Chapter 4

Restoration
4.1 Introduction

Image restoration technique deals with those images that have been recorded in the presence of one or more sources of degradation. The technique can be defined as the reconstruction of the original image or ideal image from the observable image by the effective inversion of degradation phenomenon through which the object was imaged. So restoration technique requires some from the knowledge concerning the degradation phenomenon. The image may be degraded due to atmospheric turbulence, motion blur, defocusing etc. and the knowledge may come in the form of parametric or non-parametric statistical models, or analytical (deterministic) models.

Using Gaussian models for image and noise statistics, a maximum a posteriori (Bayes') estimate of restored image can be derived [65]. Other approaches to Bayes' analysis for image restoration have been formulated in recursive form [46],[89]. Because of great success in the one-dimensional signal processing, there have been many attempts to extend Kalman filtering technique to two-dimensional image processing. In the two-dimensional cases, the enormity of data calls particularly for an efficient recursive processor. Another problem opposing this extension is that the specification of recursive model requires that the 'past', the 'present' and the 'future' be defined. It is, of course, trivial to do so for one-dimensional signal. For image data, however, there is no unique way to specify these. However, early efforts to achieve a truly recursive two-dimensional Kalman filter were of
only limited success because of both the difficulty in establishing a suitable two-dimensional recursive model and also the high dimension of the resulting state vector. Because of above mentioned factors computational load was found to be excessive. The problems surmounted by introducing some approximations like strip processor (that updates a line segment at a time), and 'reduced update' Kalman filter (that processes elements near the present point) [149]. Computational cost can also be reduced by introducing circulant matrix approximation for the banded Toeplitz structured model matrices [8] or by using semicausal model for the recursive filtering [66].

All analytical models (whether linear or non-linear, space-invariant or space-variant) are based upon the concept of a point-spread function ie, the effect on the recorded image of a point source of light in object plane. We will concentrate our discussion on linear, space-invariant degradation model. Space-variant degradation and restoration models are treated in [122],[123]. Though nonlinear methods require elaborate and costly computational procedures these cannot be overlooked, because they represent more realistic situations. Nonlinear restoration methods may come directly from the image formation and recording system [65]. A second situation in which nonlinear image restoration may arise is from constraints. For example, inclusion of a simple constraint that all the pixels of the restored image should have nonnegative value in the linear formulation leads to nonlinear restoration technique [6 (pp.188-193)].
Most of the linear restoration methods can be classified into two broad categories: (a) least square methods and (b) constrained least square methods. The problem of least squares filtering of pictures with a deterministic PSF was solved by Helstrom [58]. However, one of the most important problems namely, the ill-conditioned or singular nature of image restoration can not be dealt with by the use of the least-squares criterion. Because of ill-conditionedness or singularity of the image restoration problem, no unique inverse of degrading phenomenon exists. In such cases, one may use pseudo-inverse technique for image restoration. Computational requirements to implement this technique is quite high. A fast computational technique for pseudo-inversion is described in [103]. Another method of obtaining pseudo-inverse of PSF matrix is known as singular value decomposition [61]. The method can also be used to restore the images degraded through space-variant PSF [122]. The singularity problem can also be handled by a more powerful optimization criterion namely, minimum mean-square-error (MMSE) estimation. MMSE filter can be derived from both a priori and a posteriori knowledge. The MMSE filter derived from a priori knowledge is usually referred to as a traditional Wiener filter [6 (pp.132-140)] which assumes a constant mean. This implies that all spatial variation of image ensemble is accounted for by the covariance matrix which is Toeplitz in structure. The derivation of MMSE filter using a posteriori knowledge is dealt with in [136].

The mean-square-error minimisation criterion has been found useful in choosing specific solution from the infinite family of
solutions possible due to the ill-conditioned or singular nature of the image restoration problems. However, there is experimental evidence that the least square or Wiener filtering does not always lead to a visually good result. Of the modifications proposed to get around this difficulty, the constrained least square estimation (CLSE) has a demonstrated capability of improving the result [16], [44 (pp. 207-211)], [64]. CLSE filters are derived by optimising some criterion of goodness subject to the constraint that residual norm between the image and the reblurred estimated image be minimum. Minimisation of effective noise-to-signal power ratio leads to parametric Wiener filter [6 (pp. 150-151)]. Another possibility is to formulate a criterion of optimality based on a measure of smoothness such as minimising some function of second derivative of estimated image [64]. Many other goodness measures are also possible, for example, maximization of entropy [12]. An interesting and useful result can be obtained by minimising correlation between estimated image and noise signal, which will be discussed in the following sections. The filter function has some desirable properties and computational requirement to implement the proposed algorithm is less compared to other CLSE methods [16], [18].

4.2 Problem formulation

Let us recall the physical image formation system as discussed earlier in section 1.4. The degradation process can be
modeled as an operator $H$ which together with an additive noise $\eta(j,k)$ operates on an original image $f(j,k)$ to produce a degraded image $g(j,k)$ as shown in Fig 4.1. The input-output relationship is given by the expression

$$g(j,k) = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} f(m,n) h(j-m,k-n) + \eta(j,k)$$

for $j=0,1,2,\ldots, M-1$ and $k=0,1,2,\ldots, N-1$.

$h(j,k)$ is point-spread function of degrading process. Here we have assumed the linearity and space-invariance of operator $H$, because of simplicity in analysis. The assumption that noise is additive may also be subjected to criticism, but as the assumption makes the problem mathematically tractable, it is common to most work on image restoration. In matrix-vector notation equn.(4.1) can be represented as

$$g = [H]f + \eta$$

where $g$, $f$ and $\eta$ are all MN-dimensional column vectors formed by lexicographic ordering of the graylevels in the MxN grid structure.
and \([H]\) is the operator matrix \(MNxMN\). We are to find out the inverse operator, \(\phi\) which will give the restored image. So the digital image restoration problem can be viewed as that of obtaining an approximation to \(\hat{f}\), say \(\hat{f}\) given \(g\) and a knowledge about the operator \(H\). The knowledge about \(\hat{\xi}\) is limited by its statistical nature.

4.2.1 Ill-conditioned nature of restoration

One of the most essential problems is the fact that image restoration is an ill-conditioned problem at best and a singular problem at worst which may be explained briefly. We have already stated that the image restoration problem is to find the inverse transformation \(\phi\) such that

\[
\phi(g) \rightarrow \hat{f}
\]

In frequency domain, \(\phi\) is called the filter function \(\mathcal{P}(u,v)\). In a mathematical sense, the problem of image restoration corresponds to existence and uniqueness of the inverse transformation. Both existence and uniqueness are important. If the inverse transformation does not exist, then there is no mathematical basis for asserting that \(\hat{f}\) can be exactly recovered from \(g\). Problems for which there is no inverse transformation, i.e., \(\phi\) does not exist or in other words the denominator of \(\mathcal{P}(u,v)\) is equal to zero for some \((u,v)\), are said to be singular. On the other hand, \(\phi\) may exist but not be unique; i.e., there may be more than one \(\phi\),
the nature of which is used being dependent on \( \hat{f} \). Finally, even if \( \varphi \) exists and is unique, it may be ill-conditioned, by which we mean that a trivial perturbation in \( \hat{g} \) can produce nontrivial perturbation in \( \hat{f} \). That is, there exists \( \epsilon \), which can be made arbitrarily small such that

\[
\varphi (g + \epsilon) = f + \delta
\]

where \( \delta \gg \epsilon \); \( \delta \) is not arbitrarily small and is not negligible. This happens when denominator of \( P(u,v) \) becomes nearly equal to zero at some \( (u, v) \). The filter function, developed in the following sections, will be shown to be free from these problems.

4.2.2 Constrained least squares estimation [CLSE] approach

The CLSE technique can be defined as the optimisation of some criterion of goodness subject to the constraint that residual norm between the image and the reblurred estimated image be minimum. We define a criterion based on a measure of independence between the estimated image and the noise process. Let the correlation coefficient \( R(m,n) \) between the image and noise be given by

\[
R(m,n) = \frac{1}{MN} \sum_{k=0}^{N-1} \sum_{j=0}^{M-1} \eta(j,k) \hat{f}(j+m,k+n)
\]

(4.3)

for \( m = 0, 1, 2, \ldots, M-1 \) and \( n = 0, 1, 2, \ldots, N-1 \).
By lexicographic ordering, eqn. (4.3) can be represented in matrix-vector notation as

\[ \mathbf{y} = [A] \hat{\mathbf{x}} \]  

(4.4)

where \([A]\) is a block-circulant matrix of size \(MN \times MN\) and is composed of noise graylevels. Matrix \([A]\) consists of \(M^2\) partitions, each partition being of size \(N \times N\) and ordered according to

\[
[A] =
\begin{bmatrix}
[A_0] & [A_{M-1}] & [A_{M-2}] & \cdots & [A_1] \\
[A_1] & [A_0] & [A_{M-1}] & \cdots & [A_2] \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
[A_{M-1}] & [A_{M-2}] & [A_{M-3}] & \cdots & [A_0]
\end{bmatrix}
\]

Each partition \([A_j]\) is constructed from the \(j\)-th row of \(\eta(j,k)\), as follows:

\[
[A_j] =
\begin{bmatrix}
\eta(j,0) & \eta(j,N-1) & \eta(j,N-2) & \cdots & \eta(j,1) \\
\eta(j,1) & \eta(j,0) & \eta(j,N-1) & \cdots & \eta(j,2) \\
\eta(j,2) & \eta(j,1) & \eta(j,0) & \cdots & \eta(j,3) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\eta(j,N-1) & \eta(j,N-2) & \eta(j,N-3) & \cdots & \eta(j,0)
\end{bmatrix}
\]

Since the correlation between two independent processes should be zero, here the criterion is to minimize \(\| [A] \hat{\mathbf{x}} \| ^2 \) subject to the constraint \(\| \mathbf{y} - [H] \hat{\mathbf{x}} \| ^2 = \| \mathbf{y} \| ^2\). The addition of equality constraint in the minimization problem can be handled without difficulty by using the method of Lagrange multipliers. The
procedure is to express the constraint in the form
\[ \lambda \left[ \| g - [H] \hat{f} \|^2 - \| y \|^2 \right] \] and then append it to the function
\[ \left\| [A] \hat{f} \right\|^2. \]
We seek an \( \hat{f} \) which minimizes the objective function
\[ J(\hat{f}) \] given by
\[ J(\hat{f}) = \left\| [A] \hat{f} \right\|^2 + \lambda \left[ \| g - [H] \hat{f} \|^2 - \| y \|^2 \right] \] (4.5)
where \( \lambda \) is a 'Lagrange multiplier' and is a constant.

Differentiating eqn. (4.5) with respect to \( \hat{f} \) and setting the result equal to zero yields
\[ \frac{\partial J(\hat{f})}{\partial \hat{f}} = \Omega = 2[A]^T[A]\hat{f} - 2\lambda[H]^T(g - [H]\hat{f}) \] (4.6)
\( \Omega \) is an \( MN \)-dimensional null column vector.

Then the solution obtained for \( \hat{f} \) is given by
\[ \hat{f} = \left([H]^T[H] + \gamma[A]^T[A]\right)^{-1}[H]^Tg \] (4.7)
where, \( \gamma = 1/\lambda \) and the superscript \( T \) is used to denote the transpose of a vector or a matrix. The assumption of independence between the image and noise processes is practical in many situations. Also, as shown later, the estimation using eqn. (4.7) has other computational advantages.

4.3 Model of blurring process

Blurring in an image may be caused by several ways. For example, long-term exposure of atmospheric turbulence is one of
the reason. The PSF of this type of degrading process can be modeled by Gaussian function \( [6(pp,81')] \). Another reason for degradation in photography is relative motion between the camera and the scene. Total exposure at any point of the film can be obtained by integrating the instantaneous exposures over the time interval during which the shutter is open \([82],[129]\). Slepian \([129]\) modeled the PSF for motion blur by assuming that the shutter requires negligible time to change from closed to open and vice versa. The resultant PSF is then applied to restore the blurred image. For experimental purpose we have used here a linear model for degradation by defocussing. The PSF of a defocussed lens can be calculated, taking into account both geometrical optics and the effect of diffraction. It was shown by Stockseth \([132]\) that for low spatial frequencies the transfer function calculated on the basis of geometrical optics agrees well with that calculated when diffraction effects are also involved.

4.3.1 Blur due to defocussing

Usually defocussing causes intensity at a point of an object spread over a region. Assume for simplicity a point object at \((\xi,\delta)\) with graylevel \( f(\xi,\delta) = 1 \) in the object plane be spread over a circular region \( \delta \) of radius \( r = \frac{1}{2} |x_1-x_2| \) in the observed-image plane as shown in Fig. 4.2. Let \( h_{o,\xi,\delta} \) be the peak intensity after spreading and \( h_{\xi,\delta}(x,y) \) be the intensity at \((x,y)\) due to point object at \((\xi,\delta)\) in object plane. It is assumed that the observed-image plane has a one-to-one relative
correspondence with positions at the object plane, and no energy is lost due to blurring, ie,

$$\int \int_0 h_{x,y}(x,y) \, dx\,dy = 1$$

An image contains infinitely many point objects and space-invariant blurring distributes the intensity of each point over the circular region of radius $r$. Linearity assumption assures that the observed intensity at any point in the blurred plane is the aggregate of components of blurring distribution of intensities of points lying in the circular area around the corresponding point.

For discrete formulation, it is convenient to consider a square of side $(2r+1)$ centred at the $(j,k)$th pixel. Let each pixel has unit area so that there are $(2r+1)^2$ pixels in it as shown in Fig. 4.3. Then the observed intensity $g(j,k)$ in the

$$g(j,k) = \sum_{m=-r}^{j+r} \sum_{n=-r}^{k+r} \sum_{m=-r}^{j+r} f(m,n) h_{m,n}(j,k) \quad (4.8)$$
where, $h_{m,n}(j,k) = \begin{cases} h_{o,m,n} \left[ 1 - \frac{1}{r} \left\{ \frac{1}{2} \left( \sqrt{ (|j-m|^2 + |k-n|^2) } \right) \right\} \right] & \text{for } \left\{ \frac{1}{2} \left( \sqrt{ (|j-m|^2 + |k-n|^2) } \right) \right\} < r \quad (4.9) \\ 0 & \text{otherwise} \end{cases}$

Hence, $h_{m,n}(j,k)$ is the PSF in discrete domain and in a more general way, can be expressed as $h(|j-m|, |k-n|)$ and is space-invariant. Considering noise term, $g(j,k)$ is finally represented by

$$g(j,k) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) h(|j-m|, |k-n|) + \eta(j,k) \quad (4.10)$$

and in matrix-vector notation:

$$g = [H_{bt}] f + \eta \quad (4.11)$$

where $[H_{bt}]$ is a symmetric banded block Toeplitz matrix of size $MNxMN$ and consists of $M^2$ blocks. Each block of $[H_{bt}]$ is also a symmetric banded Toeplitz matrix of size $NxN$ with atmost $2q$ number of nonzero subdiagonals and the elements $h_{j,k}$ are

$$h_{j,k} = h_{||j-k||} \quad \text{for } 0 \leq j,k \leq N-1$$

and $h_{j,k} = 0 \quad \text{for } |j-k| > q$

where, $q = r-1$. These blocks are arranged in $[H_{bt}]$ in a fashion similar to the arrangement of their elements. Hence,
where, \( q = q_0 \geq q_1 \geq q_2 \geq \cdots \geq q_i \geq \cdots \geq q_q \). For slightly defocussed image \( g \ll N/2 \).

### 4.3.2 Circulant matrix formulation

It is known that utilization of Fourier transform processing technique permits scalar computation as contrasted to vector...
computation required to solve equn. (4.7). Since all (block) circulant matrices can be diagonalized by (two-dimensional) Fourier transform vectors, we like to modify equn. (4.11) so that blurring matrix takes the form of block circulant matrix. This requires observed image \( g(j,k) \) be extended. To justify the process of extension let us consider Fig. 4.4. The figure illustrates that

![Diagram](image_url)

**Fig. 4.4** (a) Relationship of observation field, original image field and PSF, (b) Relationship of arrays with common matrix origin.

(i) \( f_e \), i.e., the original image area from which observed image gets intensity contributions, is extracted from an infinite extent field; and (ii) area of \( g \) is less than that of \( f_e \), and \( g \) must be extended to cover the region equal to that covered by \( f_e \); otherwise there will be fewer observation points \( g \) than points \( f_e \) to be estimated leading to an underdetermined set of equations.
Some authors, e.g., [67], have extended $g$ to $g_0$ by assuming gray levels in the border region of $f_0$ be zero. This is not true always and may result some perturbations in the estimated image that cannot be ignored in many cases. Here we extend $g$ in such a way that the new values approximates the actual values more closely. In essence, here we try to extrapolate the values of $g_0$ from $g$.

Consider, for simplicity of representation, $M=N$. We define elements $g_0(j,k)$ of extended image matrix from $g(j,k)$ as follows:

$$g_0(j,k) \approx \begin{cases} 
\frac{g(j,k) + (S' - s_{jN+k}) \cdot g(N-1,N-1)}{S'} & \text{for } 0 \leq j, k \leq q-1 \text{ and } 0 \leq j, k \leq N-1 \\
g(j,k) & \text{for } q \leq j, k \leq N-1 \\
\frac{g(N-1,N-1) + (S' - s_{(j+1)N+q-1-k}) \cdot g(0,0)}{S'} & \text{for } N \leq j \leq N+q-1 \text{ and } 0 \leq k \leq N-1 \\
\frac{s_{(N+q-1-j)N+q-1-k} \cdot g(N-1,N-1) + (S' - s_{(N+q-1-j)N+q-1-k}) \cdot g(0,0)}{S'} & \text{for } 0 \leq j \leq N-1 \text{ and } N \leq k \leq N+q-1 \\
\frac{g(N-1,N-1)}{S'} & \text{for } N \leq j, k \leq N+q-1 
\end{cases}$$

where, $s_m = \sum_{n=0}^{N^2-1} h(m,n) \cdot h(m,n)$ is the $(m,n)$th element of $[H_{bt}]$ and should not be confused with $h_{j,k}$. Also, $S' = \max_m \{s_m\}$. 

\(125\)
Under the assumption that the graylevels of the pixels outside the image region of interest are uniform and are equal to the average of those of background of the image (which is more general than the assumption made in [64],[67], where these graylevels are assumed to be zero), elements \( g_\theta(j,k) \) can be defined in a more simplified form. Since in circulant matrix formulation the pixels outside the observed image field are to be considered and since the image restoration problem is ill-conditioned, it is better to assume the graylevels of pixels outside the image field equal to that of background rather than zero.

If there are at least \( r \)-rows and \( r \)-columns of pixels of equal graylevels lie on the border of observed image field [Fig.4.4(a)], we can simplify equn.(4.12) as

\[
G_\theta(j,k) = \begin{cases} 
    g(j,k) & \text{for } 0 \leq j \leq N-1 \text{ and } 0 \leq k \leq N-1 \\
    g(j,N-1) & \text{for } 0 \leq j \leq N-1 \text{ and } N \leq k \leq N+q-1 \\
    g(N-1,k) & \text{for } N \leq j \leq N+q-1 \text{ and } 0 \leq k \leq N \\
    g(N-1,N-1) & \text{for } N \leq j \leq N+q-1 \text{ and } N \leq k \leq N+q-1 
\end{cases}
\]  

(4.13)

Now the extended observed image is related to the extended original image in the following manner.

\[
G_e = \left[H_{bc}\right] \tilde{f}_e + \tilde{g}_e 
\]  

(4.14)

\( G_e \), \( f_e \) and \( g_e \) are \( N_1^2 \)-dimensional column vectors formed by lexicographic ordering of \( g_\theta(j,k) \), \( f_e(j,k) \) and \( g_e(j,k) \), respectively and \( N_1 = N+q \). Here \( f_e(j,k) \) is assumed as
\( f_g(j,k) = \begin{cases} 
R_1(j,k) & \text{for } 0 \leq j \leq N-1 \text{ and } 0 \leq k \leq N-1 \\
R_2(j,k) & \text{for } 0 \leq j \leq N+q-1 \text{ and } N \leq k \leq N+q-1 \\
R_3(j,k) & \text{for } N \leq j \leq N+q-1 \text{ and } 0 \leq k \leq N+q-1 
\end{cases} \) 

(4.15)

\([H_{bc}]\) is a symmetric block circulant matrix of size \(N_1^2 \times N_1^2\).

4.4 Diagonalization of operator

The technique, which has already been used to obtain eqn.(4.7) from eqn.(4.2), yields \(\hat{f}_g\) by minimizing \(\| [A_e] \hat{f}_g \|^2\) subject to the constraint \(\| g_e - [H_{bc}] \hat{f}_g \|^2 = \| g_e \|^2\) as

\[ \hat{f}_g = ([H_{bc}]^T[H_{bc}] + \gamma[A_e]^T[A_e])^{-1}[H_{bc}]^Tg_e \]  

(4.16)

where the matrix \([A_e]\) is constructed from the elements of \(g_e\).

Now \([H_{bc}]\), being a block circulant matrix, can be diagonalized by two-dimensional discrete Fourier transform matrix \([64]\) in the following manner.

\[ [D_H] = [U]^{-1}[H_{bc}][U] \]  

(4.17)

where, \([D_H] = \text{diag.} \{ d_0, d_1, d_2, \ldots, d_{N_1^2-1} \}\), is, \(d_1\) is the \(i\)-th eigenvalue of \([H_{bc}]\) and is given by

\[ d_1 = \mathcal{H}([\frac{1}{N_1}], i \mod N_1) \]
is used to denote the greatest integer not exceeding \( \frac{i}{N_1} \) and \( i \mod N_1 \) represents the remainder obtained by dividing \( i \) by \( N_1 \). \( h(. , .) \) are two-dimensional Fourier transform coefficients of modified PSF \( h_0(. , .) \) and are real as \( [H_{bc}] \) is symmetric. When \( [W]^{-1} \) is multiplied with a lexicographically ordered image vector we obtain the lexicographically ordered Fourier transform coefficients of that image, ie,

\[
F_{\theta} = [W]^{-1} f_{\theta}
\]

\([A_{\theta}]\) is also a block circulant matrix. So it also can be diagonalized in the same way as \( [H_{bc}] \). Let the noise terms are uncorrelated, ie,

\[
\Sigma \Sigma q_{\theta}(j,k) q_{\theta}(j+m, k+n) = 0 \quad \text{for all} \quad m, n \quad \text{but not both equal to zero. Then a straight-forward analysis leads to}
\]

\[
[A_{\theta}]^T[A_{\theta}] = \frac{1}{N_1^2} \Sigma_q [I]
\]

where \([I]\) is the identity matrix of size \( N_1 \times N_1 \) and

\[
\Sigma_q = \frac{1}{N_1^2} \Sigma_{j=0}^{N_1-1} \Sigma_{k=0}^{N_1-1} q_{\theta}^2(j,k)
\]

which is average energy or power of noise function. Let \( \mu_{q} \) is the mean value of the noise graylevels and \( \sigma_{q} \) is its variance, then

\[
\mu_{q} = \frac{1}{N_1^2} \Sigma_{j=0}^{N_1-1} \Sigma_{k=0}^{N_1-1} q_{\theta}(j,k)
\]
and \( q_\eta = \frac{1}{N_1^2} \sum_{j=0}^{N_1-1} \sum_{k=0}^{N_1-1} \left[ \eta(j,k) - \mu_\eta \right]^2 \)

\[
= \frac{1}{N_1^2} \sum_{j=0}^{N_1-1} \sum_{k=0}^{N_1-1} \eta(j,k)^2 - \frac{2}{N_1^2} \mu_\eta \sum_{j=0}^{N_1-1} \sum_{k=0}^{N_1-1} \eta(j,k) + \mu_\eta^2
\]

Therefore, \( \bar{q}_\eta = 2\mu_\eta \mu_\eta + \mu_\eta^2 \) (4.19)

That means that one can compute the average energy of the noise process if mean and variance are known.

Combining eqns. (4.16), (4.17) and (4.18) we obtain

\[
\hat{F}_\epsilon = (\mathbf{U} \mathbf{D}_H)^T \mathbf{U}^{-1} \mathbf{U} \mathbf{D}_H \mathbf{U}^{-1} + \gamma \frac{1}{N_1^2} \mathbf{U} \mathbf{E}_\eta \mathbf{U}^T \mathbf{U}^{-1} \mathbf{U} \mathbf{D}_H \mathbf{U}^{-1} \mathbf{g}_\eta
\]

or \( \mathbf{U}^{-1} \hat{F}_\epsilon = (\mathbf{D}_H)^T \gamma \mathbf{E}_\eta \mathbf{I} \mathbf{D}_H \mathbf{U}^{-1} \mathbf{U} \mathbf{D}_H \mathbf{U}^{-1} \mathbf{g}_\eta \)

In two-dimensional representation, \( \hat{F}_\epsilon(u,v) \) and \( G_\eta(u,v) \) gives the \((u,v)\)th Fourier transform coefficient of \( \hat{F}_\epsilon(j,k) \) and \( g_\eta(j,k) \), respectively. Hence

\[
\hat{F}_\epsilon(u,v) = \frac{\mathbf{U}(u,v)}{\mathbf{U}^2(u,v) + \gamma \mathbf{E}_\eta} \mathbf{G}_\eta(u,v) \quad (4.20)
\]

for \( u=0,1,2,\ldots, N_1-1 \) and \( v=0,1,2,\ldots, N_1-1 \).

The filter function \( P(u,v) \) (as described in Fig. 4.1 implicitly) is now defined as

\[
P(u,v) = \frac{\mathbf{U}(u,v)}{\mathbf{U}^2(u,v) + \gamma \mathbf{E}_\eta}
\]
As already said, image restoration using inverse filtering technique occasionally encounters computational difficulties when denominator vanishes or tends to vanish for some values of \((u,v)\). But in our case, it is evident that \(\mathcal{H}(u,v)\) is real and non-negative; and \(\xi_\eta\), being noise power, presents a considerable positive value. Now the incident, that denominator of the filter function becomes zero or very nearly equal to zero, completely depends on the value of \(\gamma\). It will be shown in the next section that if the noise power be greater than zero but less than that of observed image, which is most common in practice, then the value of \(\gamma\) becomes greater than zero. Hence the denominator is sufficiently greater than zero and \(\hat{f}_\theta(u,v)\) can be computed uniquely.

### 4.5 Estimation and algorithm

Solving eqn. (4.20) [or eqn. (4.16)] subject to the constraint \(\| \hat{g}_\theta - ([H_{bc}] \hat{f}_\theta) \|^2 = \| \eta_\theta \|^2\) we obtain the value of \(\gamma\) and \(\hat{f}_\theta(u,v)\) [or \(\hat{f}_\omega\)]. Among them the first one represents \(N^2\) number of linear equations and the constraint identity represents a nonlinear equation. Since it is difficult to solve simultaneously a nonlinear equation and a set of linear equations, we try to find \(\gamma\) by iterative technique. To avoid computations required to find inverse Fourier transform of \(\hat{f}_\theta(u,v)\) and then to
find \( [H_{bc}]_{fg} \) during the estimation of \( \gamma \), we apply Parseval's theorem to constraint identity and obtain

\[
\frac{1}{N^2} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} |G_b(u,v) - H(u,v) \hat{f}_e(u,v)|^2 \cdot \nonumber
\]

\[
= \frac{1}{N^2} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} |\hat{N}_b(u,v)|^2 = N^2 \varepsilon_\eta
\]

\( \hat{N}_b(u,v) \) is the \((u,v)\)th Fourier transform coefficient of noise \( \eta_b(j,k) \). Let us define an error function

\[
\psi(\gamma) = \frac{1}{N^2} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} |G_b(u,v) - H(u,v) \hat{f}_e(u,v)|^2
\]

\[
= \frac{1}{N^2} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \left| G_b(u,v) - \frac{\psi^2 G_b(u,v)}{\psi^2 + \gamma \varepsilon_\eta} \right|^2
\]

or

\[
\psi(\gamma) = \frac{1}{N^2} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \frac{\gamma^2 \varepsilon_\eta^2 |G_b(u,v)|^2}{[\psi^2 + \gamma \varepsilon_\eta]^2} (4.21)
\]

we now propose to adjust \( \gamma \) so that

\[
\psi(\gamma) = N^2 \varepsilon_\eta \pm \varepsilon_a (4.22)
\]

where, \( \varepsilon_a \) is an accuracy factor. It will be shown in the next subsection that \( \psi(\gamma) \) is piecewise monotonic increasing function which implies that finding \( \gamma \) by iterative method is not a difficult problem.
4.5.1 Lower bound of $\gamma$

We have yet to show the condition under which the filter, as described by eqn. (4.20), results in a minimisation rather than maximisation of the function $J\hat{\gamma}_e$. From the calculus of several variables we have the following sufficient condition:

\[
\frac{\partial^2 J}{\partial \hat{\gamma}_e^2} = \begin{bmatrix}
\frac{\partial^2 J}{\partial \hat{\gamma}_1^2} & \frac{\partial^2 J}{\partial \hat{\gamma}_1 \partial \hat{\gamma}_2} & \cdots & \frac{\partial^2 J}{\partial \hat{\gamma}_1 \partial \hat{\gamma}_N} \\
\frac{\partial^2 J}{\partial \hat{\gamma}_2 \partial \hat{\gamma}_1} & \frac{\partial^2 J}{\partial \hat{\gamma}_2^2} & \cdots & \frac{\partial^2 J}{\partial \hat{\gamma}_2 \partial \hat{\gamma}_N} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 J}{\partial \hat{\gamma}_N \partial \hat{\gamma}_1} & \frac{\partial^2 J}{\partial \hat{\gamma}_N \partial \hat{\gamma}_2} & \cdots & \frac{\partial^2 J}{\partial \hat{\gamma}_N^2}
\end{bmatrix}
\]

be a positive definite matrix, which implies that all eigenvalues must be greater than zero. Here [as in eqn. (4.6)]

\[
\frac{\partial J}{\partial \hat{\gamma}_e} = 2[A_{\hat{\gamma}}]^{T}[A_{\hat{\gamma}}] \hat{\gamma}_e - 2\lambda[H_{bc}]^{T}(g_{\hat{\gamma}} - [H_{bc}] \hat{\gamma}_e)
\]

and \[
\frac{\partial^2 J}{\partial \hat{\gamma}_e^2} = 2[A_{\hat{\gamma}}]^{T}[A_{\hat{\gamma}}] + 2\lambda[H_{bc}]^{T}[H_{bc}]
\]

\[
= \frac{2}{N_1^2} \varepsilon[I] + 2\lambda[H_{bc}]^{T}[H_{bc}]
\]
So \( \frac{2^J}{2^k} \) is also a block circulant matrix of size \( N_1^2 \times N_1^2 \).

The eigenvalues of this matrix will be

\[
2E_\eta + 2\lambda \mathcal{H}^2(u,v) > 0
\]

for \( u=0,1,2, \ldots, N_1-1 \) and \( v=0,1,2, \ldots, N_1-1 \).

Therefore, the Lagrange multiplier must satisfy the condition:

\[
\lambda > \frac{-E_\eta}{\mathcal{H}^2(u,v)}
\]

for \( u=0,1,2, \ldots, N_1-1 \) and \( v=0,1,2, \ldots, N_1-1 \).

Equivalently

\[
\lambda > (-) \min_{(u,v)} \left\{ \frac{E_\eta}{\mathcal{H}^2(u,v)} \right\}
\]

Hence, the parameter \( \gamma \) of filter function must satisfy

either \( \gamma \geq 0 \)

or \( \gamma < (-) \max_{u,v} \left\{ \frac{\mathcal{H}^2(u,v)}{E_\eta} \right\} = \gamma_1 \) (say) \hspace{1cm} (4.23)

Now if the restoration problem is singular, i.e., \( \mathcal{H}(u,v) = 0 \), for at least one pair of \((u,v)\)'s, then \( \gamma \) must be positive. Otherwise \( \gamma \) may assume negative value. In order to determine the circumstances which may lead to a negative value for \( \gamma \), the error relationship in eqn. (4.21) will be explored in some greater detail in the next section.
4.5.2 Monotonicity in the error function

Taking the derivative of equn. (4.21) with respect to $\gamma$, we get

$$
\frac{d\phi(\gamma)}{d\gamma} = \frac{1}{N_1^2} \sum_{u=0}^{N_1-1} \sum_{v=0}^{N_1-1} \frac{2\gamma E^2 \mathcal{H}^2(u,v) |G_e(u,v)|^2}{[\mathcal{H}^2(u,v) + \gamma E]^3}
$$

Now for $\gamma > 0$ we have $\mathcal{H}^2(u,v) + \gamma E > 0$ and for $\gamma < 0$ we have $\mathcal{H}^2(u,v) + \gamma E < 0$, for all $u$ and $v$.

So

$$
\frac{d\phi(\gamma)}{d\gamma} > 0 \quad \text{for} \quad \gamma > 0
$$

$$
> 0 \quad \text{for} \quad \gamma < Y_1
$$

(4.24)

Therefore, the error function is a monotonic increasing function in both the range of $\gamma$, ie, $(-\infty, Y_1)$ and $(0, \infty)$, respectively.

In order to gain further insight into the behavior of error as a function of $Y$, we can evaluate $\phi(\gamma)$ at some particular values of $\gamma$.

$$
\phi(\gamma) \bigg|_{\gamma = 0} = 0
$$

(4.25a)

$$
\phi(\gamma) \bigg|_{\gamma \rightarrow \infty} = \frac{1}{N_1^2} \sum_{u=0}^{N_1-1} \sum_{v=0}^{N_1-1} |G_e(u,v)|^2
$$

(4.25b)

$$
\phi(\gamma) \bigg|_{\gamma \rightarrow Y_1} \rightarrow \infty
$$

(4.25c)

From equns. (4.24) and (4.25) we can sketch a typical $\phi(\gamma)$ versus $\gamma$ curve as shown in Fig. 4.5. This figure implies...
that \( T \) will assume negative values if and only if (i) negative values are possible from eqn.(4.23) and (ii) noise energy is greater than the energy in received signal, i.e.,

\[
N_1^2 \varepsilon_q > \frac{1}{N_1^2} \sum_{u=0}^{N_1-1} \sum_{v=0}^{N_1-1} |G_e(u,v)|^2
\]

similar results are also derived in [33] through complex domain analysis.

### 4.5.3 A concise algorithm for restoration

In describing the algorithm for CLSE method using signal-noise correlation criterion as described in previous sections, we assumed that the noise energy satisfies the following condition:
The assumption is most common in real life situation. This enables us to consider monotonically increasing part of the plot of $f(y)$ versus $y$, where $y > 0$. The monotonicity of $f(y)$ means that there is only one unique value of $y$ such that $f(y) = N_1^2 E y$ and an iterative process on $y$ is sufficient to find that unique value. The following steps summarise the proposed algorithm.

**Algorithm 4.1**

**Step 1:** Construct and store the first row of block Toeplitz matrix $[H_{bt}]$ from point spread function defined by eqn. (4.9).

**Step 2:** Modify the first row of $[H_{bt}]$ to the first row of a block circulant matrix $[H_{bc}]$.

**Step 3:** From the elements of the first row of $[H_{bc}]$ construct $h_e(j, k)$ of size $N_1 \times N_1$ by the process just reverse to lexicographic ordering.

**Step 4:** Find $H(u, v)$ by two-dimensional Fourier transform of $h_e(j, k)$.

**Step 5:** Construct extended image matrix $g_e(j, k)$ of size $N_1 \times N_1$ using eqn. (4.13) [or in some cases, eqn. (4.12)] from observable image $g(j, k)$ of size $(N_1 - q) \times (N_1 - q)$. 
Step 6: Find \( G_e(u,v) \) by two-dimensional Fourier transform of \( g_e(j,k) \).

Step 7: If average energy \( E_q \) (or power) of noise gray-levels is not known, calculate it from the mean and variance using eqn. (4.19).

Step 8: Choose and fix some initial value of \( \gamma (>0) \). Find total estimation error as given by eqn. (4.21).

(a) Increase the value of \( \gamma \), if \( \gamma(\gamma) < N_1^2 \frac{\epsilon_x}{\eta} - \epsilon_a \).

(b) Decrease the value of \( \gamma \), if \( \gamma(\gamma) > N_1^2 \frac{\epsilon_x}{\eta} + \epsilon_a \).

Step 9: For acceptable value of \( \gamma \), if \( N_1^2 \frac{\epsilon_x}{\eta} - \epsilon_a < \gamma(\gamma) < N_1^2 \frac{\epsilon_x}{\eta} + \epsilon_a \) determine \( \hat{F}_e(u,v) \) and then \( \hat{F}_e(j,k) \) by applying two-dimensional inverse Fourier transform on \( \hat{F}_e(u,v) \). Hence find \( \hat{f}(j,k) \) [see eqn. (4.15)].

Now if \( N_1^2 \frac{\epsilon_x}{\eta} > \frac{1}{N_1^2} \sum_{u=0}^{N_1-1} \sum_{v=0}^{N_1-1} |G_e(u,v)| \), which seldom occurs in practice; the above algorithm will remain same except the step 8 which will be modified as: "Choose and fix some initial value of \( \gamma < \gamma_1 < \gamma \). Find total estimation error as given by eqn. (4.21). (a) Increase the value of \( \gamma \), if \( \gamma(\gamma) < N_1^2 \frac{\epsilon_x}{\eta} - \epsilon_a \).

(b) Decrease the value of \( \gamma \), if \( \gamma(\gamma) > N_1^2 \frac{\epsilon_x}{\eta} + \epsilon_a \)."

4.6 Experimental results of restoration algorithm and discussions

To test the validity of the present approach to a practical problem the algorithm has been implemented on a microphotograph of
Fig. 4.6(a) Original defocused image of chromosome containing small noise [S.N.R. = 24.8 dB].

Fig. 4.6(b) Restored image of Fig. 4.6(a) using Algorithm 4.1 for r=2.

Fig. 4.6(c) Restored image of Fig. 4.6(a) using Algorithm 4.1 for r=3.
Fig. 4.7(a) Degraded image of chromosome obtained by adding noise to Fig. 4.6(a) [S.N.R. ≈ 17.1 dB].

Fig. 4.7(b) Restored image of Fig. 4.7(a) using Algorithm 4.1 for r=2.
human chromosome. A small rectangular portion of the entire picture is taken by windowing, which satisfies the condition that the background of the extracted portion has uniform gray-level. The size of the extracted portion is 48x48. The portion has been extended in both dimension by adding pixels with gray-levels equal to that of the background so that the present size is N x N and N + q = 2^l, where l = 6. The value of N is chosen intentionally (or deliberately) so that we can use fast algorithm to find Fourier transform and inverse Fourier transform of the sets of datas.

The original defocused image [Fig.4.6(a)] contains small noise (S.N.R. $\approx$ 24.8 dB) due to granularity in the developing paper and digitization effect. Fig.4.6(b) and Fig.4.6(c) are restored images of Fig.4.6(a) obtained for $r=2$ and $r=3$, respectively. To see the efficiency of the present algorithm on large noise, uniformly distributed random noise has been added to Fig.4.6(a) to get Fig.4.7(a), where S.N.R. is decreased to 17.1 dB (app.). Fig.4.7(b) represents the restored image of Fig.4.7(a) for $r=2$.

It is seen that the noise has been removed fairly well, but the edges of the objects, i.e., chromosomes are more smooth than that in Fig.4.6(b). This is because of the fact that the noise power term in eqn. (4.2U) reduces heavily the value of high-frequency components of the filter function and hence the high-frequency components of the image is greatly reduced. This effect smooths the edges.
Another important thing should be noted that the algorithm developed are true for other types of degrading phenomenon also with some exceptions. Here $[H_{bc}]$ is a symmetric matrix, consequently $\mathcal{H}(u,v)$ is real for all $(u,v)$. If $[H_{bc}]$ is not symmetric, then the filter function will be expressed as

$$\hat{p}(u,v) = \frac{\mathcal{H}^*(u,v)}{|\mathcal{H}(u,v)|^2 + \gamma E_n}$$

where, $\mathcal{H}(u,v)$ will be complex quantity and $\mathcal{H}^*(u,v)$ denotes complex-conjugate of $\mathcal{H}(u,v)$. Hence in the earlier discussions all $\mathcal{H}^2(u,v)$ should be replaced by $|\mathcal{H}(u,v)|^2$.

4.6.1 Comparison with other methods

It is useful to compare the present criterion and those used previously. The unconstrained least square criterion or, in other words, inverse filtering does not utilize the properties of noise and hence its performance is poor under certain noise condition. The filter function proposed here will be recognized as inverse filter as $E_n$ approaches zero, that means, the filter parameter $\gamma$ also approaches zero. In this sense, the present CLSE filter as given in equ'n (4.20) is somewhat similar to Wiener filter [Appendix A.5]. The Wiener filter does not guarantee $\|\hat{y} - [H]\hat{f}\|^2 = \|y\|^2$ and the noise power spectrum as well as noise-free image power spectrum needed to realise the filter may not be available a priori.
Fig. 4.8(a) Restored image of Fig. 4.6(a) obtained by minimizing second difference of restored image for \( r=2 \).

Fig. 4.8(b) Restored image of Fig. 4.6(a) obtained by minimizing second difference of restored image for \( r=3 \).

Fig. 4.8(c) Restored image of Fig. 4.7(a) obtained by minimizing second difference of restored image for \( r=2 \).
Also, the restored image may not be visually acceptable although Wiener filter minimises the mean square restoration error in a statistical sense. The CLSE techniques using either smoothness criterion or the present criterion are not optimal like Wiener filter. In [64] smoothness of the restored image is emphasised and hence the sharpness of the image is affected. In the present method, the independence of noise and image is emphasised. The effect of the process is to de-correlate the noise from the picture. The method described by Hunt [64] and the present method are comparable in computer complexity. None of the above mentioned methods assures the positivity of the restored image data. So before displaying, the range of image data are to be shifted and/or scaled to make it a subset of actual dynamic graylevel range of the display device.

The results of using smoothness criterion in CLSE technique are given in Fig.4.8. Figs.4.8(a) and 4.8(b) represents the restored image of Fig.4.6(a) for \( r=2 \) and \( r=3 \), respectively. Fig.4.8(c) represents the restored image of Fig.4.7(a) for \( r=2 \). It is seen that these images [Fig.4.8] are more smooth and blurred than those in Figs.4.6(b), 4.6(c) and 4.7(b), respectively. This effect is expected because in this CLSE technique the second difference is minimised. Moreover, because the second difference minimisation has not been incorporated in our method, there is a sharp difference in graylevels at the object edges in the restored images.
Restoration through eqn.(4.20) has several advantages. Firstly, for nonzero noise energy singularity in the right hand side of eqn.(4.21) does not occur though the degrading matrix \( [H_{bc}] \) is singular [appendix (A.7)] and inverse Fourier transform can be computed uniquely. Secondly, the noise energy can be calculated through its mean and variance only. Thus a complete knowledge of first order statistics and subsequent Fourier transform are not necessary. This fact, along with eqn.(4.21) improves the speed of algorithm. The time required to process the image is less than that required by CLSE method using smoothness criterion.