \begin{array}{c}
-r, c \\
1+\alpha;
\end{array} \right] L_n^{(\alpha)}(x) (tx)^r
\end{align*}
\]

Replacing \(xt\) by \(t\) and \(y\) by \(-y\), the above reduces to the following formula due to L. Weisner [5]

\[
\begin{align*}
(1-t)^{-\alpha+c-1} (1-t+ty)^{-c} \exp \left( \frac{-xt}{1-t} \right) _1F_1 \left[ \begin{array}{c}
c \\
1+\alpha;
\end{array} \right] L_n^{(\alpha)}(x) (t)^r.
\end{align*}
\]

(b) Generalized Bessel polynomials \(Y_n(x; a-n, b)\)

can be defined by

\[
Y_n(x; a-n, b) = b^{-n} x^{2-a+n} e^{b/x} D^n (x^{n+e-2} e^{-b/x})
\]

and the generating function for the polynomials

\(Y_n(x; a-n, b)\) due to Chatterjea [6] is
Replacing \( t \) by \( t/x \) we have

\[
\sum_{n=0}^{\infty} \frac{b^n x^{-n} t^n}{n!} Y_n(x, a-n, b) = (1-xt)^{1-a} e^{bt/x}
\]

In this case

\[
W(x) = x^{e-2} e^{-b/x},
\]

\[
K_n = b^n,
\]

\[
P_n = x^{-n} Y_n(x; a-n, b),
\]

\[
a_n = \frac{b^n}{n!}
\]

and \( G(x,t) = (1-t)^{1-a} e^{bt/x} \)

Now applying our result in (2.1.7) we obtain

\[
(1-xt-xy)^{1-a} e^{bt(y+1)} = \sum_{r=0}^{\infty} b^r B_r(y) Y_r(x; a-r, b) \frac{t^r}{r!}
\]

where \( B_r(y) = \sum_{n=0}^{\gamma^r} (-1)^m (-r)^m \frac{y^m}{m!} \)

(C). Next we consider the generating function

\[
(1-t)^{-1} {}_2F_0 \left[ 1, \alpha+1; -\frac{\alpha}{1-t} \right] = \sum_{n=0}^{\infty} Y_n(\alpha-n)(x) t^n
\]
where \( Y_n^{(x)}(x) = Y_n(x, \alpha + 2, 2) \) in the notation of Krall and Frink [4]

Changing \( t \) by \( t/x \) we have

\[
(1 - t/x)^{-1} \ _2F_0 \left[ \begin{array}{c} 1, \alpha + 1; \frac{t}{1-t/x} \end{array} \right] = \sum_{n=0}^{\infty} x^{-n} Y_n^{(\alpha-n)}(x)t^n
\]

Also replacing \( a \) by \( \alpha+2 \) and \( b \) by \( 2 \) in \( Y_n(x; a, b) \)

and then writing \( \alpha-n \) for \( \alpha \),

we obtain

\[
y_n^{\alpha-n}(x) = 2^n x^{-\alpha+n} e^{2/x} D^n (x^n e^{-2/x})
\]

Here \( W(x) = x^\alpha e^{-2/x} ; a_n = 1 \)

\( K_n = 2^n ; p_n(x) = x^{-n} Y_n^{\alpha-n}(x) ; \) and

\( G(x,t) = (1-t/x)^{-1} \ _2F_0 \left[ \begin{array}{c} 1, 1 + \alpha; \frac{tx}{x-1} \end{array} \right] \)

Applying our result in (2.1.7) we obtain

\[
(1-ty)^{-1} \ _2F_0 \left[ \begin{array}{c} 1, \alpha + 1; \frac{tx}{(1-ty)(1-xt)} \end{array} \right] e^{2t} \left( \frac{1}{1-xt} \right)^{\alpha+1}
\]

\[
= \sum_{r=0}^{\infty} \frac{t^r}{r!} B_r(y) Y_r^{\alpha-r}(x) \frac{t^r}{r!}
\]

where \( B_r(y) = \sum_{m=0}^{r} \frac{(-1)^m (-r)_m y^m}{2^m} \)
(d) Further if we consider the generating function for the generalized Bessel polynomial

\[ (1-t)^{-c} \, _2F_0 \left[ c, \alpha + 1; - \frac{tx}{1-t} \right] = \sum_{n=0}^{\infty} \frac{(c)_n}{n!} t^n Y_n(\alpha - n)(x) \]

where \( Y_n(\alpha)(x) \) has the meaning as in (C).

In this case we also replace \( t \) by \( t/x \) and obtain

\[ (1-t/x)^{-c} \, _2F_0 \left[ c, \alpha + 1; - \frac{tx}{x-t} \right] = \sum_{n=0}^{\infty} \frac{(c)_n}{n!} t^n x^{-n} Y_n(\alpha - n)(x) \]

Here we notice that

\[ P_n(x) = x^{-n} Y_n(\alpha - n)(x) ; \]

\[ a_n = \frac{(c)_n}{n!}, \quad w(x) = x^{\alpha} e^{-2/x} \]

\[ k_n = 2^n, \quad G(x,t) = (1-t/x)^{-c} \, _2F_0 \left[ c, \alpha + 1; - \frac{tx}{x-t} \right] \]

Now, with the help of (2.1.7) we obtain

\[ (1-ty)^{-c} \, _2F_0 \left[ c, \alpha + 1; - \frac{txy}{(1-ty)(1-xt)} \right] e^{\frac{t}{1-xt}} = \sum_{r=0}^{\infty} \frac{(c)_r}{r!} B_r(\gamma) H_{r}^{\alpha-r}(x) \frac{t^r}{r!} \]

where \( B_r(y) = \sum_{m=0}^{r} \frac{(-1)^m (-r)_m (c)_m}{2^m m!} y^m \)
The Jacobi polynomial \( P_n^{(\alpha-n)}(x-1) \) can be defined by
\[
P_n^{(\alpha-n, \beta)}(x-1) = \frac{(-1)^n (2-x)^{-\alpha+n} - x^{-\beta} D^n}{2^n n!} \left[(2-x)x^{-\alpha}n^\beta\right]
\]
and the generating relation for \( P_n^{(\alpha-n, \beta)}(x) \) is
\[
\sum_{n=0}^{\infty} P_n^{(\alpha-n, \beta)}(x)t^n = \left(1 + t^{1/2}\right)^{-\alpha-\beta-1}
\]
Replacing \( x \) by \( x-1 \) and \( t \) by \( t/2-x \) we obtain
\[
\sum_{n=0}^{\infty} P_n^{(\alpha-n, \beta)}(x-1)\frac{t^n}{(2-x)^n} = (1 + t/2-x)^{\alpha} \left(1 + t/2\right)^{-\alpha-\beta-1}
\]
Here \( w(x) = (2-x)^{\alpha} x^{\beta} \), \( k_n = (-1)^n 2^n n! \)
\( a_n = 1 \), \( P_n(x) = (2-x)^{-n} P_n^{(\alpha-n, \beta)}(x-1) \)
\( G(x,t) = (1 + t/2-x) \cdot (1 + t/2)^{-\alpha-\beta-1} \)
Now we apply our result in (2.1.7) and obtain
\[
\sum_{\alpha=0}^{\infty} b(\alpha, \beta) \frac{(2-x)^{\alpha}}{\alpha!} \frac{(1-xt+\frac{tx}{2})^{\alpha-\beta-1}}{\alpha!} \]
\[
= \sum_{\alpha=0}^{\infty} b(\alpha, \beta) \frac{(2-x)^{\alpha}}{\alpha!} \frac{(1-xt+\frac{tx}{2})^{\alpha-\beta-1}}{\alpha!} \]
\[
= \sum_{\alpha=0}^{\infty} b(\alpha, \beta) \frac{(2-x)^{\alpha}}{\alpha!} \frac{(1-xt+\frac{tx}{2})^{\alpha-\beta-1}}{\alpha!} \]
Where \( b(\alpha, \beta) = \sum_{m=0}^{r} \frac{(-r)_m (y/2)_m}{m!} \)
(f) For the Jacobi polynomial \( P_n(\alpha, \beta, -n)(x) \)
we have

\[
P_n(\alpha, \beta, -n)(x) = \frac{(-1)^n(1-x)^{-\alpha}(1+x)^{-\beta+n}}{2^n n!} D^n[(1-x)^{\alpha}(1+x)^{\beta}]
\]

Replacing \( x \) by \( 1-x \) we obtain

\[
P_n(\alpha, \beta, -n)(1-x) = \frac{(-1)^n x^{-\alpha}(2-x)^{-\beta+n}}{2^n n!} D^n [(x^n+\alpha)(2-x)^{\beta}]
\]

The generating relation for \( P_n(\alpha, \beta, -n)(x) \) is

\[
\sum_{n=0}^{\infty} P_n(\alpha, \beta, -n)(x)t^n = (1-t)^{\beta} (1-\frac{1+x}{2})^{-\alpha-\beta-1}
\]

Changing \( x \) by \( 1-x \) and \( t \) by \( t/2-x \) we have

\[
\sum_{n=0}^{\infty} P_n(\alpha, \beta, -n)(1-x)(\frac{t}{2-x})^n = (1-\frac{t}{2-x})^\beta (1-\frac{t}{2})^{-\alpha-\beta-1}
\]

In this case, our

\[
P_n(x) = (2-x)^{-n} P_n(\alpha, \beta, -n)(1-x);
\]

\( a_n = 1 \), \( w(x) = x^\alpha (2-x)^\beta \)

\( k_n = (-1)^n 2^n n! \), \( g(x,t) = (1-\frac{t}{2-x})^\beta (1-\frac{t}{2})^{-\alpha-\beta-1} \)
with the help of (2.1.7) we obtain

\[
\left[ x^t (2t-y) \right]^\beta (1-xt - txy \frac{x}{2}) = x^\alpha - \beta - 1
\]

\[= \sum_{\nu=0}^{\infty} b_\nu(y) p_\nu(\alpha', \beta - \nu) (1-x) \left( \frac{2xt}{x-2} \right)^\nu\]

where \( b_\nu(y) = \sum_{m=0}^{\nu} (-r)_m \frac{y^m}{m!} \)

(g) Again, considering the generating relation

\[\sum_{n=0}^{\infty} \frac{t^n}{(\alpha+1)_n} p_n(\alpha', \beta - n)(x) = \frac{1}{1-F_1(-\beta; \alpha + 1; \frac{1+xt}{2})} \exp\left(\frac{1+xt}{2}\right)\]

and replacing \( x \) by \( 1-x \), \( t \) by \( \frac{t}{\beta-x} \) we have

\[\sum_{n=0}^{\infty} \frac{t^n}{(\alpha+1)_n} (2-x)^{-n} p_n(\alpha', \beta - n)(1-x) = \frac{1}{1-F_1(-\beta; \alpha + 1; \frac{xt}{2(2-x)})} \exp\left(\frac{t}{2}\right)\]

As in (f), here also

\[w(x) = x^\alpha (2-x)^{\beta}, \quad k_n = (-1)^n \frac{n!}{n!}\]

\[a_n = \frac{1}{(\alpha+1)_n}, \quad p_n(x) = (2-x)^{-n} p_n(\alpha, \beta - n)(1-x)\]

and \( G(x,t) = \frac{1}{1-F_1(-\beta; \alpha + 1; \frac{xt}{2(2-x)})} \exp\left(\frac{t}{\beta}\right)\)
Applying our result in (2.1.7) we obtain

\[ F_1 (-\beta; \alpha+1; \frac{tx^2 y}{2(1-xt)(2-2xt-x)}) \exp \left( \frac{txy}{2(1-xt)} \right) \]

\[ \cdot \left( 1 - \frac{2xt}{2-x} \right)^{-\alpha-\beta-1} \]

\[ = \sum_{r=0}^{\infty} \left( \frac{2xt}{x-2} \right)^r b_r(y) P_r(\alpha, \beta - r)(1-x) \]

Where \( b_r(y) = \sum_{m=0}^{r} \frac{(-r)_m y^m}{2^m (\alpha + 1)_m m!} \)

(h) Lastly we consider, another generating function for the Jacobi polynomials,

\[ \sum_{n=0}^{\infty} \frac{(\lambda)_n (\alpha, \beta - n)}{(c)_n} t^n P_n(\alpha, \beta - n)(x) \]

\[ = \frac{(1+x)^{-\beta}}{2} F_2 \left[ \alpha + 1; -\beta, \lambda; \alpha + 1, c; \frac{1-x}{2}, \frac{1+x}{2}, t \right] \]

where \( F_2 (a, b, b'; c, c'; x, y) \)

\[ = \sum_{\beta, k=0}^{\alpha} \frac{(a)_{n+k} (b)_k (b')_n x^n y^k}{k! n! (c)_k (c')_n} \]
Replacing $x$ by $1-x$ and $t$ by $t/2-x$ we obtain

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(c)_n} t^n (2-x)^{-n} p_n(\alpha, \beta - n) (1-x)$$

$$= \frac{(2-x)^{-\beta}}{2} F_2\left[\alpha+1; -\beta, \lambda; \alpha+1, c; \frac{x}{2}, \frac{t}{2}\right]$$

In this case our

$$w(x) = x^\alpha (2-x)^\beta, \quad k_n = (-1)^n 2^n n!$$

$$a_n = \frac{(\lambda)_n}{(c)_n}, \quad p_n(x) = (2-x)^{-n} p_n(\alpha, \beta - n) (1-x)$$

and

$$G(x,t) = \frac{(2-x)^{-\beta}}{2} F_2\left[\alpha+1; -\beta, \lambda; \alpha+1, c; \frac{x}{2}, \frac{t}{2}\right]$$

Again, with the help of (2.1.7) we have

$$(2-\frac{x}{1-xt})^{-\beta} F_2\left[\alpha+1; -\beta, \lambda; \alpha+1, c; \frac{x}{2(1-xt)}, \frac{txy}{2(1-xt)}\right].$$

$$= \sum_{r=0}^{\infty} b_r(y) P_r(\alpha, \beta - r) (1-x) \left(\frac{2xt}{x-2}\right)^y$$

Where $b_r(y) = \sum_{m=0}^{r} \frac{(-r)_m (\lambda)_m}{2^m (c)_m} \frac{y^m}{m!}$
In this section we [7 Bull. Cal. Math 66, 107-110 (1974)] shall exhibit the importance of the operational derivation of the generating functions for ultraspherical and analogous polynomials. Our derivation of generating functions is purely formal. Of course the region of convergence can be determined by examining the singularities of the generating function.

For our purpose we require the operational result

\[ e^{\lambda f(x)} = \phi(x + a) \]

The well-known ultraspherical or Gegenbauer polynomials \( P_n^\lambda(x) \) may be defined by means of the following explicit representation [8]:

\[ P_n^\lambda(x) = \sum_{p=0}^{\left\lfloor \frac{n}{\lambda} \right\rfloor} \frac{(-1)^p (\lambda)_n}{p! (n-2p)!} (2x)^{n-2p} \]

P. Humbert [9] and P. Barrucand [10] independently considered one generalization of the ultraspherical polynomials. In a paper [11], we have shown that their generalized ultraspherical
polynomials

\[ P_n^\lambda (x,k) \] may be given explicitly by

\[
(2.2.3) \quad P_n^\lambda (x,k) = \sum_{p=0}^{\lfloor n/k \rfloor} \frac{(-1)^p (\lambda)_p (n-(k-1)p)}{p! (n-kp)!} (kx)^{n-kp}; k = 1, 2, \ldots
\]

In the same paper, while considering some other properties of \( P_n^\lambda (x,k) \) we have found

\[
(2.2.4) \quad P_n^\lambda (x,k) = \frac{(-1)^kn}{k^{n(k-1)}} \frac{(\lambda)_n}{(1-\lambda-n)(k-1)n} \cdot \n^n (k-1)_{k-1}
\]

It may be of interest to point out that the formula (2.2.4) gives rise to the following useful forms for the two special cases \( P_n^\lambda (x,2) \) and \( P_n^\lambda (x,3) \), the former is the well-known ultraspherical polynomials, while the later was considered by Humbert [9].

\[
(2.2.5) \quad P_n^\lambda (x,2) \equiv P_n^\lambda (x) = \frac{(\lambda)_n}{2^n (1-\lambda-n)_n} D_n^2 F_1 \left( -n, \lambda + \frac{1}{2}; x^2 \right)
\]

\[
= \frac{(-1)^n}{2^n n!} D_n^2 \left( -n, \lambda + \frac{1}{2}; x^2 \right)
\]
(2.2.6) \[ P_n^\lambda (x, 3) = \frac{(-1)^n \binom{\lambda}{n}}{\binom{\lambda}{n} \Gamma(1 - \lambda - n)} \cdot \frac{D^{2n}}{(1 - \lambda - n)^n} \cdot _3F_2 \left( -n, \frac{\lambda - n - 1}{2}, \frac{\lambda - n + 1}{2}; \frac{1}{2}, \frac{3}{2}; 4x^3 \right) \]

(A) Operational derivation of generating functions:

First we consider the polynomials \( P_n^\lambda (x) \) defined by (2.2.5)

We have

\[
\sum_{n=0}^{\infty} P_n^\lambda (x) t^n = \sum_{n=0}^{\infty} \frac{(-t)^n}{2^n n!} D^n \cdot \sum_{r=0}^{\infty} \frac{(-n)_r (\lambda)_r}{\left( \frac{1}{2} \right)_r r!} \cdot \frac{x^{2r}}{r!}
\]

\[
= \sum_{n=0}^{\infty} \frac{(-t/2)^n}{n!} D^n \cdot \sum_{r=0}^{\infty} \frac{(\lambda)_r}{\left( \frac{1}{2} \right)_r r!} (t/2)^r D^{2r} X
\]

\[
= e^{-\frac{tD}{2}} \sum_{r=0}^{\infty} \frac{(-1)^r (2xt)^r}{r!} \cdot \frac{(\lambda)_r}{\left( \frac{1}{2} \right)_r r!}
\]

(2.2.7) \[ = e^{-\frac{tD}{2}} (1 - 2xt)^{-\lambda} \]

Thus it follows from (2.2.7) and (2.2.1) that

\[
\sum_{n=0}^{\infty} P_n^\lambda (x) t^n = \left[ 1 - 2(x - t/2) \right]^{-\lambda} = (1 - 2xt + t^2)^{-\lambda} \]
Where \[ |2xt - t^2| < 1 \].

Which is the usual generating function of the ultraspherical polynomial \( P_n^\lambda(x) \).

Next we have from (2.2.6)

\[
\sum_{n=0}^{\infty} P_n^\lambda(x,3) t^n = \sum_{n=0}^{\infty} \frac{P_k(x)t^n}{3^{2n}(1-\lambda)_n n!} \cdot 3F_2\left(-n, \frac{\lambda-n}{2}, \frac{\lambda-n+1}{2}; \frac{1}{3}, \frac{2}{3}; 4x^3\right)
\]

\[
= \sum_{n=0}^{\infty} \frac{t^n P_{2n}}{6^n (1-\lambda)_n n!} \sum_{m=0}^{n} \frac{(-n)_m(\frac{\lambda-n}{2})_m(\frac{\lambda-n+1}{2})_m}{\left(\frac{1}{3}\right)_m \left(\frac{2}{3}\right)_m} (4x^3)_m
\]

\[
= \sum_{n=0}^{\infty} \frac{(t/6)^n P_{2n}}{n! (\lambda-n)_n} \sum_{m=0}^{\infty} \frac{(-n)_m}{m!} (3xt)^m
\]

\[
= \sum_{n=0}^{\infty} \frac{(-t/3)^n P_{2n}}{n! (\lambda-n)_n} (1-3xt)^{-\lambda+n}
\]

\[
= \sum_{n=0}^{\infty} \frac{(-t^2/3 D)^n}{n!} (1-3xt)^{-\lambda}
\]

(2.2.8) \( \left[1 - 3(x - \frac{t^2}{3}) - \frac{t^2}{3}\right]^{-\lambda/3} \) = \( (1-3xt + t^3)^{-\lambda} \)

where \( |3xt - t^3| < 1 \).
This gives the generating function for the Humbert polynomials [5].

Lastly we have from (2.2.4)

\[
\sum_{n=0}^{\infty} \binom{\lambda}{n} \frac{d^{n(k-1)}}{k^{n(k-1)}} \frac{(-1)^{kn}}{(1-kn-n)_{(k-1)n}} \frac{2^{n+\lambda-n_k}}{k^{1-n} (k-1)} \left[ -n, \frac{2n+\lambda-n_k}{k-1}, \ldots, \frac{2n+\lambda-kn+k-2}{k-1} \right] t^n
\]

\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \binom{\lambda}{n} \binom{\lambda-kn}{m} \frac{(-1)^{n} d^{n(k-1)}}{k^{n(k-1)}} \frac{(-t)^{m}}{m!} \frac{(\lambda-kn+2n)^m}{(\lambda-kn+2n)_{(k-2)n}} \frac{x^m}{n!} t^{n+m}
\]

\[
= \sum_{n=0}^{\infty} \binom{\lambda}{n} \frac{(-t/k)^{n}}{n!} \frac{d^{n(k-1)}}{(\lambda-kn+2n)_{(k-2)n}} \frac{(-1)^k}{(1-kxt)^{\lambda} (kxt)^{m}}
\]

\[
= \sum_{n=0}^{\infty} \frac{(-t/k)^{n}}{n!} \frac{d^{n(k-1)}}{(\lambda-kn+2n)_{(k-2)n}} \frac{(1-kxt)^{-\lambda} (kxt)^{m}}{m!}
\]

\[
= \sum_{n=0}^{\infty} \frac{(-t/k)^{n}}{n!} \frac{d^{n(k-1)}}{(1-kxt + tk)^{-\lambda}}
\]

\[
= \sum_{n=0}^{\infty} \frac{(-t/k)^{n}}{n!} \frac{d^{n(k-1)}}{(1-kxt + tk)^{-\lambda}}
\]

\[
= \left[ 1-k \left( x - \frac{t}{k} \right)^{k-1} \right]^{-\lambda}
\]

\[
= (1-kxt + tk)^{-\lambda}
\]
Thus

\[
\sum_{n=0}^{\infty} P_n^\lambda (x,k) t^n = (1-kxt + t^k)^{-\lambda}
\]

where \( |kxt - t^k| < 1 \),

\( k \) being a positive integer.

Which is the well-known generating function for the generalized ultraspherical polynomial defined by (2.2.3) or (2.2.4).

Thus we remark that our formulas (2.2.4), (2.2.5) and (2.2.6) enable us to derive the corresponding generating relations in a natural manner with the use (2.2.1).
In this section we \cite{13, Mathematics Balkanica 6:4(1976) 21-29} shall derive an explicit statement of theorems (given below) in the case of a set of functions of several variables. Moreover, we like to show various applications of these theorems some of which are believed to be new in the field of special functions.

Our theorems are stated below:

**Theorem 1.**

For a set of functions $S_\alpha (x_1, x_2, \ldots, x_n)$ of order $\alpha$ generated by

\begin{equation}
\sum_{n=0}^{\infty} a_n S_n + \alpha (x_1, x_2, \ldots, x_n) t^n
\end{equation}

where the sequence of the coefficients $a_n$ is selected in such a way that the series on the left of (2.3.1) gives rise to a generating function $S_\alpha (x_1, x_2, \ldots, x_n, t)$ separated like the right member of (2.3.1).

Let

\begin{equation}
F(x_1, x_2, \ldots, x_n, t) = \sum_{n=0}^{\infty} a_n S_n + m (x_1, x_2, \ldots, x_n) t^n
\end{equation}
Where $F \left( x_1, x_2, \ldots, x_n \right)$ is of arbitrary nature then the following bilateral generating relation for $s_m \left( x_1, x_2, \ldots, x_n \right)$ holds

\[
\sum_{n=0}^{\infty} s_{n+m} \left( x_1, \ldots, x_n \right) \phi_n \left( z \right) t^n.
\]

where

\[
\phi_n \left( z \right) = \sum_{k=0}^{n} a_k A_{n-k} z^k.
\]

We shall now show that our theorem 1 can be modified in the case when $A_n$ and $a_n$ are functions of $\alpha$ and $m$ respectively. The modified theorem can be stated as follows:

**Theorem II**

For a set of functions $s_\alpha \left( x_1, x_2, \ldots, x_n \right)$ of order $\alpha$ generated by

\[
\sum_{n=0}^{\infty} A_n \left( \alpha \right) s_{n+\alpha} \left( x_1, x_2, \ldots, x_n \right) t^n
\]

\[
= f \left( x_1, x_2, \ldots, x_n, t \right) \frac{S_{\alpha} \left( h_1 \left( x_1, \ldots, x_n, t \right), \ldots, h_n \left( x_1, \ldots, x_n, t \right) \right)}{\left[ g \left( x_1, x_2, \ldots, x_n, t \right) \right]^\alpha}
\]
where the sequence of the coefficient $A_n(\alpha')$ is selected in such a way that the series on the left of (2.3.5) gives rise to a generating function separated like the right member of (2.3.5).

Let

\begin{equation}
F(x_1, x_2, \ldots, x_n, t) = \sum_{n=0}^{\infty} a_n(m) S_{n+m}(x_1, x_2, \ldots, x_n) t^n
\end{equation}

where $F(x_1, x_2, \ldots, x_n)$ is of arbitrary nature then the following bilateral generating relation for $S_m(x_1, x_2, \ldots, x_n)$ holds.

\begin{equation}
f(x_1, x_2, \ldots, x_n, t) = \sum_{n=0}^{\infty} S_{n+m}(x_1, x_2, \ldots, x_n) \sigma_{n,m}(Z) t^n
\end{equation}

where

\begin{equation}
\sigma_{n,m}(Z) = \sum_{k=0}^{n} a_k(m) A_{n-k}(m+k) Z^k
\end{equation}

The new bilateral generating relations derived in this paper are given below:
(a) \((1-t-y)^{-1-\alpha-m} \exp \left( \frac{x(t(1+y))}{t+y-1} \right) L_m^\alpha \left( \frac{x}{1-t-y} \right)\)

\[= \sum_{n=0}^{\infty} L_{n+m}^\alpha (x) \sigma_{n,m} (y) t^n\]

where \(\sigma_{n,m} (y) = \sum_{k=0}^{n} \binom{k+m}{k} \binom{n-m}{n-k} y^k = \binom{m+n}{n} (1+y)^n\)

and \(L_n^\alpha (x)\) is the well-known Laguerre polynomial.

(b) \(\frac{1}{2} \left[ (1-2xt + t^2) -2(x-t) y + y^2 t^2 \right]^{\frac{n-\gamma}{2}}\)

\[= \sum_{k=0}^{\infty} C_{n+k}^\gamma (x) \sigma_k (y) t^k\]

where \(\sigma_k (y) = \sum_{m=0}^{k} \binom{n+m}{m} \binom{n+k}{k-m} y^m\)

\[= \binom{n+k}{k} (1+y)^k\]

and \(C_{n+k}^\gamma (x)\) is the well-known Gegenbauer polynomial.
\[(c) \ (1 - xt - zxt)^{-\alpha} (1 - yt - yzt)^{-\beta} g_m \left( \frac{x}{1 - xt - zxt}, \frac{y}{1 - yt - yzt} \right) \]

\[= \sum_{n=0}^{\infty} g_n + m (x, y) \delta_{n, m} (z) t^n .\]

where \(\delta_{n, m} (z) = \sum_{k=0}^{n} \binom{k+m}{k} \binom{n+m}{n-k} z^k\)

\[= \binom{m+n}{n} (1 + z^n) \]

and \(g_n (x, y)\) is the Lagrange polynomial in two variables defined by

\[(1 - xt)^{-\alpha} (1 - yt)^{-\beta} = \sum_{n=0}^{\infty} g_n (x, y) t^n .\]

In a recent paper [14] S.K.Chatterjea made an attempt to prove the following theorem in connection with the unification of a class of bilateral generating relations for certain special functions:

**Theorem:**

For a set of functions \(S_\alpha (x)\) of order \(\alpha\) (in particular, for a sequence of functions \(S_\alpha (x)\) of degree \(\alpha\)) generated by

\[(2.3.9) \sum_{n=0}^{\infty} A_n S_n +\alpha (x) t^n = \frac{f(x,t)}{g(x,t)} S_\alpha (h(x,t)) \]

\]
Where the sequence of coefficients $A_n$ is selected in such a way that the series on the left of (2.3.9) gives rise to generating functions separated like the right member of (2.3.9).

Let

$$F(x, t) = \sum_{n=0}^{\infty} a_n S_{n+m}(x) t^n$$  \hspace{2cm} (2.3.10)

where $F(x, t)$ is of arbitrary nature, then the following bilateral generating relation for $S_m(x)$ holds:

$$f(x, t) = \frac{F(x, t)}{g(x, t)}$$

$$= \sum_{n=0}^{\infty} S_{n+m}(x) \phi_n(y) t^n.$$  \hspace{2cm} (2.3.11)

where

$$\phi_n(y) = \sum_{k=0}^{n} a_k A_{n-k} y^k$$  \hspace{2cm} (2.3.12)

It may be of interest to point out that when $A_n$ and $a_n$ are functions of $\alpha$ and $m$ respectively, then the theorem holds provided we replace $\phi_n(y)$ by $\sum_{k=0}^{n} a_k (m) A_{n-k} (m + k) y^k.$
It may be noted that the above theorem of Chatterjea serves as a novel extension of a theorem of J.P. Singh and H.M. Srivastava [15]

(A) **Proof of theorem I.**

We have

\[ \sum_{n=0}^{\infty} s_{n+m}(x_1, x_2, \ldots, x_n) z^n = \sum_{n=0}^{\infty} s_{n+m}(x_1, x_2, \ldots, x_n) \sum_{k=0}^{n} a_k A_n-k z^k t^n \]

\[ = \sum_{k=0}^{\infty} a_k (zt)^k \sum_{n=0}^{\infty} A_n s_{n+m+k}(x_1, x_2, \ldots, x_n) t^n \]

with the help of (2.3.1) above reduces to

\[ = \sum_{k=0}^{\infty} a_k (zt)^k \frac{f(x_1, x_2, \ldots, x_n, t)}{g(x_1, x_2, \ldots, x_n, t)} \frac{s_{m+k}(h_1(x_1, \ldots, x_n, t), \ldots, h_n(x_1, \ldots, x_n, t))}{m+k} \]

\[ = \frac{f(x_1, x_2, \ldots, x_n, t)}{g(x_1, x_2, \ldots, x_n, t)} \sum_{k=0}^{\infty} a_k s_{m+k}(h_1(x, \ldots, x_n, t), \ldots, h_n(x, \ldots, x_n, t)) \frac{(zt)^k}{g(x, \ldots, x_n, t)} \]

Again using (2.3.2) we finally obtain,

\[ \frac{f(x_1, x_2, \ldots, x_n, t)}{g(x_1, x_2, \ldots, x_n, t)} \frac{h_1(x_1, \ldots, x_n, t), \ldots, h_n(x_1, \ldots, x_n, t)}{g(x, \ldots, x_n, t)} \frac{(zt)^k}{g(x, \ldots, x_n, t)} \]

which is (2.3.3)
Proof of theorem II.

We have
\[ \sum_{n=0}^{\infty} S_{n+m} (x_1, x_2, \ldots, x_n) \alpha_{n,m} (z) t^n \]
\[ = \sum_{k=0}^{\infty} a_k (m) (zt)^k \sum_{n=0}^{\infty} A_k (m+k) S_{n+m+k} (x_1, \ldots, x_n) t^n \]

with the help of (2.3.5) and (2.3.6) we finally obtain (2.3.7) in precisely the same manner as hinted in the proof of theorem I.

(B) Applications:

For Laguerre polynomial, we have the generating relation

\[ \sum_{n=0}^{\infty} \binom{n+m}{m} L_n^{(\alpha)} (x) t^n = (1-t)^{-1-\alpha-m} \exp\left(\frac{-xt}{1-t}\right) L_m^{(\frac{x}{1-t})} \]

By virtue of theorem, (2.3.13) may be combined with itself. To this end, we observe that

\[ f(x,t) = (1-t)^{-1-\alpha} \exp\left(\frac{-xt}{1-t}\right), \quad g(x,t) = (1-t) \]

\[ h(x,t) = x/1-t, \quad S_{n+m}(x) = L_{n+m}^{(\alpha)} (x), \quad S_m(x) = L_m^{(\alpha)} \left(\frac{x}{1-t}\right) \]

\[ A_n(m) = \binom{n+m}{n}, \quad a_n(m) = \binom{n+m}{n} \quad \text{and} \]

\[ F(x,t) = (1-t)^{-1-\alpha-m} \exp\left(\frac{-xt}{1-t}\right) L_m^{(\alpha)} \left(\frac{x}{1-t}\right) \]

Now using (2.3.7) we obtain
(2.3.14) \[ \left[ 1-t(1+y) \right]^{1-x-m} \exp \left( \frac{xt(1+y)}{1-t(1+y)} \right) L_m^{(\alpha)} \left( \frac{x}{1-t(1+y)} \right) \]

= \sum_{n=0}^{\infty} L_{n+m}^{(\alpha)}(x) \sigma_{n,m}(y) t^n.

where

(2.3.16) \[ \sigma_{n,m}(y) = \sum_{k=0}^{n} \binom{k+m}{k} \binom{n+m}{n-k} y^k = \binom{m+n}{n} (1+y)^n \]

which does not seem to appear in any other earlier investigation.

In particular putting \( y = 0 \) (2.3.14) becomes

(2.3.16) \[ (1-t)^{1-x-m} \exp \left( \frac{xt}{1-t} \right) L_m^{(\alpha)} \left( \frac{x}{1-t} \right) \]

= \sum_{n=0}^{\infty} \frac{(n+m)!}{n! m!} L_{n+m}^{(\alpha)}(x) t^n.

which is (2.3.13).

Again, for \( m = 0 \) (2.3.14) reduces to

(2.3.17) \[ \left[ 1-t(1+y) \right]^{1-x} \exp \left( \frac{xt(1+y)}{t(1+y)-1} \right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) (1+y)^n \]

which is identical with the well-known generating relation

\[ (1-t)^{1-x} \exp \left( \frac{-xt}{1-t} \right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n. \]
Hermite polynomial.

For Hermite polynomial, we have the generating relation

\[(2.3.18) \sum_{n=0}^{\infty} \frac{H_{n+k}(x)t^n}{n!} = \exp (2xt-t^2) H_k(x-t)\]

uniting (2.3.18) with itself and noticing that in this case

\[f(x,t) = \exp (2xt-t^2),\; g(x,t) = 1,\; h(x,t) = x-t\]

\[s_\alpha = H_k(x-t),\; s_{n+\alpha} = H_{n+k},\; A_n = 1/n!,\; a_n = \frac{1}{n!}\]

and \(F(x,t) = \exp (2xt-t^2) H_k(x-t)\)

Then applying our result (2.3.5) we obtain

\[(2.3.19) \exp (2xt-t^2 + 2(x-t)yt - y^2t^2) H_k(x-t(1+y))\]

\[= \sum_{n=0}^{\infty} \frac{H_{n+k}(x)(t(1+y))^n}{n!}\]

which is same as (2.3.18) so that the application of theorem I does not give us a new bilateral generating relation.

Ultraspherical polynomial.

We know that the generating relation for the ultraspherical polynomial is
\begin{equation}
\sum_{k=0}^{\infty} \binom{n+k}{k} c_{n+k}(x) t^k = (1-2xt+t^2)^{-\frac{n}{2}}.
\end{equation}

Uniting (2.3.20) with itself, we see that in this case \( f(x,t) = (1-2xt+t^2)^{-\frac{n}{2}} \), \( g(x,t) = (1-2xt+t^2)^{-\frac{n}{2}} \).

\begin{align*}
h(x,t) &= \frac{x-t}{\sqrt{1-2xt+t^2}}, \quad S_{n+\alpha} = c_{n+k}(x), \\
S_n &= c_n\left(\frac{x-t}{\sqrt{1-2xt+t^2}}\right), \quad A_k(n) = \binom{n+k}{k}, \\
ak(n) &= \binom{n+k}{k} \quad \text{and} \quad F(x,t) = (1-2xt+t^2)^{-\frac{n}{2}}.
\end{align*}

Applying our result in (2.3.7) we obtain,

\begin{equation}
(2.3.21) \quad \left[(1-2xt+t^2)-2(x-t)y+yt^2\right]^{-\frac{n}{2}}.
\end{equation}

\begin{align*}
\sum_{k=0}^{\infty} c_n^{(\alpha)}(x) s_{k,n}(y) t^k
&= \sum_{k=0}^{\infty} c_{n+k}(x) k_{n+k}(y) t^k.
\end{align*}
Where

\[ \sum_{m=0}^{k} \binom{n+m}{m} \binom{n+k}{k} y^m = \binom{n+k}{k} (1+y)^k \]

We notice that \( y = 0 \) in (2.3.20) gives rise to (2.3.21) and also taking \( n = 0 \) in (2.3.21) we have

\[ (\cdot l-2x+t') -2 (x-t) y + y^2 t^2 \]

\[ = \sum_{k=0}^{\infty} c_k^j (x) (l+y) t^k \]

which is identical with the generating relation

\[ (\cdot l-2x+t') = \sum_{k=0}^{\infty} g_k^j (x) t^k \]

Lagrange's polynomial of two variables.

Generating relation for Lagrange's polynomial of two variables is

\[ \sum_{n=0}^{\infty} \binom{m+n}{n} g_{n+m} (x,y)t^n = (1-xt) \alpha (1-yt) \beta, \]

uniting (2.3.24) with itself and observing that

\[ f(x,y,t) = (1-xt)^\alpha (1-yt)^\beta, \ g(x,y,t) = 1 \]

\[ S_n \alpha (x,y) = g_{n+m} (x,y), \ S_n \beta (x,y) = g_m \frac{x}{l-xt}, \frac{y}{l-yt} \]

\[ A_n (m) = \binom{m+n}{n}, \ a_n (m) = \binom{m+n}{n} \]

and

\[ F(x,y,t) = (1-xt)^\alpha (1-yt)^\beta g_m \frac{x}{l-xt}, \frac{y}{l-yt} \]
Now applying our result in (2.3.7) we obtain

\[(2.3.25) \quad (1-xt-zxt)^\alpha (1-yt-yzt)^\beta g_m(\frac{x}{1-xt-zxt}, \frac{y}{1-xt-zxt})\]

\[= \sum_{n=0}^{\infty} g_{n+m}(x,y) \beta_{n,m}(z) t^n\]

where

\[(2.3.26) \quad \beta_{n,m}(z) = \sum_{k=0}^{n} \binom{k+m}{k} \binom{n+m}{n-k} z^k \]

\[= \binom{m+n}{n} (1+z)^n\]

We observe that \( z = 0 \) in (2.3.25) yields (2.3.24)

Again, for \( m = 0 \) in (2.3.25) we obtain the following generating function

\[(2.3.27) \quad (1-xt-zxt)^\alpha (1-yt-yzt)^\beta\]

\[= \sum_{n=0}^{\infty} g_n(x,y)(1+z)t^n\]

which is identical with the generating relation

\[(1-xt)^\alpha (1-yt)^\beta = \sum_{n=0}^{\infty} g_n(x,y) t^n\]

Bessel function

For Bessel function of first kind, we have the generating relation
(2.3.28) \[ \sum_{n=0}^{\infty} \frac{J_{n+\alpha}(x)}{n!} t^n = (1 - \frac{2t}{x})^{-\frac{\alpha}{2}} J_{\alpha}(\sqrt{\frac{x^2}{2} - 2xt}) \]

Uniting (2.3.28) with itself and noticing that
\[ g(x,t) = (1 - \frac{2t}{x})^{\frac{1}{2}} \]
\[ f(x,t) = 1, \quad h(x,t) = \sqrt{x^2 - 2xt} \]
\[ s_{n+\alpha}(x) = J_{n+\alpha}(x), \quad s_{\alpha}(x) = J_{\alpha}(\sqrt{x^2 - 2xt}), A_n = \frac{1}{n!} \]
\[ a_n = \frac{1}{n!} \quad \text{and} \quad F(x,t) = (1 - \frac{2t}{x})^{-\frac{\alpha}{2}} J_{\alpha}(\sqrt{x^2 - 2xt}) \]

Applying the result in (2.3.5) we have
\[ (2.3.29) \quad \left(1 - \frac{2t}{x}\right)^{-\frac{\alpha}{2}} \left(1 - \frac{2yt/\sqrt{1-2t/x}}{\sqrt{x^2 - 2xt}}\right)^{-\frac{\alpha}{2}} \]
\[ J_{\alpha} \left(\sqrt{x^2 - 2xt - 2\sqrt{x^2 - 2xt} \frac{yt}{\sqrt{1-2t/x}}} \right) = \]
\[ \sum_{n=0}^{\infty} J_{n+\alpha}(x) \left(\frac{(1+yt)^n}{n!}\right) \]

**Bessel polynomial.**

For Bessel polynomial the generating relation is
(2.3.30) \[
\sum_{n=0}^{\infty} \frac{y_{n+m}(x)t^n}{n!} = (1-2xt)^{-\frac{m}{2}} \exp\left[ \frac{1}{x} \left( 1 - \sqrt{1-2xt} \right) \right].
\]

\[
\cdot y_m \left( \frac{x}{\sqrt{1-2xt}} \right)
\]

When it is united with itself, we notice that

here \( f(x,t) = (1-2xt)^{-\frac{1}{2}} \exp\left[ \frac{1}{x} \left( 1 - \sqrt{1-2xt} \right) \right]; g(x,t) = \sqrt{1-2xt} \)

\( h(x,t) = \frac{x}{\sqrt{1-2xt}} \), \( S_{n+\alpha}(x) = y_{n+m}(x) \)

\( S_{\alpha}(x) = y_m \left( \frac{x}{\sqrt{1-2xt}} \right) \), \( A_n = \frac{1}{n!} \), \( a_n = \frac{1}{n!} \)

and \( F(x,t) = (1-2xt)^{-\frac{m}{2}} \exp\left[ \frac{1}{x} \left( 1 - \sqrt{1-2xt} \right) \right] y_m \left( \frac{x}{\sqrt{1-2xt}} \right) \)

with the help of (2.3.5) we obtain

(2.3.31) \[
\text{Exp} \left( \frac{1}{x} \left( 1 - \frac{1-2xt-2xyn}{x} \right) (1-2xt-2xyn)^{-\frac{m}{2}} \right).
\]

\[
y_m \left( \frac{x}{\sqrt{1-2xt-2xyn}} \right) = \sum_{n=0}^{\infty} y_{n+m}(x) \left( \frac{1+y}{n!} \right)^n
\]

which is identical with (2.3.30).
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