CHAPTER V

A CLASS OF MIXED TRILATERAL GENERATING RELATIONS
FOR Tchebychev POLYNOMIALS : A NOVEL
EXTENSION OF Lee's TRILATERAL
GENERATING RELATION INVOLVING
Tchebychev AND Charlier
POLYNOMIALS
Introductions

Recently P.A. Lee [1] has considered some generating functions involving the Charlier polynomials -
\[ C_n(x; a) = \binom{2}{a} \binom{-n, -x}{-a} \] from the view-point of differential operators. An interesting result of his observation is a trilateral (which he calls trilinear) generating function of Charlier polynomials with the Tchebychev polynomials. In order to extend his interesting result we propose to consider the Lie algebra for Tchebychev polynomials. Although Tchebychev polynomials are particular cases of Gegenbauer polynomials, yet we have made a direct calculation about the construction of Lie algebra for Tchebychev polynomials. In course of our investigation we have found a class of bilateral as well as mixed trilateral generating function for Tchebychev polynomials. In fact, we have formed the following theorems:

**Theorem I**

If there exists a unilateral generating relation of the form
\[ G(x, t) = \sum_{m=0}^{\infty} \frac{a_m}{m!} T_m(x) t^m \]
where \( a_m \) is an arbitrary constant, then there exists a bilateral generating relation of the form
(5.1.1) \( G \left( \frac{x - wy}{\sqrt{1 - 2xwy + w^2y^2}}, \frac{twy}{\sqrt{1 - 2xwy + w^2y^2}} \right) \)

\[ = \sum_{p=0}^{\infty} \frac{T_p(x) (wy)^p}{p!} b_p(t) \]

where \( b_p(t) = \sum_{m=0}^{p} a_m \binom{p}{m} p_m(t) \).

**Theorem II**

If there exists a bilateral generating relation of the form

(5.1.2) \( G(x,t,u) = \sum_{m=0}^{\infty} \frac{a_m}{m!} T_m(x) p_m(u) t^m \)

where \( p_m(u) \) is an arbitrary classical polynomial or function of degree \( m \) and \( a_m \) is an arbitrary constant, then there exists a mixed trilateral generating relation of the form

(5.1.3) \( G \left( \frac{x-wy}{\sqrt{1-2xwy+w^2y^2}}, \frac{twy}{\sqrt{1-2xwy+w^2y^2}}, u \right) \)

\[ = \sum_{r=0}^{\infty} \frac{T_r(x)(wy)^r}{r!} b_r(t, u) \]

where \( b_r(t, u) = \sum_{m=0}^{r} a_m \binom{r}{m} p_m(u) t^m \).
The importance of the theorem is that whenever one knows a generating relation of the type (5.1.2) for a particular value of \( a_m \), the corresponding mixed trilateral generating relation (5.1.3) can at once be derived from our theorem.

P.A. Lee has proved the following trilateral generating relation with Tchebychev polynomials:

\[
(5.1.4) \sum_{n=0}^{\infty} z^n T_n(x) C_n(k; \alpha) C_n(l; \beta) / n!
\]

\[
= e^{\bar{\rho}} (1 - \frac{\bar{\rho}}{\alpha})^k (1 - \frac{\bar{\rho}}{\beta})^l C_k(l; -\frac{(\alpha - \beta)(\beta - \bar{\rho})}{\bar{\rho}})
\]

\[
+ e^{\bar{\rho}'} (1 - \frac{\bar{\rho}'}{\alpha})^k (1 - \frac{\bar{\rho}'}{\beta})^l C_k(l; -\frac{(\alpha - \beta')(\beta' - \bar{\rho}')}{\bar{\rho}'}),
\]

where \( \bar{\rho} = (x + \sqrt{x^2-1})z \)

\( \bar{\rho}' = (x - \sqrt{x^2-1})z \)

It is interesting to remark that we have found a quadrilateral generating relation involving Tchebychev polynomials and Charlier polynomial from Lee's result.
\[
(5.1.5) \sum_{p=0}^{\infty} \frac{T_p(x)w^p}{p!} \sum_{p=0}^{b} \binom{p}{n}(n)_{p-n} C_n(k; \lambda) C_n(\lambda; \beta) z^n
\]

\[
= \frac{1}{2} \left\{ e^{\lambda_1} \left( 1 - \frac{\lambda_1}{2} \right)^{k} \left( 1 - \frac{(\alpha - \lambda_1)(\beta - \lambda_1)}{\lambda_1} \right) \right\}
\]

\[
+ e^{\lambda_2} \left( 1 - \frac{\lambda_2}{2} \right)^{k} \left( 1 - \frac{(\alpha - \lambda_2)(\beta - \lambda_2)}{\lambda_2} \right) \right\}
\]

where \( \lambda_1 = \frac{x - w + \sqrt{x^2 - 1}}{zw} \frac{zw}{1 - 2xw + w^2} \)

and \( \lambda_2 = \frac{x - w - \sqrt{x^2 - 1}}{zw} \frac{zw}{1 - 2xw + w^2} \)
THE Tchebychev POLYNOMIALS \( T_n(x) \) SATISFY THE FOLLOWING ORDINARY DIFFERENTIAL EQUATION:

\[
\begin{align*}
(5.2.1) \quad (1-x) D^2 T_n(x) - x D T_n(x) + n^2 T_n(x) &= 0
\end{align*}
\]

L. Weisner's group-theoretic method consists in constructing a partial differential equation from the ordinary differential equation by giving a suitable interpretation to \( n \) and then finding a non trivial continuous transformations group which is admitted by the partial differential equation. So we replace \( D \) by \( x \frac{\partial}{\partial x} \) and \( n \) by \( y \frac{\partial}{\partial y} \) and put
\[
A = y \frac{\partial}{\partial y}.
\]

Let us now seek two first order linear differential operators \( B \) and \( C \) such that

\[
(5.2.2) \quad B \left[ T_n(x) y^n \right] = b_n T_{n-1}(x) y^{n-1}, \quad n \geq 1
\]

where \( b_n \) are functions of \( n \) which are independent of \( x \) and \( y \).

Let \( B = B_1(x,y) \frac{\partial}{\partial x} + b_2(x,y) \frac{\partial}{\partial y} + b_3(x,y) \).
and using the relation

\[(1-x^2) \frac{d}{dx} T_n(x) = n \left[ T_{n-1}(x) - x T_n(x) \right] \]

we have

\[B \left[ T_n(x) y^n \right] = (B_1(x,y) \frac{\partial}{\partial x} + B_2(x,y) \frac{\partial}{\partial y} + B_0(x,y)) \left[ T_n(x) y^n \right] \]

\[= B_1(x,y) \frac{\partial}{\partial x} T_n(x) y^n + B_2(x,y) n T_n(x) y^{n-1} + B_0(x,y) T_n(x) y^n \]

\[= B_1(x,y) \frac{\partial}{\partial x} T_n(x) y^n + B_2(x,y) n T_n(x) y^{n-1} + B_0(x,y) T_n(x) y^n \]

In order to make the coefficients of \(T_{n-1}(x) y^{n-1}\) independent of \(x\) and \(y\) we choose \(B_1 = (1-x^2) y^{-1}\). so, above becomes

\[B \left[ T_n(x) y^n \right] = n T_{n-1}(x) y^{n-1} n x T_n(x) y^{n-1} + B_2 T_n(x) ny^{n-1} + B_0 T_n(x) y^n \]

\[= n T_{n-1}(x) y^{n-1} + T_n(x) y^n - n x y^{-1} + B_2 ny^{-1} + B_0 \]
We make the coefficient of $T_n(x)y^n = 0$ by choosing $B_2 = x$, $B_0 = 0$.

So, we have $B = (1-x^2)y^{-1} \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$ and $b_n = n$.

Again let $C \left[ T_n(x)y^n \right] = C_n T_{n+1}(x)y^{n+1}$, $n\neq 0$.

Where $C_n$ is a function of $n$ which are independent of $x$ and $y$.

In a similar manner we easily obtain

$C = y(x^2-1) \frac{\partial}{\partial x} + xy^2 \frac{\partial}{\partial y}$ and $C_n = n$.

So, we have $C \left[ T_n(x) y^n \right] = nT_{n+1}(x) y^{n+1}$

**Group of operators.**

Let $A = y \frac{\partial}{\partial y}$ we shall use the commutator relation $[A, B]$ with $[A, B] u = (AB-BA) u$.

We find that $[A, B] u = -Bu$; $[A, C] u = Cu$

$[B, C] u = 2A(u)$

These commutator relations show that the operators $A, B, C$ generate a Lie group.
We now choose new variables $x$ and $Y$ so that $C$ will be transformed into $K \frac{\partial^2}{\partial x^2}$ where $K^2 = 1$.

In the relation

$$Cu = (x^2 - 1) y \frac{\partial u}{\partial x} + xy^2 \frac{\partial u}{\partial y}$$

$$= (x^2 - 1) y \left( \frac{\partial u}{\partial x} \frac{\partial x}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial x} \right)$$

$$+ xy^2 \left( \frac{\partial u}{\partial x} \frac{\partial y}{\partial y} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial y} \right)$$

Setting coefficients of $\frac{\partial u}{\partial y} = 0$ we get

$$\begin{align*}
(x^2 - 1) y \frac{\partial^2}{\partial x^2} + xy^2 \frac{\partial^2}{\partial y} &= 0
\end{align*}$$

The corresponding subsidiary equations are

$$\frac{x - 1}{x^2 - 1} \frac{dx}{y} = \frac{dy}{y} ; \quad dy = 0$$

$$Y = \zeta \left( \frac{x^2 - 1}{y^2} \right) , \quad \zeta \text{ is an arbitrary function} .$$

We take the special case $Y = \frac{x^2 - 1}{y^2}$.
Again, setting the coefficients of \( \frac{\partial u}{\partial x} = k \), we get,

\[
(x^2 - 1) y \frac{\partial x}{\partial x} + xy^3 \frac{\partial x}{\partial y} = k
\]

or,

\[
(x^2 - 1) \frac{\partial x}{\partial x} + xy \frac{\partial x}{\partial y} = \frac{k}{y}
\]

The corresponding subsidiary equations are

\[
\frac{dx}{x^2 - 1} = \frac{dy}{xy} = \frac{dx}{k/y}
\]

\[
\eta(x^2 - 1, -x - \frac{xy}{y}) = 0,
\]

where \( \eta \) is arbitrary.

Choosing \( k = -1 \) and \( \eta = -x - k \frac{x}{y} \),

\[
X = \frac{x}{y} \quad (\because \eta \neq 0)
\]

So, we have \( X = \frac{x}{y}, \ Y = \frac{x^2 - 1}{y^2} \)

Solving for \( x, \ y \) we obtain

\[
x = \frac{x}{\sqrt{x^2 - y}}, \quad y = \frac{1}{\sqrt{x^2 - y}}
\]
Now we are in a position to find $e^{cC} f(x,y)$
the extended form of the transformed group generated by $C$.

By substitution $x = \frac{x}{\sqrt{x^2 - Y}}$, $y = \frac{1}{\sqrt{x^2 - Y}}$
will transform $C$ into $-D$ where $D = \frac{\partial}{\partial x}$

Making this substitution and applying Taylor's theorem we obtain

$$e^{cC} f(x,y) = e^{-cD} f \left( \frac{x}{\sqrt{x^2 - Y}}, \frac{1}{\sqrt{x^2 - Y}} \right)$$

$$= f \left( \frac{x - C}{(x-C)^2 - Y}, \frac{1}{(x-C)^2 - Y} \right)$$

Finally we use the inverse substitution

$$X = \frac{x}{y}, \quad Y = \frac{x^2 - 1}{y^2}$$

$$e^{cC} f(x,y) = f \left( \frac{x-cy}{\sqrt{1-2xcy + c^2 y^2}}, \frac{y}{\sqrt{1-2xcy + c^2 y^2}} \right)$$
We now choose new variables $X$ and $Y$ so that $B$ will be transformed into $k \frac{\partial}{\partial x}$ where $k^2 = 1$. If we substitute

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X} \frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y} \frac{\partial Y}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X} \frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y} \frac{\partial Y}{\partial y}$$

In the relation $Bu = (1-x^2)y^{-1} \frac{\partial u}{\partial x} + x \frac{\partial u}{\partial y}$

$$= (1-x^2)y^{-1} \left( \frac{\partial u}{\partial X} \frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y} \frac{\partial X}{\partial y} \right)$$

$$+ x \left( \frac{\partial u}{\partial X} \frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y} \frac{\partial Y}{\partial y} \right)$$

Setting coefficients of $\frac{\partial u}{\partial y} = 0$ we get

$$(1-x^2)y^{-1} \frac{\partial X}{\partial x} + x \frac{\partial Y}{\partial y} = 0$$

The corresponding subsidiary equations are

$$\frac{x^2}{1-x^2} dx = \frac{dy}{y}, \quad dy = 0.$$ 

$$Y = \xi \left( (1-x^2) y^2 \right), \xi \text{ is arbitrary function.}$$
We take the special case \( Y = (1 - x^2)y^2 \)

Again setting the coefficient of \( \frac{\partial u}{\partial x} = K \)

we get \((1-x^2)y^{-1} \frac{\partial x}{\partial x} + x \frac{\partial x}{\partial y} = \dot{x}\)

the corresponding equations are

\[
\frac{dx}{1-x^2} = \frac{dy}{xy} = \frac{dx}{Ky}
\]

the solution is

\[
\eta \left( (1-x^2)y^2, -x + Kxy \right) = 0.
\]

Choosing \( K = -1 \), and \( \eta = x + Kxy \),

\[X = -xy \quad (\because \eta = 0)\]

We have \( X = -xy, Y = (1-x^2)y^2 \)

Solving for \( x \) and \( y \) we obtain

\[x = \frac{X}{\sqrt{X^2 + Y}}, \quad y = -\sqrt{X^2 + Y}\]

We are now in a position to find \( e^{bB} f(x,y) \).

By substitution \( x = \frac{X}{\sqrt{X^2 + Y}}, \quad y = -\sqrt{X^2 + Y}\)

will transform \( B \) into \(-D\) where \( D = \frac{\partial}{\partial X} \).
Making this substitution and applying Taylor's theorem, we have
\[ e^{bB} f(x, y) = e^{-bD} f \left( \frac{X}{X^2 + Y}, -\sqrt{X^2 + Y} \right) \]
\[ = f \left( \frac{X - b}{\sqrt{(X-b)^2 + Y}}, -\sqrt{(X-b)^2 + Y} \right) \]

Finally we use the inverse substitution
\[ X = -xy, \quad Y = (1-x^2) y^2 \]
\[ e^{bB} f(x, y) = f \left( \frac{-xy-b}{\sqrt{y^2 + 2xyb + b^2}}, -\sqrt{y^2 + 2xyb + b^2} \right) \]
§ 5.3.

A CLASS OF BILATERAL GENERATING RELATION:

(5.3.1) Let \( G(x, t) = \sum_{m=0}^{\infty} \frac{a_m}{m!} T_m(x) t^m \)

Replacing \( t \) by \( twy \) we have

\[ G(x, twy) = \sum_{m=0}^{\infty} \frac{a_m}{m!} T_m(x) t^m w^m y^m. \]

Operating both sides by \( \exp(wc) \)

\[ \exp(wc)G(x, twy) = \exp(wc) \sum_{m=0}^{\infty} \frac{a_m}{m!} T_m(x) t^m w^m y^m. \]

L.H.S. of the last equation

\[ = \exp(wc) G(x, twy) \]

\[ = G\left(\frac{x-wy}{\sqrt{1-2xwy + w^2 y^2}}, \frac{twy}{\sqrt{1-2xwy + w^2 y^2}}\right) \]

R.H.S. of the last equation

\[ = \sum_{p=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_m}{m!} \frac{w^p}{p!} (wt)_m \frac{m(m+1)(m+2)\ldots(m+p-1)}{m+p} T_{m+p}(x) y^{m+p} \]
\[ \sum_{p=0}^{\infty} \sum_{m=0}^{\infty} \frac{w^p}{p!} \frac{a_m}{m!} (wt)^m (m)_p T_{m+p}(x) y^{m+p} \]

\[ = \sum_{p=0}^{\infty} \sum_{m=0}^{p} \frac{w^{p-m}}{(p-m)!} \frac{a_m}{m!} (wt)^m (m)_{p-m} T_p(x)y^p. \]

\[ = \sum_{p=0}^{\infty} \frac{T_p(x)(yw)^p}{p!} b_p(t) \]

Where \( b_p(t) = \sum_{m=0}^{p} a_m (\frac{p}{m})(m)_{p-m} t^m. \)

Equating we obtain

\[ (5.3.2) \quad G \left( \frac{x-w}{\sqrt{1-2xw+w^2}}, \frac{tw}{\sqrt{1-2xw+w^2}} \right) \]

\[ = \sum_{p=0}^{\infty} \frac{T_p(x)(w)^p}{p!} b_p(t) \]

Where \( b_p(t) = \sum_{m=0}^{p} a_m (\frac{p}{m})(m)_{p-m} t^m. \)

on putting \( y = 1 \).
A CLASS OF MIXED TRILATERAL GENERATING RELATIONS.

(5.4.1) Let \( G(x, t, u) = \sum_{m=0}^{\infty} \frac{a_m}{m!} T_m(x)p_m(u)t^m \)

Where \( p_m(u) \) is an arbitrary classical polynomial function of degree \( m \) and \( a_m \) is an arbitrary constant.

Replacing \( t \) by \( twy \) we have

\[ G(x, twy, u) = \sum_{m=0}^{\infty} \frac{a_m}{m!} T_m(x)(twy)^m p_m(u) \]

Operating both sides of the last equation by \( \exp(WC) \) we have

\[ \exp(WC)G(x, twy, u) = \exp(WC) \sum_{m=0}^{\infty} \frac{a_m}{m!} T_m(x)p_m(u)(twy)^m \]

L.H.S. of the last equation

\[ = \exp(WC)G(x, twy, u) \]

\[ = G \left( \sqrt{\frac{x-wy}{1-2xwy+w^2y^2}}, \sqrt{\frac{twy}{1-2xwy+w^2y^2}} \right), u \)

R.H.S. of the last equation

\[ = \sum_{r=0}^{\infty} \frac{(WC)^r}{r!} \sum_{m=0}^{\infty} \frac{a_m}{m!} T_m(x) t^{m+r} w^r y^m p_m(u) \]

\[ = \sum_{r=0}^{\infty} \frac{w^r}{r!} \sum_{m=0}^{\infty} \frac{a_m(m)_r}{m!} T_{m+r}(x) y^{m+r} t^m w^r p_m(u) \]
\[
= \sum_{r=0}^{\infty} \sum_{m=0}^{r} \frac{y^{r-m}}{(r-m)!} \frac{a_m}{m!} (m)_{r-m} T_r(x) y r_m w^m p_m(u)
\]

\[
= \sum_{r=0}^{\infty} \frac{T_r(x) (wy)^r}{r!} b_r(t,u)
\]

Where \( b_r(t,u) = \sum_{m=0}^{r} a_m \binom{r}{m} (m)_{r-m} p_m(u)t^m \)

Equating we have

\[
(5.4.2) \quad G\left(\frac{x-w}{\sqrt{1-2xw+w^2}}, \frac{tw}{\sqrt{1-2xw+w^2}}, u\right)
\]

Where \( b_r(t,u) = \sum_{m=0}^{r} a_m \binom{r}{m} (m)_{r-m} p_m(u)t^m \).

on putting \( y = 1 \).
EXTENSION OF LEE'S RESULT.

A trilateral generating function with the Techebichev polynomials, due to P.A.Lee, is as follows:

$$(5.5.1) \quad \sum_{n=0}^{\infty} z^n T_n(x) C_n(k; \alpha) C_n(\lambda; \beta) / n!$$

$$= \frac{1}{2} \left\{ e^{P} (1 - \frac{P}{\alpha})^{k} (1 - \frac{P}{\beta})^{\lambda} C_k(\lambda; \frac{\alpha - P}{P}) \right.$$  
+ $$e^{P'} (1 - \frac{P'}{\alpha})^{k} (1 - \frac{P'}{\beta})^{\lambda} C_k(\lambda; \frac{\alpha - P'}{P'}) \right\}$$

where $P = (x + \sqrt{x^2 - 1}) z$

and $P' = (x - \sqrt{x^2 - 1}) z$.

Replacing $z$ by $zyw$ we have

$$\sum_{n=0}^{\infty} w^n z^n y^n T_n(x) C_n(k; \alpha) C_n(\lambda; \beta) / n!$$

$$= \frac{1}{2} \left\{ e^{P} (1 - \frac{P}{\alpha})^{k} (1 - \frac{P}{\beta})^{\lambda} C_k(\lambda; \frac{\alpha - P}{P}) \right.$$  
+ $$e^{P'} (1 - \frac{P'}{\alpha})^{k} (1 - \frac{P'}{\beta})^{\lambda} C_k(\lambda; \frac{\alpha - P'}{P'}) \right\}$$
where \( \rho = (x + \sqrt{x^2 - 1}) \, zyw \)

and \( \rho' = (x - \sqrt{x^2 - 1}) \, zyw \)

Operating both sides by \( \exp WC \), L.H.S. of the last equation becomes

\[
\exp(WC) \sum_{n=0}^{\infty} w^{n} z^{n} y^{n} T_{n}(x) C_{n}(k; \alpha) C_{n}(l; \beta) / n!
\]

\[
= \sum_{p=0}^{\infty} \frac{w^{p} c^{p}}{p!} \sum_{n=0}^{\infty} z^{n} y^{n} T_{n+p}(x) C_{n+p}(k; \alpha) C_{n}(l; \beta) / n!
\]

\[
= \sum_{p=0}^{\infty} \frac{w^{p}}{(p-n)!} \sum_{n=0}^{\infty} \frac{n!}{(p-n)!} w^{n} z^{n} y^{p} T_{n+p}(x) C_{n+p}(k; \alpha) C_{n}(l; \beta) / n!
\]

\[
= \sum_{p=0}^{\infty} \frac{T_{p}(x)(wy)^{p}}{p!} \sum_{n=0}^{p} \binom{n+p}{n} c_{n}(k; \alpha) c_{n}(l; \beta) z^{n}
\]

R.H.S. of the last equation

\[
= \exp(wc) \left\{ e^{\rho} (1 - \frac{\rho}{\alpha})^{k} (1 - \frac{\rho}{\beta})^{l} c_{k} (x; \alpha; \frac{\rho}{\beta} (\beta - \rho)) + e^{\rho'} (1 - \frac{\rho'}{\alpha'})^{k} (1 - \frac{\rho'}{\beta'})^{l} c_{k} (x; \alpha'; \frac{\rho'}{\beta'} (\beta' - \rho')). \right\}
\]
where \( p = (x + \sqrt{x^2 - 1}) zyw \)

and \( p' = (x - \sqrt{x^2 - 1}) zyw \).

\[
\frac{1}{8} \left( e^{\lambda_1 (1 - \frac{\lambda_1}{\alpha})} \left( 1 - \frac{\lambda_1}{\beta} \right) \frac{1}{C_k(l; \frac{\lambda_1}{\alpha} \frac{\beta - \lambda_1}{\lambda_2})} + e^{\lambda_2 (1 - \frac{\lambda_2}{\alpha})} \left( 1 - \frac{\lambda_2}{\beta} \right) \frac{1}{C_k(l; \frac{\lambda_2}{\alpha} \frac{\beta - \lambda_2}{\lambda_2})} \right)
\]

Where \( \lambda_1 = \frac{(x - w y + \sqrt{x^2 - 1}) z w}{1 - 2x w + w^2 y^2} \)

and \( \lambda_2 = \frac{(x - w y - \sqrt{x^2 - 1}) z w}{1 - 2x w + w^2 y^2} \)

Equating we obtain

\[
(5.5.2) \sum_{p=0}^{\infty} \frac{T_p(x) w^p}{p!} = \sum_{n=0}^{b} \left( \frac{p}{n} \right)_{p-n} C_n(k; \alpha) C_n(l; \beta) z^n
\]

\[
= \exp(wC)\frac{1}{8} \left( e^{\lambda_1 (1 - \frac{\lambda_1}{\alpha})} \left( 1 - \frac{\lambda_1}{\beta} \right) \frac{1}{C_k(l; \frac{\lambda_1}{\alpha} \frac{\beta - \lambda_1}{\lambda_2})} + e^{\lambda_2 (1 - \frac{\lambda_2}{\alpha})} \left( 1 - \frac{\lambda_2}{\beta} \right) \frac{1}{C_k(l; \frac{\lambda_2}{\alpha} \frac{\beta - \lambda_2}{\lambda_2})} \right)
\]

where \( \lambda_1 = \frac{(x - w + \sqrt{x^2 - 1}) z w}{1 - 2x w + w^2 y^2} \)

and \( \lambda_2 = \frac{(x - w - \sqrt{x^2 - 1}) z w}{1 - 2x w + w^2 y^2} \)

on putting \( y = 1 \).
REFERENCES
