In this chapter, we present some optimization models at the farm level to clarify some of the issues raised in Chapters I and II.

A few words regarding the notation used in this Chapter might be helpful. As usual suffix notation is used for partial derivatives of a function of more than one variables, reserving dash and double dashes for first order and second order derivatives of functions of one variable. In the case of Model V, a departure from the usual convention has been made. Usually $u_i$ refers to the partial derivative of $u$ with respect to its $i$th argument. Since in this model, the consumption alternative is indicated by $C_3$ and the utility function in the model is $u(C_1, C_3)$, the derivative of $u$ with respect to $C_3$ should be indicated by $u_2$. We have, however, used $u_3$ in place of $u_2$ to maintain a symmetry in the equilibrium conditions. Parameters are indicated by a bar over the symbol. Where parameters are subjected to change, the bar is withdrawn.

For functions, small letters are used to differentiate them from determinants which are indicated by capital letters. Double suffixes attached to a capital letter indicate minor determinants derived from the indicated determinants.

We shall start with a simplified version of Nakajima's model of a pure commercialized farm to show how even for such a farm, the price elasticity of marketing could be negative if the
income effect is stronger than the substitution effect, both of typo one.

\[ \text{MODEL: } u = u(A, Y) \]

Subject to \[ Y = Y_o + PQ(A) \]

\[ \text{NOTATION} \]

\( A \) is the amount of family labour, \( Y \), the level of family income, \( Y_o \), money income from other sources, \( Q \), the level of farm output, \( u \), the utility function of the family.

\[ \text{ASSUMPTIONS} \]

\( u_1 < 0, u_2 > 0 \)

\[ \begin{bmatrix} u_{11} & u_{12} & u_1 \\ u_{21} & u_{22} & u_2 \end{bmatrix} > 0 \]

and

\[ \begin{bmatrix} w_{31} \\ w_{11} \\ w_{21} \end{bmatrix} = \begin{bmatrix} u_{12} & u_{11} \\ u_{22} & u_2 \end{bmatrix} < 0 \]

The Lagrangian \( L = u(A, Y) + \lambda \left[ Y - Y_o - PQ(A) \right] \)

The equilibrium conditions are (assuming \( A < A \))

\[ \begin{align*}
(60.1) & \quad u_1 = \lambda \cdot PQ'(A) = 0 \\
(60.2) & \quad u_2 + \lambda = 0 \\
(60.3) & \quad Y - Y_o - PQ(A) = 0
\end{align*} \]

They give

\[ \frac{PQ'(A)}{u_2} = \frac{-u_1}{u_2} = \frac{dY}{dA} \quad \text{u = constant.} \]
Define $D$ as

$$
D = \begin{vmatrix}
  u_{11} & u_{12} & \dot{P}Q''(A) \\
  u_{21} & u_{22} & \dot{P}Q'(A) \\
  \dot{P}Q'(A) & 1 & 0
\end{vmatrix}
$$

$$
D = \frac{\mu}{u_2^2} + \lambda \dot{P}Q''(A)
$$

Hence, $D > 0$, since $\lambda < 0$.

The second order condition for a minimum is therefore satisfied. The following results could be derived.

$$
\frac{\partial}{\partial Y} \left( \frac{-u_1}{u_2} \right) = \frac{-u_{31}}{u_2^2} > 0
$$

$$
\frac{\partial}{\partial A} \left( \frac{-u_1}{u_2} \right) \bigg|_{u = \text{constant}} = \frac{\dot{w}}{u_2^3} > 0
$$

$$
\frac{d A}{d Y_0} = \frac{u_{31}}{u_2 D} < 0
$$

$$
\frac{d A}{d \bar{P}} = \frac{-\lambda}{D} \frac{\dot{P}Q'(A)}{u_2} + \frac{Q(A)}{D} \frac{u_{31}}{u_2} = \frac{-\lambda \dot{P}Q'(A) + Q(A) \frac{u_{31}}{D}}{D}
$$

**ECONOMIC IMPLICATIONS**

If we plot $A$ on the horizontal axis and $Y$ on the vertical axis the indifference curve between labour and money income will
have a positive slope.

\[ \frac{dY}{dA} \bigg|_{u = \text{const.}} = \frac{-u_1}{u_2} > 0 \]

In equilibrium this will be equal to value of marginal productivity of labour.

Equation (64) implies that as we move along the indifference curve to the right and upwards (like a movement from A to C in the diagram) the slope will become steeper and steeper since \( w \) is positive. Similarly equation (63) implies that if we move vertically upwards from one point of an indifference curve to another point on another indifference curve (say, like a movement from A to B), the slope will also get steeper since \( w_{31} \) is negative. The amount \( (A - A) \) is the leisure consumed by the family as a whole.

Equation (65) implies that the effect of a change in the price on the family labour will depend on the relative strengths of the substitution effect of type one which is positive and \( Q(A) \frac{w_{31}}{u_2} \), the income effect of type one, which is negative.

The effect on output of a price change will be in the same direction as its effect on family labour since
\[
\frac{d Q}{d P} = Q'(A) \cdot \frac{d A}{d P}
\]

**MODEL - II**

Max \( R_1, R_2, Y = \bar{p}_1 Q_1(R_1) + \bar{p}_2 Q_2(R_2) \)

\( R_1 + R_2 = \bar{R} = \text{given}. \)

For convenience, let us write; \( Q_1 = f(R_1) \), and \( Q_2 = g(R_2) \).

**NOTATION**

\( Y \) is the total gross income of the farm which is also equal to total net income since the resources used have no opportunity cost, so to say, the total resources used being \( \bar{R} \), which is given. \( R_1 \) is the amount of resources devoted to the production of \( i \)th commodity, \( i = 1, 2 \). Resources are non-disposable in the sense that the entire amount \( \bar{R} \) has to be used.

**ASSUMPTIONS**

\( f' > 0, \quad g' > 0 \)

The Lagrangean is

\[
= \bar{p}_1 f'(R_1) + \bar{p}_2 g'(R_2) + (\bar{R} - R_1 - R_2)
\]

The equilibrium conditions are (assuming \( 0 \leq R_i \leq \bar{R}, i = 1, 2 \))

\( (68.1) \quad \bar{p}_1 f'(R_1) - \lambda = 0 \)

\( (68.2) \quad \bar{p}_2 g'(R_2) - \lambda = 0 \)

\( (68.3) \quad R - R_1 - R_2 = 0 \)

The second order condition for a maximum will be satisfied if (but not only if)
Note that
\[
\frac{d Q_1}{d P_2} = \frac{d Q_1}{d R_1} \cdot \frac{d R_1}{d P_2}
\]
But,
\[
\frac{d R_1}{d P_2} = \frac{g'}{f'g'' + p_2 g''}
\]
Hence,
\[
\frac{d Q_1}{d P_2} = \frac{g' f'}{f'g'' + p_2 g''}
\]
\[
\frac{d Q_2}{d Q_1} \bigg|_{R = \text{const.}} = \frac{\frac{p_2}{F_1}}{g'' + \frac{g'}{f'^2}}
\]
and
\[
\frac{d^2 Q_1}{d Q_1^2} \bigg|_{R = \text{Constant.}} = \frac{\frac{F_1 g'' + p_2 f''}{F_1 f'^2}}
\]

**Economic Interpretation**

If the production possibility curve is strictly concave to the origin, then
\[
P_1 f'' + p_2 g'' < 0
\]
and the second-order condition for a maximum will be satisfied. But condition (69) implies \(\frac{d R_1}{d P_2} < 0\) and \(\frac{d Q_1}{d P_2} < 0\).

Hence the resource transfer effect would be strictly positive.
MODEL - III

\[
\begin{align*}
\text{Max } & \quad u( C_1, C_2 ) \\
\text{Subject to } & \quad \bar{P}_1 Q_1 (\bar{P}_1) - \bar{P}_1 C_1 - \bar{P}_2 C_2 + \bar{Y}_0 = 0
\end{align*}
\]

NOTATION

The farm is producing \( Q_x \) amount of the first commodity and home-consumption \( C_x \) amount of the same commodity. The proceeds of sale of the first commodity at a given price, \( \bar{P}_1 \), supplemented by income from other sources, are used for the purchase of the consumption alternative at a given price \( \bar{P}_2 \).

ASSUMPTIONS

\( u_1 > 0, \quad \forall i = 1, 2 \), and

\( U > 0 \) where

\[
U = \begin{bmatrix}
  \ u_{11} & \ u_{12} & \ u_1 \\
  \ u_{12} & \ u_{22} & \ u_2 \\
  \ u_1 & \ u_2 & \ 0
\end{bmatrix}
\]

The Lagrangian is

\[
u(C_1, C_2) + \lambda \sqrt{P_1 Q_1 (P_1) - P_1 C_1 - P_2 C_2 + Y_0}
\]

The equilibrium conditions are (assuming \( C_1 < Q_1 \))

\[
\begin{align*}
(75) \quad & u_i - \lambda \bar{P}_i = 0, \quad i = 1, 2 \\
(76) \quad & \bar{P}_1 Q_1 (\bar{P}_1) - \bar{P}_1 C_1 - \bar{P}_2 C_2 + \bar{Y}_0 = 0
\end{align*}
\]

Define \( D = \begin{bmatrix}
  u_{11} & u_{12} & -P_1 \\
  u_{12} & u_{22} & -P_2 \\
  -P_1 & -P_2 & 0
\end{bmatrix} \)

\[
(77) \quad D = \frac{1}{\lambda^2} \cdot u > 0.
\]
The second order condition for a maximum is, therefore, satisfied,

\[(78) \quad \frac{d^2 C_2}{d C_1^2} \quad \text{if } u = \text{const.} = \frac{u_1}{u_2} \]

\[(79) \quad \left( \frac{u_1}{u_2} \right) = \frac{U_{32}}{u_2} \]

\[(80) \quad \left( \frac{u_1}{u_2} \right) = -\frac{U_{31}}{u_2} \]

\[(81) \quad \frac{d^2 C_2}{d C_1^2} \quad \text{if } u = \text{const.} = \frac{U}{u_2} \]

Define $M_1 = Q_1 - C_1$ as marketable surplus of first commodity, hence we derive,

\[(82) \quad \frac{\partial C_1}{\partial P_1} = \frac{\lambda P_1^2}{D} + \frac{M_1 U_{31}}{\lambda D} + \frac{P_1 Q_1^i (P_1)}{D} U_{31} \]

\[(83) \quad \frac{\partial C_2}{\partial P_1} = \frac{\lambda P_1 P_2}{D} - \frac{M_1 U_{32}}{\lambda D} - \frac{P_1 Q_1^i (P_1)}{D} U_{32} \]

\[(84) \quad \frac{\partial C_1}{\partial P_2} = \frac{\lambda P_1 P_2}{D} - \frac{M_1 U_{31}}{\lambda D} \]

\[(85) \quad \frac{\partial C_2}{\partial P_2} = \frac{\lambda P_1^2}{D} + \frac{M_1 U_{32}}{\lambda D} \]

When $Q_1$ is predetermined, i.e., when $Q_1^i (P_1) = 0$

\[(82.1) \quad \frac{\partial C_1}{\partial P_1} = \frac{\lambda P_1^2}{D} + \frac{M_1 U_{31}}{\lambda D} \]

\[(83.1) \quad \frac{\partial C_2}{\partial P_1} = \frac{\lambda P_1 P_2}{D} = \frac{M_1 U_{32}}{\lambda D} \]

\[(86) \quad \frac{\partial M_1 P_i}{\partial P_1} = \frac{\partial M_1}{\partial P_1} \cdot \frac{P_i}{M_1} = \frac{\partial C_1}{\partial P_1} \cdot \frac{P_i}{M_1} \quad i = 1, 2 \]
We also have, for once for all change in $Q_1$

\[ \frac{dC_i}{dQ_1} = \frac{P_1 U_{3i}}{P_i}, \quad i = 1, 2. \]

When $Y_0 = 0$, and $Q_1$ predetermined: we have

\[ e_{M_1} P_1 + e_{M_2} P_2 = 0. \]

Hence

\[ (87.1) \quad \begin{vmatrix} e_{M_1} P_1 \\ e_{M_2} P_2 \end{vmatrix} = \begin{vmatrix} e_{M_1} P_2 \\ e_{M_2} P_1 \end{vmatrix} \]

\[ e_{C_1} P_2 + e_{C_2} P_1 = 0. \]

Thus:

\[ (88.1) \quad \begin{vmatrix} e_{C_1} P_2 \\ e_{C_2} P_1 \end{vmatrix} = \begin{vmatrix} e_{C_2} P_1 \\ e_{C_1} P_2 \end{vmatrix} \]

\[ \frac{dC_1}{dP_1} = \frac{\sum C_1}{\delta P_1} \frac{dP_1}{dP_2} + \frac{\sum C_1}{\delta P_2} \frac{dP_2}{dP_1} \]

When (i) $P_1$ rises in the same proportion, i.e., $dP_1/dP_2 = P_1/P_2$, and (ii) $Y_0 = 0$, we have,

\[ (90) \quad \frac{dC_1}{dP_1} = \frac{P_1 Q_1 (P_1) U_{31}}{dP_1} dP_1 \]

\[ (90.1) \quad = 0, \]

Lastly, when $Y_0 = 0$, we have, both in the case of $Q_1$

being predetermined and in the case it is a function of $P_1$,

\[ (91) \quad e_{M_1} P_1 = e_{C_2} P_1^{-1} \]
Let us first consider what could happen if $U_{31} \geq 0$ and $U_{32} > 0$. The above two conditions would be trivially satisfied under classical assumptions, namely,

\[ u_{ij} = 0, \quad i \neq j \]

\[ < 0, \quad i = j \]

Besides the above two conditions are also sufficient to make $U$ positive, since,

\[ U = u_1, \quad U_{31} = u_2 U_{32} \]

Geometrically, $U_{31} > 0$, implies that as one moves from one point of an indifference curve to another point on an indifference curve to its right like $A$ to $B$ in the diagram parallel to the $x$-axis, (assuming the first commodity is represented on the $x$-axis and the second commodity on the $y$-axis), the slope of the indifference curve gets flatter \[ \text{equation (79)} \] while $U_{32} < 0$ implies that as one moves vertically upwards from one point of an indifference curve to a point on a higher indifference curve ($A$ to $C$) the slope gets steeper. \[ \text{Equation (80)} \]. The positiveness of $U$, implies that as one moves from $C$ to $B$ on the same indifference curve, the slope gets flatter and flatter. It also indicates that the indifference curve is convex to the origin, or the function is quasi-concave. \[ \text{Equation (81)} \].

Diagram — (7)
Now $U_{31} > 0$ and $U_{32} < 0$ also imply by virtue of equations (82) to (85).

\[ (94) \frac{\delta C_2}{\delta P_1} > 0 \text{ and } \frac{\delta C_2}{\delta P_2} < 0 \text{ but } \]

\[ (95) \frac{\delta C_1}{\delta P_1} \text{ and } \frac{\delta C_1}{\delta P_2} \text{ are indeterminate.} \]

The equation (91) suggests that so long as the only source of income is the production of the crop, the sign of the price elasticity of marketable surplus can be derived from the magnitude of the cross-price elasticity of the consumption alternative alone, whether output is pre-determined or not.

A simple economic explanation of equation (91) can be given. Suppose $P_1$ has increased by one per cent, $P_2$ remaining the same. As a consequence, let us assume that $C_2$ has increased by $1 + k$ per cent, where $k$ is positive. Then total expenditure on $C_2$ has also increased by $1 + k$ per cent and to finance it $M_1$ must increase by $k$ per cent. If $k$ is negative then $M_1$ must decrease by $k$ per cent.

Thus the elasticity of marketable surplus is positive, zero or negative, according as the cross price-elasticity of consumption alternative is greater than, equal to, or less than unity. Besides when output itself can be adjusted to the change in price, a comparison of equation (83) and (83.1) suggests that the magnitude of the above cross price-elasticity, (hence, the possibility of its becoming more than unity) increases if $Q_1'(P_1)$ is positive and $U_{32}$ is negative.
In the short-run situation, i.e., when $Q_1$ is pre-determined and assuming, once again, $V_0 = 0$, we get

1. Any increase in the prices $P_1$ and $P_2$ by the same proportion will have no impact on consumer's equilibrium. This follows from equation (86.1), which tells us $d C_1$, and hence by virtue of equation (76), $d C_2$ will also be zero. Hence $M_1$ will also remain unchanged, so long as the relative price remain the same.

2. The sign of the elasticity of marketable surplus of the first commodity, with respect to the absolute price of the absolute price of the first commodity will be opposite of that of the elasticity of marketable surplus with respect to the absolute price of the second commodity.

3. The elasticity of marketable surplus with respect to $P_1$ will be equal in magnitude to the elasticity of marketable surplus with respect to $P_2$ since the above two elasticities are of opposite signs and add up to zero. In view of the above three propositions, it makes sense to speak of the relative price-elasticity of the marketable surplus since how the relative price changes (i.e., through a change in $P_1$ or $P_2$ or both) is immaterial. In fact, we can take any of the commodity as the numeraire and set its price equal to unity. Lastly the elasticity of marketable surplus with respect to the relative price of commodity one (i.e., $P_1 / P_2$) is positive, zero, or negative, according as the own price-elasticity of demand for the consumption alternative is greater than equal to or less than unity, in magnitude.

This is because when $Q_1$ is predetermined and $V_0 = 0$, if the cross-price elasticity of the consumption alternative is
greater than unity, the own-price elasticity of the consumption alternative should also be, in magnitude, greater than unity, since the above two elasticities and up to zero.

A simple economic explanation of it is when only \( P_2 \) has decreased, \( P_1 \) remaining the same if the demand for \( C_2 \) is own-price elastic, total expenditure on \( C_2 \) will increase with a decrease in the price of \( P_2 \). To finance this increased expenditure, marketed surplus of the first commodity has to be increased because of the budget constraint. Note that when \( Q \) is predetermined and \( \bar{y}_0 = 0 \), any change in \( P_1 \) by \( K \) proportion is equivalent to a change to \( P_2 \) by \( 1/K \) proportion.

When \( Q_1 \) is not pre-determined, the impact of a change in the relative price of the two commodities will also depend on the absolute changes in the price-levels of the two commodities, and the marketable surplus may change even when the relative price has not changed.

Let us give a numerical example to illustrate the above points.

Suppose, to start with, we have the situation

\[
\begin{align*}
P_1 &= 2 \\
P_2 &= 3 \\
C_1 &= 10 \\
C_2 &= 10 \\
Q_1 &= 40 \\
M_1 &= 30
\end{align*}
\]

(A)

As a result of an increase in \( P_1 \), we now have situation B, assuming \( Q_1 \) is pre-determined.

\[
\begin{align*}
P_1 &= 3 \\
P_2 &= 3 \\
C_1 &= 12 \\
C_2 &= 28 \\
Q_1 &= 40 \\
M_1 &= 28
\end{align*}
\]
Comparing situations (A) and (B), we find that while \( P_1 \) has increased by 50%, \( C_2 \) has increased by less than 50% so the cross-price-elasticity of \( C_2 \) is less than unity. The own-elasticity of marketable surplus of the first commodity is negative.

Let us now consider situation (B₁) assuming once again that \( Q_1 \) is pre-determined.

\[
\begin{align*}
(B_1) & \quad P_1 = 2 \quad C_1 = 12 \quad Q_1 = 40 \\
& \quad P_2 = 2 \quad C_2 = 28 \quad M_1 = 28
\end{align*}
\]

The values of \( C_1 \) and \( C_2 \) are derived by noting that the relative price of the two commodities is the same in situations (B) and (B₁).

Comparing situations (A) and (B₁) we find that the own-price elasticity of \( C_2 \) is also less than unity and the total expenditure on \( C_2 \) in situation (B₁) is greater than that in situation (A).

Now, if \( Q_1 \) is not pre-determined, then, we may have, as a result of an increase of \( P_1 \) by 50%, instead of situation (B), situation (B₂) so that,

\[
\begin{align*}
(B_2) & \quad P_1 = 3 \quad C_1 = 18 \quad Q_1 = 50 \\
& \quad P_2 = 3 \quad C_2 = 32 \quad M_1 = 32
\end{align*}
\]

Since as between situations (A) and (B₂), the cross-price elasticity of \( C_2 \) is greater than unity, (demand for \( C_2 \) has risen by more than 50%), the own-price elasticity of \( M_1 \) is also positive. But situation (B₁) may still continue to happen as a result of a decrease of price of \( P_2 \) from situation (A).

So if the relative price of the first commodity increases due to a rise in the price of the first commodity, the marketable
surplus increases, on the other hand, if it increases due to a decrease in the price of the consumption alternative, marketable surplus would decrease.

If we put alternative values of \( C_2 \) in situation (B), such as \( C_2 = 30 \) or \( C_2 = 32 \), it can be checked that \( M_1 \) would also be equal to 30 (or 32) so that equation (91) is verified. Correspondingly in situation (\( B_1 \)), also, \( C_2 = 30 \) or 32, so that the own-price elasticity of \( C_2 \) as between situations (A) and (\( B_1 \)) would be zero or positive respectively.

Since, we are dealing with discrete data, the magnitude of cross-price elasticity of \( C_2 \) would, however, be not exactly equal to the own-price elasticity of \( C_2 \). Assuming that \( Q_1 \) is predetermined, one can give a simple economic explanation of the inequalities (94) under classical assumptions of additive utility function and diminishing marginal utilities.

From equation (75), we may write

\[
(75.1) \quad u_1 = u_2 \cdot \frac{P_1}{P_2}
\]

i.e., in equilibrium, the marginal utility of home consumption is equal to the marginal utility of sale (a term used by Khusrro \( \mathcal{J} \)).

Now, suppose, \( P_1 \) has increased (or \( P_2 \) has decreased). The marginal utility of sale is, thereby, increased. Let us cut down \( C_1 \) by imposing a levy on the consumer to the extent of \( q_1 \) amount (say), so that the marginal utility of home consumption increases in such a way that the above equality (75.1) is restored. At this point we have the equilibrium situation corresponding to the changed levels of prices and a predetermined output of the level of
C₃ - q₁. Let us now restore q₁ to the consumer. By virtue of equation (86), C₂ would now increase from its former level.

From the above, it may appear that as q₁ is restored, the increase in C₁ out of it could not be greater than q₁. But this is not correct, since the increase in P₁ (or the decrease in P₂) enables the consumer to increase C₁ from its former level even when C₂ is increased from its former level so long as the total expenditure on the consumption alternative divided by the price of the second commodity in the new equilibrium situation is less than that in the former equilibrium situation.

Alternatively, if the marginal utility of home consumption is higher in the new equilibrium, C₁ would also be lower in the new equilibrium. But that means, the marginal utility of sale will also be higher in the new equilibrium. But the latter has two components. One of that (i.e., P₁/P₂) is definitely higher. But u₂ must be lower as C₂ in new equilibrium situation is higher. So nothing definite emerges about C₁ in the new equilibrium situation.

From the above it is thus apparent that classical assumptions of separable utility function and diminishing marginal utilities are not sufficient to make the price-elasticity of marketable surplus positive even when the output is predetermined, as Khusro tried to prove geometrically.

For the sake of completeness, it may be noted that if P₁ decreases (or P₂ increases) we could lend the consumer q₁ amount(say) to increase his consumption so that the equality (73.1) is restored. Now as Q₁ is taken away from the consumer C₂ would decrease from its former level.
**MODEL - IV**

Max \( u (C_1, C_3) \)

\[
\bar{P}_1 C_1 + \bar{P}_3 C_3 = \bar{P}_1 Q_1 + \bar{P}_2 Q_2
\]

\[
Q_1 = f (R_1)
\]

\[
Q_2 = g (R_2)
\]

\[
R_1 + R_2 = \bar{R}
\]

The model can be decomposed into two sub-models

**MODEL - IVa**

Max \( Y = \bar{P}_1 Q_1 (R_1) + \bar{P}_2 Q_2 (R_2) \)

Subject to \( R_1 + R_2 = \bar{R} \)

This is Model II and hence from (68.1 to 68.3).

(68.1) \[ \bar{P}_1 f' = \bar{P}_2 g' \] and

(68.2) \[ \frac{d R_1}{d \bar{P}_1} + \frac{d R_2}{d \bar{P}_1} = 0 \]

From the above equations, we derive \( Y^0 = Y^0 (P_1, P_2, R) \)

and by virtue of equation (68.5), \( \frac{\partial Y^0}{\partial \bar{P}_1} = Q_1 \)

**MODEL - IVb**

Max \( u (C_1, C_3) \)

Subject to \( P_1 C_1 + P_3 C_3 = \bar{Y}^0 (P_1, P_2, R) \).

It is very similar to Model III.

\[
D = \frac{1}{\lambda^2} \cdot U - \lambda \bar{P}_1 Q_1^t (A) \bar{P}_3 > 0, \text{ since }
\]

\[
\lambda = u_3 / \bar{P}_3 > 0.
\]
The second order condition is therefore satisfied. We derive

\[ \frac{\delta^2 c_3}{\delta p_1} = \frac{\lambda q_1^i (A) \int p_1 p_3}{D} + \frac{d_{32} m_1}{D} \]  

\[ \frac{\delta^2 c_3}{\delta p_3} = -\frac{\lambda q_1^i (A) \int p_1 p_3}{D} + \frac{d_{32} c_3}{D} \]

where \( m_1 = q_1(A) - \bar{c}_1 \)

\[ e_{c_3} p_1 = \frac{\delta c_3}{\delta p_1} \cdot \frac{p_1}{c_3} = \frac{\lambda q_1^i (A) \int p_1 p_3}{c_3} + \frac{d_{32} p_1 m_1}{D c_3} \]

\[ e_{c_3} p_3 = \frac{\delta c_3}{\delta p_3} \cdot \frac{p_3}{c_3} = -\frac{\lambda q_1^i (A) \int p_1 p_3}{c_3} + \frac{d_{32} p_3}{D} \]

\[ e_{c_3} p_1 + e_{c_3} p_3 = 0 \]

\[ \frac{\delta A}{\delta p_1} = \frac{\lambda q_1^i (A) p_3^2}{D} + \frac{m_1 d_{31}}{D} \]

\[ \frac{\delta A}{\delta p_3} = -\frac{\lambda p_3 p_1 q_1^i (A)}{D} + \frac{c_3 d_{31}}{D} \]

Hence

\[ e_A p_1 + e_A p_3 = 0 \]

Similarly since \( \frac{\delta m_1}{\delta p_i} = q_1^i (A) \frac{\delta A}{\delta p_i} \) (i = 1, 3)

\[ e_{m_1} p_1 + e_{m_1} p_3 = 0. \]
The above model is very similar to Ghatak's model except for some minor points of difference. We first present a table to translate our notation in terms of Ghatak's notation.

<table>
<thead>
<tr>
<th>Ghatak's notation</th>
<th>$u_a$</th>
<th>$a$</th>
<th>$i$</th>
<th>$L$</th>
<th>$P_a$</th>
<th>$P_i$</th>
<th>$M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Notation in this Paper</td>
<td>$u$</td>
<td>$c_1$</td>
<td>$c_3$</td>
<td>$A-A$</td>
<td>$P_1$</td>
<td>$P_3$</td>
<td>$M_1$</td>
</tr>
</tbody>
</table>

The major points of departure of the Ghatak model are:

1. In Ghatak's model, $a$ stands for consumption of agricultural goods and $i$, for consumption of industrial goods. $P_a$ and $P_i$ are their respective price levels. In our model, $C_1$ is the consumption of produced goods and $C_3$ for consumption of all other non-produced, cash-purchased goods. It may be noted, it is no longer appropriate to interpret $C_3$ as the consumption alternative since $C_1$ is now being predetermined, it has no alternative or substitute.

2. In Ghatak's model, $M$ is defined directly as $M = f(L)$ and in our case $M_1 = Q_i(A) - c_1$.

3. The budget constraint in Ghatak's model is also different, namely,

$$p_a M = p_i i + T + R$$

where $T =$ taxes and $R =$ rents.

It may be noted that the equilibrium conditions equations 100.1 to 100.3 are such that if both prices rise in the same proportion, $C_1$, $C_3$ and $A$ would remain unchanged. So, in our model only relative prices matter. Ghatak's model, because of the incorporation of $T$ and $R$ terms has not the above property since while gross money income will change in the same proportion as prices, net income will
rise more than proportionately. Equation (109) will therefore not be satisfied for Ghatak's model. In our model, it is meaningful to speak of relative price elasticity of marketable surplus of the first commodity, since how the relative price changes (i.e., through a change in \(P_1\) or \(P_3\) or both) will have no impact on the sign or magnitude of the elasticity of marketable surplus with respect to a change in the relative price. Unfortunately this is not true in the case of Ghatak's model.

Yet Ghatak makes the demand for industrial goods as well as marketable surplus of agricultural goods a function of relative price, \(p_a/p_i\). Furthermore, he has assumed that the demand for industrial goods is an increasing function of relative price \(p_i/p_a\) and derives from it the positive price elasticity of the marketable surplus function.

Even if we set \(T = R = 0\) in Ghatak's system, the positive relative price elasticity of the demand for industrial goods will be necessarily true if \(D_{32}\) is positive. Under classical assumptions, as stated before, i.e., if we have \(u_{23} = 0\) and \(u_{22} < 0\), \(D_{32}\) would, however, be positive.

But even when the demand for industrial goods is an increasing functions of \(p_a/p_i\) it would not be sufficient to make the relative price elasticity of marketable surplus function positive as Ghatak tries to prove geometrically, unless the own price elasticity (or cross-price elasticity) of demand for industrial goods is greater than unity in magnitude.
Ghatak's geometric proof is based on his supposition that an increase in the demand for industrial goods will lead to a decrease in leisure.

Now, a sufficient condition for the relative price elasticity of the marketable surplus being positive in our system (or in Ghatak's system, setting $T = R = 0$) is demand for leisure increases as the relative price of the first commodity, in relation to other consumption goods increases.

The sufficiency of the condition should be intuitively obvious. As $A$ increases, $Q_1$ should increase and since $C_1$ is fixed, $N_1$ should also increase.

Once again, equations (106 to 107) suggest that a sufficient condition for the demand for leisure to increase with an increase in $P_1$ (or a decrease in $P_3$) is that $D_{31}$ should be negative. But under classical assumptions ($u_{23} = 0$, $u_{33} < 0$), $D_{31}$ should be positive.

If we assume $D_{31}$ to be positive, model IV turns into a modified version of Nakagima's Model of a pure commercialized farm (Model I). Instead of $C_1$ being zero, it is set at a constant level. The sign of the price elasticity of marketable surplus will thus be the same as the sign of the price elasticity of output response, which again would depend on the relative strengths of the substitution effect between leisure and money income (to be spent on non-produced consumption goods) and the income effect on more consumption of leisure.

Ghatak's geometric proof, can be represented in the following way:
From what has been said above, it is apparent that the weakest chain in the link is $i \uparrow \Rightarrow L \downarrow \Rightarrow M \uparrow$.

What Ghatak has failed to recognize is that a relative increase in the price of agricultural goods (in his model) in relation to industrial goods in quite consistent with more consumption of leisure (as a result of income effect of type one) and this again may happen even when more industrial goods are being consumed. To give a numerical example, suppose to start with, we have $D_1$.

\[
P_1 = 3, \quad P_3 = 3, \quad A = 10, \quad Q_1 = 22
\]
\[
c_1 = 15, \quad c_3 = 7, \quad M_1 = 7
\]

As the price of $P_3$ has fallen to say, 2, we may have, $c_1$ being predetermined, $D_1$ when

\[
P_1 = 3, \quad P_3 = 2, \quad A = 0, \quad Q_1 = 21
\]

\[
(D_1)
\]
\[
c_1 = 15, \quad c_3 = 9, \quad M_1 = 6
\]

Similarly if $P_1$ increase to $\frac{9}{2}$, $P_3$ remaining the same, we shall have

\[
P_1 = \frac{9}{2}, \quad P_3 = 3, \quad A = 9, \quad Q_1 = 21
\]

\[
(D_2)
\]
\[
c_1 = 15, \quad c_3 = 9, \quad M_1 = 6
\]

where all the values other than the prices are the same as in $(D_1)$, the relative prices in $(D_1)$ and $(D_2)$ being identical.

So as the relative price $P_1 / P_3$ increases due to a rise in $P_1$ or a fall in $P_3$, $c_3$ increases but $A$ decreases hence $A - A$ increases.