Chapter 5

OUTLOOK AND CONCLUDING REMARKS

The question of characterizing and classifying possible excitations of non-linear lattice systems is a fascinatingly complex one. Confining our attention to a 1D lattice with a FHN type dynamics involving a fast and a slow variable, for instance, while we have obtained the single pulse solution and the 1-parameter family of periodic pulse train solutions, it remains to investigate the possibility of $N$-pulse solutions for finite integers $N$, similar to those in 1D continuum models [63–66]. Another related question is the possibility of chaotic pulse trains where the sequence of spacings between successive pulses in a pulse train emerges as a random or a pseudorandom sequence. Yet a third area of possible interest involves excitations on a Nagumo type or FHN type ring where the variables satisfy a periodic boundary condition. Each of these questions requires a careful analysis of the slow dynamics relating to the recovery variable.
While a singular perturbation approach similar to that in chapter three with slow dynamics in between fast level transitions representing kink and antikink solutions may not be of direct use in looking for such new multiple pulse solutions, one may still fruitfully adopt it in conjunction with bifurcation analysis involving heteroclinic and saddle node structures in reaction-diffusion systems (see, e.g., [170, 171]).

In this context, it may be of some interest to look into the following aspect of travelling pulse trains on a FHN ring. The wavelength and time-period of such a pulse train are given by

\[ \Delta n = Z, \]
\[ T = \frac{1}{X} Z, \]

where \( Z \) is the period in terms of the propagation variable \( (\zeta) \) depending on the parameters of the systems in accordance with eq.(3.3.2d). The wavelength \( \Delta n \) (number of lattice sites after which the values of \( v_n, w_n \) are repeated at any given instant of time) in turn is related to the size \( N \) of the ring as

\[ N = k \Delta n, \]

where \( k (\geq 1) \) is an integer.

For sufficiently small values of \( \epsilon \), the results of chapter three show that \( Z \) varies by a large amount for relatively small variation of \( W \) (see chapter three) or \( X \), and a lattice ring of given size \( N \) can support a travelling pulse train with a given integer number \((k)\) of pulses filling up the lattice only for a specific value \((X_k)\) of the speed. While one expects that a pulse train with this specific speed constitutes an excitation on the lat-
tice that can circulate indefinitely without any alteration of shape, it is not immediately apparent as to what happens to perturbations imposed on the pulse profile, the range of the perturbations being comparable to the ring size. Though there exist speeds \((\chi_{k\pm1})\) close to the speed \((\chi_k)\) of the original pulse train, the perturbation may not necessarily evolve towards a pulse train with \(k \pm 1\) pulses filling up the ring since this would involve some kind of a drastic change of shape of a pulse train that the initial perturbation may not allow for. In other words, the question of stability of such pulse trains (periodic structures close to a heteroclinic connection) seems to be a delicate one and may imply highly complex spatio-temporal structures.

A related question pertains to excitations in a discrete Nagumo type system on a ring where one has, once again, periodic boundary conditions, but now without the slow change of the recovery variable \(w\). As in the case of the open lattice \((N \to \infty)\) (see sec. 1.3.3, sec. 1.3.4 and [82, 107]) there exists a large number of Turing type stationary patterns on the ring, these being precisely those stationary patterns in the open lattice that satisfy the periodic boundary condition of the ring. Because of the periodic boundary conditions, none of these stationary patterns is monotonic, and results in [15, 83] tell us that corresponding stable travelling patterns cannot exist in the open lattice. However, it seems worthwhile to explore whether stable travelling patterns obtained by the dressing (see chapter four) of these stationary ones can exist on the ring, with their propagation speed going to zero as the ring size \((N)\) goes to infinity. Such travelling solutions would then constitute a complex set of spatio-temporal structures on a spatially discrete Nagumo type ring.
Finally, the technique of construction of travelling front solutions in the PWL Nagumo system adopted in chapter two, can be employed to work out travelling front solutions in higher dimensions and their features like pinning and anisotropy of propagation (see, e.g., [57]).

These and related investigations on moving excitations in lattice systems made up of bi-stable (or, more generally, excitable) units and also of coupled units following a conservative Hamiltonian dynamics are in the process of being taken up as fruitful consequences of the basic approach adopted in the present thesis.