"To have learned to open the mind to hitherto unknown and even inconceivable states of thought and feelings, is to have undergone a permanent change. It is like learning to swim. Once a swimmer always a swimmer."

- Alfred E. Zimmern.
CHAPTER IV

FLUID-FILLED POROUS ELASTIC MEDIUM

In the present chapter we shall derive the equations of equilibrium as well as the constitutive equations for a medium of a two-phased system of the solid and the fluid. The system consists of a porous elastic body filled with viscous compressible fluid. Several special problems will be solved to illustrate the general theory in both the cases when the elastic body is isotropic as well as anisotropic. A natural extension of the elastic theory to a visco-elastic body filled with fluid will also be illustrated with an example. It will be shown that there exists a complete analogy between theory of the fluid-filled isotropic elastic body and the coupled quasi-static theory of thermoelasticity as considered in the previous chapter III.

All the problems referred to above fall under the category of the so-called quasi-static theory in which no account is taken of the inertia forces. This does not mean that the time factor does not enter the theory. In fact we do get the time-history through the law governing the flow of fluid through the porous structure.

If on the other hand the inertia forces are taken into account, there arises the Dynamical theory of the deformation of the porous elastic medium filled with fluid. This theory introduces not only the densities of the solid and...
the fluid, but also a third coupling density corresponding
to the solid-fluid system. Occurrence of terms involving
this coupling density of the two-plased system complicates
further the solution of particular problems, in comparison
with the dynamical theory of a single-phased system of
either the solid or the fluid. We have considered a prob-
lem of semi-infinite medium loaded by an impulsive force
to illustrate this dynamical theory of the porous elastic
medium filled with a fluid.

4.1. Concept of the Solid-Fluid System and the
Constitutive Equations

We consider a medium which is the combination of an
elastic solid material and a fluid, viscous or non-viscous,
compressible or incompressible. The solid material is
supposed to form the skeleton or frame-work of the body,
having a statistical distribution of infinitesimally small
pores filled with the fluid. It will be assumed that the
pores are all interconnected, whereas pores with walls com-
pletely sealed off will be considered as a part of the
solid. Now if forces are applied on the surface of the
medium the solid part will be deformed and the fluid will
start flowing. To have a rough idea of the physical situa-
tion we may think of the seepage of water from a mass of
clay which a solid-fluid system of loose earth and water.
In fact the present theory is well applicable to engineer-
ing problems such as the seepage of water under a dam and
the stress calculation of the underlying mass, flow of oil
and water in petroleum reservoirs, settlement and consolidation of clay in foundations and many others.

In order to give the mathematical formulation of the theory of such a two-phased medium we begin with the definition of porosity, stress and strain, and other state variables.

Porosity

If \( V_1 \) denotes the volume of the interconnected pores or void spaces contained in a sample of bulk volume \( V \) of the solid-fluid system then porosity \( f \) is defined as

\[
f = \frac{V_1}{V}.
\]

Thus it is the void space in a unit volume of the bulk material. It is also defined as

\[
f = \frac{s'}{s}.
\]

where \( s' \) is the pore area in a cross-sectional area \( s \) of the bulk material. That these two definitions are the same can be shown by integrating the ratio \( s'/s \) over a unit length in a direction normal to the cross sectional area \( s' \). The value of the integral then represents the fraction \( f \) of the volume occupied by the pores.

Stress and Strain. Equations of Equilibrium

Let us consider a portion of the bulk volume of the solid-fluid system in the form of a rectangular parallelepiped whose edges have lengths \( \Delta x, \Delta y, \) and \( \Delta z \) parallel to the coordinates axes \((x, y, z)\) (Fig. 4.1).
Let $T_x, T_y, \ldots$ etc. denote the stress components per unit area on the solid part of the bulk volume and $p$ be the pressure of the fluid. We shall make, at this stage, an assumption that the shearing stresses in the fluid due to viscosity is small enough to be neglected in stress equations of equilibrium in comparison with the pressure $p$ of the fluid. (The possibility of inclusion of the shearing stress is of course not completely ruled out in literature [9].) On the face of area $\partial y \partial z$ of the element of volume the pore area is $\frac{\partial y \partial z}{\partial y \partial z}$ by the definition (2), and hence the total normal stress on the fluid is

$$-p \frac{\partial y \partial z}{\partial y \partial z},$$

the negative pressure of the fluid being defined as the fluid-stress. Again, since $(1-f) \frac{\partial y \partial z}{\partial y \partial z}$ represents the area of the solid part on the same face, the total normal stress on the solid is

$$(1-f) T_{xx} \frac{\partial y \partial z}{\partial y \partial z}.$$

The shearing stress on the fluid have been assumed to be negligible, while those on the solid are

$$(1-f) T_{xy} \frac{\partial y \partial z}{\partial y \partial z}, \quad (1-f) T_{xz} \frac{\partial y \partial z}{\partial y \partial z}.$$

We define

$$\sigma = -fp,$$

$$\sigma_{xx} = (1-f) T_{xx}, \quad \sigma_{xy} = (1-f) T_{xy}, \quad \sigma_{xz} = (1-f) T_{xz}.$$
Hence the total stress components on the face of area $d_y d_z$ are

$$
\sigma_{xx} \ , \ \sigma_{xy} \ , \ \sigma_{xz}
$$

respectively parallel to the axes. Similarly on the face of area $d_z d_x$, the total stress components are

$$
\sigma_{yx} \ , \ \sigma_{yy} + \sigma \ , \ \sigma_{yz}
$$

while those of the face of area $d_x d_y$ are

$$
\sigma_{zx} \ , \ \sigma_{zy} \ , \ \sigma_{zz} + \sigma.
$$

Taking the moments of these stress components about the axes it is easily seen that

$$
\sigma_{yz} = \frac{\partial \tau_{xy}}{\partial z}, \ \sigma_{zx} = \sigma_{xz}, \ \sigma_{xy} = \sigma_{yx}.
$$

Considering the equilibrium of the element of volume, we deduce, by usual method, the stress equations of equilibrium as

$$
\frac{\partial (\sigma_{xx} + \sigma)}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + \rho \chi = 0
$$

$$
\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial (\sigma_{yy} + \sigma)}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + \rho \chi = 0
$$

$$
\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial (\sigma_{zz} + \sigma)}{\partial z} + \rho \\zeta = 0
$$

(4)

where $\rho$ is the mass per unit volume of the bulk material, and $(\chi, \zeta, \xi)$ are the components of external forces per
unit mass. The equations of equilibrium are of the same form as those for any continuous medium made up of a single material, except that the normal stress components are increased by \( \sigma \). The stress components \( \sigma_{xx}, \ldots, \sigma_{yy}, \ldots \) however do not have the same significance as in purely elastic bodies. In purely elastic bodies these stress components represent forces per unit area of a cross section of the solid. In the present case \( \sigma_{xx}, \ldots, \sigma_{yy}, \ldots \) are forces applied on the solid portion of a cross section of unit area, as seen from the definitions (3) above, while \( \sigma \) is the total normal tension (negative pressure) applied to the fluid part of a cross-section of unit area.

In tensor notation the equations of equilibrium (4) can be written as

\[
\left( \sigma_{ij} + \sigma \delta_{ij} \right)_{ij} a + \rho \dot{x}_i = 0,
\]

(5)

**Average Values**

The solid stresses \( \sigma_{xx}, \ldots, \sigma_{yy}, \ldots \) and the fluid stress \( \sigma \) as defined above are certainly functions of the coordinates \( x, y, z \). Now if a point happens to be within a fluid element, the solid stresses at that point have no significance. Similarly, if the point is taken within the solid element, the fluid stress at that point is without any meaning. In order to avoid any confusion in interpreting the physical quantities it is better to think of the solid and fluid stresses as average values over a unit
cross section of the bulk material. In fact, not only the stress components but also the displacement components which will be introduced just now, will be thought of as the average values over a cross-section of unit area.

Strain and Displacement Components

The average displacement components of the solid will be denoted by $u_i$ and that of the fluid by $U_i$. The strain components for the solid are defined as

$$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i})$$

while those of the fluid are

$$\varepsilon_{ij} = \frac{1}{2} (v_{i,j} + v_{j,i})$$

and the respective dilatations are

$$\varepsilon = \varepsilon_{ii}, \quad \varepsilon = \varepsilon_{ii}.$$

Equivalent Fictitious Body of a Single Phase

We are dealing with a two-phased system of solid and fluid. It will be explained now that this two-phased system can be replaced by a single-phased fictitious body from a purely mathematical point of view. The idea of introducing an imaginary body of a single phase in place of the multiple-phased system has been fruitfully utilised in
other fields also \[10\]. In the case of a single-phased elastic system there are six stress components \( \sigma_{ij} \) and six strain components \( \varepsilon_{ij} \), and these are connected by a certain law, for example, by Hooke's law in the case of the linearly elastic bodies. In the present discussion of the two-phased system, we have, besides the six solid-stress components \( \sigma_{ij} \), the fluid stress \( \sigma_f \), and there are six solid-strain components \( \varepsilon_{ij} \) and the fluid dilation \( \varepsilon \) which alone can create the change in the fluid stress \( \sigma_f \) in the absence of the solid deformation. Consequently we may think of a fictitious elastic body which can have the seven stress components \( (\sigma_{ij}, \sigma_f) \) and the seven strain components \( (\varepsilon_{ij}, \varepsilon) \). With this understanding, the theory of fluid-filled porous elastic medium may be treated as the generalisation of the classical theory of elasticity. In fact, all the ideas of the classical theory will be utilised in the present case with the only modification that the stress components (and also the strain components) are seven in number instead of six.

**Stress-Strain Relations**

In what follows we shall be concerned with the fictitious elastic body with seven stress components and seven strain components. The deformation will be assumed to be completely reversible with all conservation properties and the strain components small, as in the classical linear theory of elasticity. We assume (being guided by Hooke's law) that the seven stress components are linear
The elastic potential energy $V$ has the expression

$$2V = \sigma_{ij} e_{ij} + \sigma \varepsilon,$$  \hspace{1cm} (7)

From (6) and (7) it follows that $2V$ is a homogeneous quadratic function of the strain components $(e_{ij}, \varepsilon)$ and is of course a positive definite form. The stress components are then given by

$$\frac{\partial V}{\partial e_{ij}} = \sigma_{ij}, \quad \frac{\partial V}{\partial \varepsilon} = \sigma.$$ \hspace{1cm} (8)

Since

$$\frac{\partial V}{\partial e_{ij}} = \frac{\partial V}{\partial e_{ij}},$$

we have the symmetry property $C_{ij} = C_{ji}$, and the number of the coefficients $C_{ij}$ reduces to twenty eight. In particular, if the body happens to be isotropic the number of coefficients $C_{ij}$ can be shown \[5\] to reduce to four, and the stress-strain relations take the form
where \( A, N, A, R \) are the four material constants characterising the medium. The appropriate physical interpretation can be given to these constants [11]. The constants \( \kappa \) and \( N \) corresponds to Lamé constants \( \lambda \) and \( \mu \), while \( A \) and \( R \) are called jacketed and unjacketed compressibility.

If the stress components in the equation (5) are replaced in terms of the strain components with the help of (9), and then the solid strain components are expressed in terms of the solid displacement components, we obtain

\[
N \nabla^2 \mathbf{u} + (A + N + A) \nabla \mathbf{e} + (Q + R) \nabla \mathbf{e} + \mathbf{p} = 0.
\]

The above vector equation involves two unknowns - the solid displacement vector \( \mathbf{u} \) and the fluid dilatation \( \mathbf{e} \). For their unique determination, a second equation is required and this is supplied by Darcy's law governing the flow of fluid through porous media. This we discuss below.

Darcy's Law

The flow of fluid through an isotropic porous medium, when the deformation of the solid skeleton is not taken into account is governed by the Darcy's law which states that in the absence of any body force the discharge velocity of the fluid is proportional to the pressure gradient.
responsible for the flow. If the body force is included, the law can be written as \[17,20\]

\[
\vec{V_d} = \frac{k}{\mu} \left( - \nabla \bar{p} + \rho' \vec{R}' \right)
\]

where \(\vec{V_d}\) is the discharge velocity vector of the fluid, \(\bar{p}\) the pressure, \(\rho'\) the mass per unit volume, \(\vec{R}'\) the body force vector and \(\mu\) the viscosity. The coefficient \(k\) is called the Darcy's coefficient of permeability for the medium. It is determined only by the structure of the medium and is entirely independent of the nature of the fluid. The discharge velocity \(\vec{V_d}\) in the Darcy's law is however smaller than the actual average velocity \(\bar{V}\) of the fluid \[20\]. For, if we consider an area \(A\) of the bulk material, the pore area \(A'\) is equal to \(A' = fA\), where \(f\) is the porosity. Since the total mass of fluid flowing per unit time (the total discharge) is the same, we have \(A \vec{V_d} = A' \bar{V}\), giving \(\vec{V_d} = f \bar{V}\). Darcy's law then takes the form

\[
\bar{V} = \frac{k}{\mu f} \left( - \nabla \bar{p} + \rho' \vec{R}' \right).
\]

In the present case, since the solid skeleton has also a certain velocity \(\vec{\omega}\), the term \(\vec{\nabla}\) in the above law is to be replaced by the relative velocity \(\vec{\nabla} - \vec{\omega}\). Thus Darcy's law for the deformable porous isotropic medium is
\[ \nabla \cdot \mathbf{v} = \frac{K}{\mu_0} ( - \nabla p + \rho' \mathbf{X}' ) \]

from which we get

\[ \nabla \cdot \sigma - \rho_1 \mathbf{X}' = \lambda \frac{\partial^2}{\partial t^2} ( \mathbf{v} - \mathbf{X} ) \]  \hspace{1cm} (11)

where

\[ \lambda = \rho \frac{\varepsilon}{E} \]  \hspace{1cm} (12)

and \( \rho_1 = \rho' \) is the mass of fluid per unit volume of the bulk material. Taking the divergence of both sides of (11) it is obtained that

\[ \nabla \cdot \sigma' + \nabla \cdot ( \rho_1 \mathbf{X}' ) = \lambda \frac{\partial^2}{\partial t^2} ( \varepsilon - \varepsilon ) \]  \hspace{1cm} (13)

When \( \sigma' \) is eliminated with the help of the second equation of (9) the equation (13) provides the required second equation to determine \( \mathbf{u} \) and \( \varepsilon \). The equation (11) can then be utilised to find the unknown \( \mathbf{u} \).

In case the medium is anisotropic, the law (11) can be modified to give

\[
\begin{bmatrix}
\frac{\partial \sigma}{\partial x} + \rho_1 x' \\
\frac{\partial \sigma}{\partial y} + \rho_1 y' \\
\frac{\partial \sigma}{\partial z} + \rho_1 z'
\end{bmatrix}
= \begin{bmatrix}
x \sigma_{xx} + \rho_1 y z & x \sigma_{xy} + \rho_1 z x & x \sigma_{xz} + \rho_1 y z \\
x \sigma_{yx} + \rho_1 z x & y \sigma_{yy} + \rho_1 z y & y \sigma_{yz} + \rho_1 z y \\
x \sigma_{zx} + \rho_1 y z & x \sigma_{zy} + \rho_1 z y & z \sigma_{zz} + \rho_1 y z
\end{bmatrix}
\begin{bmatrix}
\dot{u}_x - \dot{u}_x \\
\dot{u}_y - \dot{u}_y \\
\dot{u}_z - \dot{u}_z
\end{bmatrix}\]  \hspace{1cm} (14)
with the symmetry property
\[ \lambda_{ij} = \lambda_{ji}, \]
and a dot denoting derivative with respect to the time variable \( t \). The matrix \( \lambda_{ij} \) are called the resistance matrix for the porous medium.

**Quasi-static Theory**

In the theory advanced above, the inertia effect has not been included and hence the stress equation of equilibrium (5) has been used instead of the stress equation of motion. But in the Darcy's law (11) the time derivative is present explicitly. Consequently all the state variables—the stresses, strains and displacements—are functions not only of the position but also of the time. The time history enters through Darcy's law. Because of the omission of inertia forces, the theory is a quasi-static one.

The derivation of the above equations as well as the underlying concepts are given by Jin [1,5]. The corresponding Dynamic theory has also been formulated [30,15] and applied to wave motions [31]. We shall not deduce here the fundamental equations for the dynamic theory. These will be assumed to solve a transient problem of a half space loaded by an impulsive force [16], and the mathematical difficulties introduced further will be evident from the solution itself.
Experimental Data:

It has been shown above that the mechanics of an isotropic poroelastic medium filled with a fluid, is completely described with the help of the four poroelastic constants (A, N, ε, R), the porosity (f), the permeability (K) and the viscosity (μ) of the fluid. The four poroelastic constants (A, N, ε, R) can be expressed [11] in terms of the porosity (f) and four other directly measurable coefficients μ*, K*, δ*, and γ* which are defined as

\[ \mu^* = N \], \hspace{1cm} \frac{1}{K^*} = \frac{2}{3} N + (A - \frac{\delta^*}{R}), \hspace{1cm} \delta^* = K^*[1 - (\frac{\delta^*}{R})^2] \cdot \gamma^* = \frac{f}{R} [f - (A + R)\delta^*]. \]

Here μ* is the shear modulus, K* and δ* are the jacketed and unjacketed compressibilities respectively, and γ* is a coefficient of the fluid content.

Fatt [34] conducted an experiment for flow of kerosene through Boise sandstone. In the sample considered, the porosity was about 26 percent, the permeability was about 800 millidarcies, and the usual value of the viscosity of kerosene was taken. From his experimental results we may pick up the following numerical values.

\[ K^* = 1.5 \times 10^{-6} \text{ poise}^{-1}, \hspace{1cm} \delta^* = 0.22 \times 10^{-6} \text{ poise}^{-1}, \]

\[ \gamma^* = 1.2 \times 10^{-6} \text{ poise}^{-1}, \]

which are valid under pressures of about 2000 to 7000 psi. The shear modulus μ* was not measured with the special experimental arrangement, because, it is almost the same as in the elastic body.
Analogy Between the Quasi-static Theories of Thermoelasticity and Poro-elasticity

There is the strikingly close similarity between the (coupled) Quasi-static Theory of Thermoelasticity discussed in section 3.2 and the Quasi-static Theory of Poroelasticity as formulated above. In fact, as stated already in section 3.2, if a problem is solved for one of these branches of study, the result can be easily interpreted for a corresponding problem of the other branch by simply interchanging the temperature and fluid pressure, and noting a certain correspondence rule among the physical constants. To show this, we write the equations of poro-elasticity in a slightly different form.

Eliminating \( \varepsilon \) between the equations in (9) we obtain

\[
\sigma_{ij} + \sigma \delta_{ij} = 2N \varepsilon_{ij} + \left[ (A - \frac{g}{\alpha}) e + (1 + \frac{\phi}{\alpha}) \sigma \right] \delta_{ij}.
\]

Which, by (3), is equivalent to

\[
\sigma_{ij} + \sigma \delta_{ij} = 2N \varepsilon_{ij} + \left[ (A - \frac{g}{\alpha}) e - (1 + \frac{\phi}{\alpha}) f \right] \delta_{ij}.
\]

Elimination of \( \varepsilon \) from (10) and (13) with the help of the second equation of (9) gives respectively

\[
N \nabla^2 \omega + (A + N - \frac{g^2}{\alpha}) \nabla \varepsilon - (1 + \frac{\phi}{\alpha}) f \nabla \phi + \phi \nabla^2 \omega = 0.
\]

\[
\frac{f^2}{c^2} \nabla^2 \phi - \frac{c}{L} \frac{d}{dt} \left( \frac{c}{L} \frac{d}{dt} \right) = \frac{f^2}{c^2} \frac{\alpha b}{\alpha c} + f \left( 1 + \frac{\phi}{\alpha} \right) \frac{\phi}{c^2}.
\]
Now we rewrite the thermoelastic equations (1) of section 3.1 and (3) of section 3.2 as

\[ \sigma_{ij} = 2\mu \epsilon_{ij} + (\lambda + \mu) \gamma^{ij} \delta \]

\[ \mu \nabla^2 \mathbf{u} + (\lambda + \mu) \nabla \cdot \mathbf{e} - \rho \nabla^2 \mathbf{T} + f_0 \mathbf{T} = 0 \]

\[ \frac{K}{\gamma} \nabla^2 \mathbf{T} + \frac{\alpha}{\gamma \gamma T_0} = \frac{\rho}{\gamma \gamma T_0} \frac{\partial \mathbf{T}}{\partial t} + \beta \frac{\partial^2 \mathbf{T}}{\partial t^2} \]

Here \( \mathbf{q} \) represents the heat source.

The equations (19), (20) and (21) can be obtained from equations (16), (17) and (18) respectively by replacing the fluid pressure \( \mathbf{p} \) and the body force \( -\frac{1}{\gamma \gamma T_0} \nabla \cdot (\mathbf{q} \mathbf{T}) \) by the temperature \( \mathbf{T} \) and the heat source \( \frac{\rho}{\gamma \gamma T_0} \), provided we remember that the total stress in porous medium is \( \sigma_{ij} = \sigma_{ij}^T \), whereas \( \sigma_{ij}^T \) represents the total stress in thermoelasticity. The physical constants have the following correspondence, namely,

\[ N \leftrightarrow \mu \]

\[ \frac{\alpha}{\gamma \gamma T_0} \leftrightarrow \lambda \]

\[ (1 + \frac{\gamma}{\gamma R}) f \leftrightarrow \beta \]

\[ \frac{\rho}{\gamma \gamma T_0} \leftrightarrow \frac{\gamma^2}{\gamma \gamma T_0} \leftrightarrow \frac{\gamma}{\gamma \gamma T_0} \]

The above five correspondence-relations connect the six poroelastic constants \( A, N, \alpha, K, \gamma \) with the five thermoelastic constants \( \lambda, \mu, \beta, \frac{K}{\gamma \gamma T_0} \) and \( \frac{\rho}{\gamma \gamma T_0} \). To complete the analogy we arbitrarily assume a sixth correspondence relation.
where \( \psi \) is a non-dimensional thermoelastic parameter calculated from the equation

\[
\frac{A}{R} = \psi \frac{A}{\mu}.
\]  

(24)

From (22) and (23) we obtain

\[
\begin{align*}
A & \leftrightarrow A + \left( \psi \frac{A}{\mu} \right)^2 \frac{T_0 \beta^2}{C_v (1 + \psi \frac{A}{\mu})^2} \\
N & \leftrightarrow \mu \\
Q & \leftrightarrow \psi \frac{A}{\mu} \frac{T_0 \beta^2}{C_v (1 + \psi \frac{A}{\mu})^2} \\
R & \leftrightarrow \frac{T_0 \beta^2}{C_v (1 + \psi \frac{A}{\mu})^2} \\
f & \leftrightarrow \frac{\beta}{1 + \psi \frac{A}{\mu}} \\
\lambda & \leftrightarrow \frac{T_0 \beta^2}{K (1 + \psi \frac{A}{\mu})^2}
\end{align*}
\]  

(25)

As a consequence of the analogy, the results obtained from the solution of a poroelastic problem can be interpreted for the corresponding results holding for a problem of thermoelasticity.
4.2. Axisymmetric Stresses in a Porous Elastic Isotropic Material Containing a Fluid

Introduction

We have given an outline of the theory of deformation of an elastic porous material containing a compressible fluid as proposed by Biot [1-6] for both isotropic and anisotropic cases. Biot worked out several problems, mainly one dimensional and two dimensional. In a recent paper [6], it has been shown that the general three dimensional problem can be reduced to the solution of two sets of equations. The two constituents of the first set satisfy the Laplace and the heat conduction equations respectively, while those two of the second set satisfy the biharmonic and heat conduction equations. In the present section the theory has been developed for axisymmetric deformation by the use of cylindrical coordinates. The problem of the semi-infinite medium loaded by surface forces has been solved with the help of Laplace-Hankel transforms. As an illustration of the calculation of stresses and displacements, the axial normal stress has been dealt with in detail.

Fundamental Equations

For the axisymmetric deformation of a porous elastic material containing a fluid the stress tensor is [13]

\[
\begin{pmatrix}
\sigma_r + \sigma & \\ \\
\sigma_r + \sigma & \sigma_z \\
\sigma_z & 0 & \sigma_r + \sigma \\
\end{pmatrix}
\]
where $\sigma$ is related to the porosity $\phi$ and the fluid pressure $p$ by

$$\sigma = -\phi p.$$  

(1)

In the absence of body forces the equations of equilibrium are

$$\frac{\partial (\sigma_{xx} + \sigma_{yy})}{\partial x} + \frac{\partial (\sigma_{zy})}{\partial y} + \frac{\partial (\sigma_{yx})}{\partial x} + \frac{\partial (\sigma_{yy} + \sigma_{zz})}{\partial z} + \frac{\partial (\sigma_{yz})}{\partial z} = 0$$

(2)

The stress strain relations are (in the isotropic case)

$$\sigma_{xx} = 2N\varepsilon_{xx} + A\varepsilon + B\varepsilon^2$$
$$\sigma_{xy} = N\varepsilon_{xy}$$
$$\sigma_{xz} = N\varepsilon_{xz}$$

(3)

where $A, N, B, R$ are four elastic constants, and

$$\varepsilon_{xy} = \frac{\partial u_x}{\partial y}, \quad \varepsilon_{yx} = \frac{\partial u_y}{\partial x}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}, \quad \varepsilon_{yz} = \frac{\partial u_y}{\partial z}$$

(4)

$(u_x, u_z)$ and $(u_y, u_z)$ being the components of displacements of the solid and the fluid respectively. The equations of flow for the fluid in the porous solid (by Darcy's law) are

$$\frac{\partial (\sigma_{xx} - \sigma_{yy})}{\partial y} = \frac{1}{\phi} \left( u_x - u_y \right)$$
$$\frac{\partial (\sigma_{xy} + \sigma_{yx})}{\partial x} = \frac{1}{\phi} \left( u_y - u_z \right)$$

(5)
where \( \lambda \) is a coefficient characterising the permeability of the material. From (3),
\[
\theta = \sigma_{\tau} + \sigma_{\rho} + \sigma_{z} = (2N + 3A)e + 3q \epsilon
\]
\[
e = \kappa (R^2 - 3q^2), \quad \epsilon = \kappa \left\{ (2N + 3A) \sigma - q \theta \right\}
\]
(6)

where
\[
\frac{1}{K} = R(2N + 3A) - 3q^2 = 3 (P, R - q^2), \quad 3P = 2N + 3A.
\]

Again from (3) and (6)
\[
2N e_{rr} = \sigma_{\tau} - \kappa_1 \theta - 2k_2 \sigma
\]
\[
2N e_{r\theta} = \sigma_{\theta} - \kappa_1 \theta - 2k_2 \sigma
\]
\[
2N e_{zz} = \sigma_{z} - \kappa_1 \theta - 2k_2 \sigma
\]
(7)

The compatibility relations between the strain components are
\[
\epsilon_{rr} = \frac{\partial}{\partial r} \left( ne_{\theta\theta} \right)
\]
(8)
\[
\frac{\partial^2 \epsilon_{rr}}{\partial r^2 \partial z} = \frac{\partial^2 \epsilon_{rr}}{\partial z^2} + \frac{\partial^2 \epsilon_{zz}}{\partial r^2}
\]
(9)

**Stress Function**

The second relation in (2) is satisfied by taking the stress function \( \phi (r, z, t) \) as
\[
\sigma_{\tau z} = - \frac{\partial^2 \phi}{\partial r \partial z}
\]
(10)
From (8),
\[ e_{rr} - e_{\theta \theta} = \gamma \frac{\partial e_{\theta \theta}}{\partial r} \]

Substitution of the values of \( e_{rr} \) and \( e_{\theta \theta} \) from (7) in this equation gives
\[ \sigma_r - \sigma_\theta = \gamma \frac{\partial e_{\theta \theta}}{\partial r} (\sigma_\theta - \kappa \theta - 2 \kappa \sigma) \]. \[(8a)\]

If
\[ \sigma_r + \sigma = \frac{\partial^2 \phi}{\partial \theta^2} + F (\gamma, \phi, t) \] \[(12)\]
the first relation of (2) will be satisfied, by using (8a) and (10), provided
\[ \sigma_\theta = \kappa \sigma + 2 \kappa \sigma \sigma - F \]
that is,
\[ \sigma_\theta + \sigma = \frac{K_1 \sigma}{1 - K_1} \nabla^2 \phi - F + \frac{K_3 \sigma}{1 - K_1} \]. \[(13)\]

where
\[ \nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2} ; \quad K_3 = 1 + 2 \kappa_2 - 3 \kappa_1 \].
The function $F$ is determined from (8) and (9) which give
$$
\frac{\partial \varepsilon_{zz}}{\partial z} = \frac{\partial (\varepsilon_{zz})}{\partial z} + \frac{\partial \varepsilon_{zz}}{\partial \tau}.
$$

Expressing $\varepsilon_{zz}, \varepsilon_{\varphi\varphi}, \varepsilon_{zz}$ in terms of the stresses by (7), and then expressing the stresses in terms of $\varphi$ and $F$, one gets
$$
\frac{1}{1-K_1} \left[ (1-2K_1) \nabla^2 \varphi - K_3 \sigma \right] + \frac{\partial^2 \varphi}{\partial \varphi \partial z} = \rho \frac{\partial^2 F}{\partial z^2}.
$$

If
$$
\rho F = \frac{\partial}{\partial z} (\varphi + \mathcal{N}),
$$
then the above relation reduces to
$$
\frac{\partial^2 \mathcal{N}}{\partial z^2} = \frac{1}{1-K_1} \left[ (1-2K_1) \nabla^2 \varphi - K_3 \sigma \right],
$$
(14a)

The relations (5) with the help of (4) can be written in the combined form as
$$
\nabla^2 \varphi = -\lambda \frac{\partial}{\partial z} (\varepsilon - \bar{e}).
$$

Substituting the values of $\varepsilon$ and $\bar{e}$ from (6) in the above relation, and then expressing $\varphi$ in terms of $\sigma$, one gets,
\[ \nabla^2 \sigma = \alpha \frac{\partial}{\partial t} (\nabla^2 \varphi) + \beta \frac{\partial \varphi}{\partial t} \]  
(15)

where
\[ \alpha = \frac{\kappa (q + R)}{k_1 - 1} \]
\[ \beta = \kappa \left[ \frac{2(q + R)(k_2 - 1)}{k_1 - 1} + \left( 3q + 2n + 3a \right)^2 \right]. \]

Again, substituting the values of \( \varepsilon_{rr} \) and \( \varepsilon_{\theta \theta} \) from (7) in (8) and then expressing the stresses in terms of \( \varphi \) and \( F \), one gets,
\[ \frac{\partial^2 \varphi}{\partial x^2} + F - \sigma = \frac{k_1}{1 - k_1} \left( \nabla^2 \varphi + 2 \left( k_2 - 1 \right) \sigma - 2k_2 \varphi \right) = -\frac{2}{\beta \sigma} (\varphi F). \]

Differentiation of both sides with respect to \( x \) twice and then elimination of \( F \) with the help of (14) and (14a) give,
\[ \nabla^4 \varphi = \frac{k_2}{1 - 2k_1} \nabla^4 \sigma. \]  
(16)

Hence it is seen from (15) and (16) that the functions and \( \varphi \) satisfy the equations
\[ \nabla^4 \sigma = L \nabla^2 \frac{\partial \sigma}{\partial t}, \]  
(17)
\[ \nabla^6 \varphi = L \nabla^4 \frac{\partial \varphi}{\partial t}, \]  
(18)
\[ L = \rho + \alpha \frac{k_2}{1 - 2k_1}. \]  
(19)
The solution of the relation (17) consistent with the relation (15) or (16) satisfying given boundary and initial conditions gives \( \varphi \), and hence stresses and displacements can be obtained. In the present case, the semi-infinite medium will be treated. Cylindrical coordinates will be used by taking the \( \tau \) axis into the medium.

**Boundary and Initial Conditions**

Let a distributed load of constant magnitude \( \varphi_0 \) per unit area be applied on the surface through a porous smooth circular slab of radius \( a \). It will be assumed that the system is at rest and unstrained initially and the load is applied suddenly. The boundary and initial conditions are then as follows:

\[
\begin{align*}
\sigma_z & = -\varphi_0 \quad \gamma < a, \quad z = 0, \quad t \geq 0 \\
\tau_{r \gamma} & = 0 \quad r > a, \quad z = 0, \quad t \geq 0 \\
\end{align*}
\]

(20)

\[
\sigma_z = 0, \quad z > 0, \quad r > 0, \quad t > 0
\]

(21)

\[
\sigma = 0, \quad z = 0, \quad r > 0, \quad t > 0
\]

(22)

\[
\varphi (r, z, t) = \sigma (r, z, t) = 0, \quad z > 0, \quad r > 0, \quad t = 0
\]

(23)

\[
\varphi (r, z, t) = \sigma (r, z, t) = 0, \quad z \to \infty, \quad r \to \infty, \quad t \to 0
\]

(24)
Transform Solution

The transforms of Laplace-Hankel will be adopted to solve the problem. A function with a bar over it will denote Hankel transform of order zero while a function with a prime on it will be taken as its Laplace transform. The function with both the bar and the prime will mean that both the transforms have been applied on it one after another. Thus,

\[ \Phi'(\eta, z, t) = \int_0^\infty \Phi(\eta, z, t) e^{-it} dt \]  

(25)

\[ \Phi'(\eta, z, t) = \int_0^\infty \Phi'(\eta, z, t) r J_\nu(\eta r) dr , \]  

(26)

\[ \nu \] denoting the Bessel function of order \( \nu \). Using the condition (23) and assuming further that

\[ r \nu^2 \Phi = r \nu^2 \Phi = r \Phi = 0 \]  

(27)

\[ r \frac{\partial}{\partial r} (\nu^2 \Phi) = r \frac{\partial}{\partial r} (\nu^2 \Phi) = r \frac{\partial \Phi}{\partial r} = 0 \]  

(28)

as \( r \to 0 \) and \( r \to \infty \), which are required for the application of Hankel transform and may be verified when the solutions have been obtained, one gets from the relation (18),

\[ (D^2 - \eta^2)(D^2 - \eta^2 - L^2) \Phi'(\eta, z, t) = 0 , \quad D = \frac{1}{\sqrt{\nu}} \]  

(29)
With similar assumptions as (27) and (28) for the function \( \sigma^* \), the relations (15), (16) and (17) reduce to

\[
\left( D^2 - \eta^2 \right) \bar{\sigma}' = \alpha \xi \left( D^2 - \eta^2 \right) \bar{\Phi}' + \beta \xi \bar{\Phi}'
\]

(30)

\[
\left( D^2 - \eta^2 \right) \left( D^2 - \eta^2 - \frac{4}{3} \right) \bar{\sigma}' = 0
\]

(31)

\[
\left( D^2 - \eta^2 \right)^2 \bar{\sigma}' = \frac{k_3}{1 - 2k_1} \left( D^2 - \eta^2 \right) \bar{\Phi}'.
\]

(32)

The boundary conditions (20) to (22) can be written as

\[
\bar{\sigma}_z' = - \frac{b_0 a}{k \eta} J_1 (k \eta), \quad z = 0
\]

(33)

\[
\int_0^\infty \sigma_{rz} \bar{r} J_1 (k \bar{r}) \, dr = 0, \quad z = 0
\]

(34)

\[
\bar{\sigma}_z' = 0, \quad z = 0.
\]

(35)

The stress components (10) and (11) transform to

\[
\int_0^\infty \sigma_{\\eta z} \bar{r} J_1 (k \bar{r}) \, dr = \eta D \bar{\Phi}'
\]

(36)

\[
\bar{\sigma}_z' + \bar{\sigma}' = - \eta^2 \bar{\Phi}'.
\]

(37)

Adding (12) and (13) and then taking the transform, one gets

\[
2 \bar{\sigma}' + \bar{\sigma}_z' + \bar{\sigma}_\eta' = \frac{1}{1 - k_1} \left( D^2 \bar{\Phi}' - k_1 \eta^2 \bar{\Phi}' + k_3 \bar{\sigma}' \right)
\]

(38)

while
The Inverse Hankel transform of (36) and (38) give

\[ \frac{\vartheta'(\tau, \rho)}{\lambda_1} = \int_0^\infty (\lambda_2^2 - \lambda_3^2) \varphi' d\rho \]

(39)

The inverse Hankel transform of (36) and (38) give

\[ \tau'(\tau, \rho) = \int_0^\infty \eta^2 J_1(\eta \rho) \varphi' d\eta \]

(40)

\[ 2\sigma' + \tau' + \sigma' = \frac{1}{\lambda_1} \int_0^\infty (\lambda_2^2 \varphi' - \lambda_3^2 \varphi' + \lambda_3 \tau') J_0(\eta \rho) d\eta \]

(41)

The first relation of (2) can be written as, after applying Laplace transform,

\[ \int_0^\infty \left[ \eta^2 (\tau' + \sigma') \right] = \tau' (\tau' + \sigma' + 2 \sigma') - \tau^2 \frac{d^2 \tau}{d\tau^2} \]

Substituting in the right hand side of the above equation from (40) and (41), integrating both sides with respect to \( \tau \) for the limits zero to \( \tau \), and interchanging the order of integration, one gets,

\[ \tau' + \sigma' = \frac{1}{(1 - \lambda_3)} \int_0^\infty \left\{ (2\lambda_1 - 1) \varphi' - \lambda_3 \varphi' + \lambda_3 \varphi' \right\} J_0(\eta \rho) d\eta \]

(42)

The relations (41) and (42) serve to determine \( \tau' \) and \( \sigma' \).

Complete Solution

The solutions of the differential equations (29) and (31) consistent with the conditions (24) and (35) are
The coefficients $A_2$, $B$, $C$ and $A_1$ are functions of $\xi$ and $\gamma$, but independent of $\zeta$. Now (30) reduces to

$$\frac{d\varphi}{dz} = \varphi + \varphi' = \rho \left( e^{\gamma z} - e^{-\gamma z} \right)$$

where $\zeta = \gamma^2 + \xi \xi$.

Since this is true for all values of $\zeta$, the coefficients of $\exp(-\gamma z)$ and $\exp(-\xi z)$ must be separately zero. Thus

$$\rho A_1 - 2\alpha \eta B = 0$$

$$\rho A_1 - \lambda A_1 - \alpha \xi L C = 0$$

It can be verified that the relation (32) is automatically satisfied, it being only another form of expressing the dependence of $\varphi'$ on $\varphi$ as the relation (30) is. The boundary conditions (33) and (34) with the help of (36) and (37) give

$$A_2 + C = \frac{b_0 a}{\delta \gamma^2} \frac{\tilde{J}_1 (\alpha \eta)}{\delta \gamma}$$

$$\eta A - B + C \xi = 0$$
From (45) to (48)

\[ B = \frac{\rho_0 \alpha \eta \xi}{\eta^2 \xi \left( \xi + \eta \xi^2 \right)} \left\{ 2\lambda \beta \eta (\eta + \xi) + L \xi^2 \right\} \]  \hspace{1cm} (49)

with

\[ \beta \lambda = \beta - L. \]  \hspace{1cm} (50)

The coefficients \( A_2, \) \( C \) and \( A_1 \) can now be easily found in terms of \( B \) as

\[ A_2 = \frac{L \rho \xi^2 - 2(\beta - L)\eta \xi}{L \rho \xi \eta} \]

\[ C = \frac{2\eta(\beta - L)}{L \rho \xi} \]

\[ A_1 = \frac{2\alpha \eta}{\rho} \]

**Evaluation of the Stresses**

Substitution of the values of \( \varphi' \) and \( \varphi' \) from (43) and (44) in (37) gives

\[ -\sigma_{zz}' = \sigma_{\xi} + \eta^2 \varphi' = \frac{\rho_0 \alpha \eta \xi}{\beta \eta^2 \xi \left( \xi + \eta \xi^2 \right)} \left\{ 2\lambda \beta \eta (\eta + \xi) + L \xi^2 \right\} \]

\[ \times \left\{ \left\{ L(2\alpha + \rho + \beta \eta z) - 2\lambda \beta \eta \frac{\xi}{\xi} \right\} e^{-\eta z} \right. \]

\[ + 2\left( \frac{\alpha^2 \eta^2}{\xi} - \alpha L \right) e^{-\frac{\xi}{2} z} \]  \hspace{1cm} (51)

By the inverse Laplace transform of (51)

\[ -\sigma_z = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} -\sigma_{zz}' e^s ds, \quad (Re's = C > 0) \]  \hspace{1cm} (51a)
The integral in (51a) can be evaluated by a contour integration. It is seen from (51) that the integrand in (51a) has poles at the points \( \xi = \infty \) and \( \xi = -\frac{n\pi}{L} \) and a branch point at \( \xi = 0 \), that is, at \( \xi = -\frac{\eta}{L} \). The curve of integration is chosen as that which encloses the entire space to the left of the line \( \xi = c \). If \( R_0 \) and \( R_1 \) be the residues at the poles respectively,

\[
R_0 = \frac{b^2 - \alpha \beta}{L} J_1(\eta) (1 + \eta^2) e^{-\eta^2} \tag{52}
\]

\[
R_1 = -\frac{b^2 - \alpha \beta}{4\beta \eta^2} \left[ (\delta_2 + 2\rho L \eta^2) e^{-\eta^2} - \delta_3 e^{-\eta^2} \right] e^{-\eta^2} \tag{53}
\]

where

\[
\delta_1 = \rho (\alpha + \beta) - 2L, \quad \delta_2 = 2L (2\alpha + \beta) + \mu^2 \beta^2 - 4\lambda L \tag{54}
\]

While calculating \( R_0 \), it is to be noticed in (51) that the expressions \( (\eta^2/\xi) e^{\eta/\xi} \) and \( (\eta^2/\xi) e^{\eta/\xi} \) are equal as \( \xi \to 0 \) and they cancel each other with no contribution to the residue. The branch point contributes nothing. Thus,

\[-\frac{e^{-\eta^2}}{\eta} = R_0 + R_1 \tag{55}\]

The inverse Hankel transform of (55) gives
\[-\sigma_z = \int_0^\infty (R_0 + R_1) J_0(\eta^*) \, d\eta \]
\[= \frac{p_0 a}{\eta_0 L} \left[ \frac{4\rho^2 L}{I_0 + z I_1} - \delta_1 \left( \frac{\delta_2 p_0 + \delta_3 p_1}{z \rho L} \right) \right] \]  

(56)

where,

\[I_0 = \int_0^\infty J_0(\eta^*) J_1(\eta^*) e^{-\eta^2} \, d\eta \]  

(57)

\[I_1 = \int_0^\infty \eta J_0(\eta^*) J_1(\eta^*) e^{-\eta^2} \, d\eta \]  

(58)

\[s_0 = \int_0^\infty J_0(\eta^*) J_1(\eta^*) e^{-\eta^2 - 4\lambda^2 \eta^2} \, d\eta \]  

(59)

\[s_1 = \int_0^\infty \eta J_0(\eta^*) J_1(\eta^*) e^{-\eta^2 - 4\lambda^2 \eta^2} \, d\eta \]  

(60)

and $s_0$ is obtained from $s_0$ by simply replacing $z$ by $1 - 2z$. The first two of the above integrals may be expressed in terms of elliptic functions while the numerical evaluation of the last two will be a little lengthy. Other stress components may be found in the same way. Thus,

\[\varphi = \frac{-2\lambda \alpha \beta J_1(\alpha \eta) \frac{2\lambda \alpha \beta (\eta + i)}{4\lambda^2 \eta} + \frac{1}{2} f}{4\lambda^2 \eta \left( \frac{1}{2} + 4\lambda^2 \eta^2 \right)} \left( e^{-\eta^2} - e^{-\frac{3}{2} \eta^2} \right) \]

By Inverse Laplace Transform

\[q(t) = \frac{1}{2\pi} \int_{-\alpha}^{\alpha} \varphi e^{it} \, dt \]
which is evaluated as before by contour integration. The residue at $\xi = 0$ vanishes. Calculating the residue at $\xi = -\nu \alpha \eta^2$ we have

$$\sigma = -\frac{\rho_0 \alpha \xi \delta \pi_j (\alpha \eta)}{\beta L} \left( e^{-\eta^2} - e^{-\nu \eta^2} \right) e^{-\lambda \eta^2}.$$ 

The Inverse Hankel transform gives

$$\sigma = -\frac{\rho_0 \alpha \xi \delta \pi_j (\alpha \eta)}{\beta L} (\nu e - \phi e)$$

where $\nu_0$ is given by (59), and $\nu_0'$ is obtained from $\nu_0$ by replacing $\nu$ by $\nu \eta$. 

Concentrated Force

The case of a concentrated force will be obtained from the previous formulae by taking

$$p = \rho_0 \pi \alpha^2$$

where $\rho_0$ is large and $\alpha$ is to be considered as small, so that

$$\pi_j (\alpha \eta) = \frac{1}{2} \alpha \eta.$$ 

Hence the integrals (57) - (60) reduce to

$$I_0 = \frac{1}{2} \alpha \int_0^\infty \eta \pi_j (\eta \nu) e^{-\eta^2} d\eta = \frac{1}{2} \frac{\alpha \nu}{(\nu^2 + \nu^2)}$$

$$I_1 = \frac{1}{2} \alpha \int_0^\infty \eta^2 \pi_j (\eta \nu) e^{-\eta^2} d\eta = \frac{1}{2} \frac{\alpha (\nu^2 + \nu^2)}{(\nu^2 + \nu^2)^{\frac{3}{2}}}.$$
\[ S_0 = \frac{1}{2} a \sum_{\eta} \mathcal{J}_0(\eta \tau) e^{-\eta^2 - 4 \tau \lambda \eta^2} d\eta. \]

\[ S_1 = \frac{1}{2} a \sum_{\eta} \eta^2 \mathcal{J}_0(\eta \tau) e^{-\eta^2 - 4 \tau \lambda \eta^2} d\eta. \]

(63)

On the axis \( \eta = 0 \), since \( \mathcal{J}_0(\eta \tau) = 1 \),

\[ I_0 = \frac{1}{2} \frac{a}{\tau^2} \quad I_1 = \frac{a}{\tau^3} \]

\[ \mathcal{S}_0 = \frac{1}{2} a \sum_{\eta} \eta e^{-\eta^2 - 4 \tau \lambda \eta^2} d\eta \]

\[ \mathcal{S}_1 = \frac{1}{2} a \sum_{\eta} \eta^2 e^{-\eta^2 - 4 \tau \lambda \eta^2} d\eta. \]

(64)

From the identity

\[ \sum_{\eta} e^{-\eta^2 - 4 \tau \lambda \eta^2} d\eta = \frac{1}{2} \sqrt{\frac{\pi}{\lambda \tau}} \text{erf} \left( \frac{z}{\sqrt{4 \lambda \tau}} \right) e^{\frac{z^2}{16 \lambda \tau}} \]

(65)

where

\[ \text{erf} (y) = 1 - \frac{2}{\sqrt{\pi}} \int_0^y e^{-u^2} du, \]

Differentiation of both sides with respect to \( z \), gives

\[ k\!(z,t) \equiv \sum_{\eta} \eta e^{-\eta^2 - 4 \tau \lambda \eta^2} d\eta = \frac{1}{8 \lambda \tau} - \frac{1}{32 \lambda \tau} \sqrt{\frac{\pi}{\lambda \tau}} z \text{erf} \left( \frac{z}{\sqrt{4 \lambda \tau}} \right) e^{\frac{z^2}{16 \lambda \tau}}. \]

(66)

A second differentiation gives
From (56), (64) - (67), one gets, for a concentrated force $P^*$,

$$-8\rho L^2 \pi \left( \frac{\sigma^2 P}{\eta} \right)_{r=0} = 12\rho L^2 \frac{1}{2z} - \delta_1 \delta_2 \delta_3 \mathcal{K}_1(z,t) + 2\rho L^2 k_2(z,t) - 8 \mathcal{K}_1(\mu z, t)$$

(68)

where $\mathcal{K}_1(\mu z, t)$ is obtained from $\mathcal{K}_1(z, t)$ by replacing $z$ by $\mu z$.

Similarly, the fluid stress, for the concentrated force, is given by

$$-2\pi \delta_1 \frac{L}{P} (\sigma)_{r=0} = \mathcal{K}_1(z, t) - \mathcal{K}_1(\mu z, t).$$

(69)

Also from (68) and (69)

$$-2\pi \delta \frac{L^2}{P} (\sigma_{z} + \sigma)_{r=0} = \frac{12\rho L^2}{2z} - \delta_1 \delta_2 \left( 2L + \mu \delta \right) \mathcal{K}_1(z, t) - \delta \mathcal{K}_1(\mu z, t) + 2L^2 k_2(z, t).$$

(70)

**Non-dimensional Form**

To put the above results in dimensionless form we write the expressions for different physical constants in terms of the fundamental constants as,
\[
K_1 = \frac{A - \frac{a^3}{R^2}}{2N + 3(A - \frac{a^{3'}}{R'})}
\]
\[
K_2 = \frac{A}{R} \cdot \frac{N}{2N + 3(A - \frac{a^{3'}}{R'})}
\]
\[
\alpha = \frac{1}{K_1} \cdot \frac{\lambda_0 (1 + \frac{a}{R})}{2N + 3(A - \frac{a^{3'}}{R'})}
\]
\[
\rho = \left\{ \frac{2(K_2-1)}{K_1-1} + \frac{2 \frac{N}{R} + 3 \frac{A+a}{R}}{1 + \frac{a}{R}} \right\} \frac{\lambda_0 (1 + \frac{a}{R})}{2N + 3(A - \frac{a^{3'}}{R'})}
\]

so that:
\[
\frac{\alpha}{\rho} = -\frac{1}{2} \left[ (1-K_2) + (1-K_1) \right] \frac{\frac{N}{R} + 3(A+a)}{1 + \frac{a}{R}}
\]

It is to be noticed that \( K_1 \) and \( K_2 \) are dimensionless quantities whereas \( \alpha \) and \( \rho \) are not. But the ratio \( \alpha/\rho \) is non-dimensional. In terms of \( K_1, K_2, \alpha \) and \( \rho \) we express other physical constants as:

\[
L = \rho \left\{ 1 + \frac{\alpha}{\rho} \cdot \frac{1+2K_2-3K_1}{1-2K_1} \right\}
\]
\[
\lambda = -\frac{1}{\rho} \cdot \frac{1+2K_2-3K_1}{1-2K_1}
\]
\[
\mu = 1 + 4 \frac{\alpha}{\rho} \cdot \frac{1+2K_2-3K_1}{1-2K_1} \left\{ \frac{1 + \frac{\alpha}{\rho} \cdot \frac{1+2K_2-3K_1}{1-2K_1}}{1+2K_2-3K_1} \right\}
\]
\[
\delta_1 = \rho \left[ \mu - 2 \frac{\alpha}{\rho} \cdot \frac{1+2K_2-3K_1}{1-2K_1} \right]
\]
\[
\delta_2 = \rho^2 \left[ \mu + 2 \left( 1+2 \frac{\alpha}{\rho} \right) \left\{ 1 + \frac{\alpha}{\rho} \cdot \frac{1+2K_2-3K_1}{1-2K_1} \right\} \right]
\]
\[
\delta_3 = \rho^2 \left[ 1 + 4 \frac{\alpha}{\rho} \left\{ 1 + \frac{\alpha}{\rho} \cdot \frac{1+2K_2-3K_1}{1-2K_1} \right\} \right]
\]
Here again the quantity \( \lambda \) is dimensionless.

If we put
\[
z = \frac{z}{z_l}, \quad 16 \lambda t = \frac{z}{z_l^2}
\]

(73)

where \( z \) represents a certain characteristic length, then the results (69) and (70) can be written in the non-dimensional form
\[
-\frac{\pi \beta}{a \delta_1} \left( \frac{4 \lambda^2 \rho}{\sigma} \right) = K_3 (\lambda z_l, \tau) - K_3 (z_l, \tau)
\]

(74)

\[
-\frac{4 \pi L \Delta \rho^{1/3}}{\delta_1 \rho} = \frac{1}{P} (\sigma_0 v + \sigma) \frac{z_l}{z_l^2} = \left[ \left( \mu_1 + \frac{2 \rho}{\sigma_0} \right) \frac{z_l}{z_l^2} \right] K_3 (z_l, \tau)
\]

(75)

where
\[
K_3 (z_l, \tau) = \frac{1}{z_l} \left[ 1 - z_l \frac{1}{\sqrt{\pi}} \text{erfc} \left( \frac{z_l}{\sqrt{\pi}} \right) e^{-\frac{z_l^2}{\tau}} \right]
\]

(76)

\[
K_4 (z_l, \tau) = \frac{1}{z_l} \left[ (1 + \frac{1}{\sqrt{\pi}}) \text{erfc} \left( \frac{z_l}{\sqrt{\pi}} \right) e^{-\frac{z_l^2}{\tau}} - \frac{2 z_l}{\sqrt{\pi}} \right]
\]

(77)

and \( K_3 (\lambda z_l, \tau) \) is obtained from \( K_3 (z_l, \tau) \) by replacing \( z_l \) by \( \lambda z_l \). It is to be noticed that \( \tau K_3 (z_l, \tau) \) and \( \tau^{-1/2} K_4 (z_l, \tau) \) depends upon a single variable \( z_l / \sqrt{\pi} \). Table 4.1 gives the values of \( \tau K_3 (z_l, \tau) \), \( \tau^{-1/2} K_4 (z_l, \tau) \), and \( \tau K_3 (\lambda z_l, \tau) \) for different values of \( z_l / \sqrt{\pi} \) taking \( \lambda = 0.5 \). Table 4.2 gives the values of \( -\frac{\pi \beta}{a \delta_1} \left( \frac{4 \lambda^2 \rho}{\sigma} \right) \) for different values of \( \tau \) when \( \tau = 1 \) and \( \lambda = 0.5 \).
The numerical evaluation has been made easier for large values of \( z, \sqrt{z} \) by using the asymptotic formula [21] \[
\text{erfc} z = \frac{1}{\sqrt{\pi}} \left( \frac{1}{2} - \frac{1}{2z^2} + \frac{1.3}{2^2z^4} - \ldots \right).
\]

Fig. 4.2 represents the values of \(-\frac{\pi aL}{\sqrt{\sigma}} \left( \frac{z^2 + \mu}{\rho} \right)\) against \( z \) when \( \gamma = 1 \) and \( \mu = 0.5 \).

If we take a particular value \[
\alpha = \frac{1 + 2k_2 - 3k_1}{1 - 2k_1} = 0.75
\]
it is seen from (72) that \[
\frac{L}{\rho} = 0.25, \quad \mu = 0.50, \quad \frac{\delta_1}{\delta} = 1.0,
\]
so that \( \frac{L}{\delta_1} = 0.25 \). Table 4.3 gives the values of \[-\frac{\mu\alpha L^2}{\delta_1} \left( \frac{z^2 + \mu}{\rho} \right)\] for different values of \( z \) when \( \gamma = 1 \), the values of \( \mu, \frac{L}{\delta_1} \), and \( \frac{L}{\rho} \) being given as above. Fig. 4.3 gives the graphical representation.
<table>
<thead>
<tr>
<th>$z_l/\sqrt{\tau}$</th>
<th>$\gamma K_3(z_l, \tau)$</th>
<th>$\gamma^{3/4} K_8(z_l, \tau)$</th>
<th>$\gamma K_3(\mu z_l, \tau)$</th>
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</thead>
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<td>1.0</td>
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<tr>
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<td>0.02000</td>
<td>-</td>
<td>0.05721</td>
</tr>
<tr>
<td>6.0</td>
<td>0.01389</td>
<td>-</td>
<td>0.02746</td>
</tr>
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Table 4.2
\[
\begin{array}{ccc}
\gamma = 1 \quad \mu = 0.5 & \\
\begin{array}{ccc}
z_i & \frac{\pi \beta L}{\alpha_0} & (\frac{E}{\alpha_0})^\tau = 0 & z_i & \frac{4\pi I^2}{\alpha_0} (\sigma_2 + \tau)\tau = 0 \\
0.0 & 0.0 & 0.2 & 37.05513 \\
0.2 & 0.12950 & 0.4 & 9.17900 \\
0.4 & 0.18879 & 0.6 & 4.11825 \\
0.6 & 0.21713 & 0.8 & 2.37335 \\
0.8 & 0.21796 & 1.0 & 1.57521 \\
1.0 & 0.21215 & 1.2 & 1.14139 \\
1.2 & 0.20133 & 1.8 & 0.58374 \\
1.8 & 0.16776 & 2.0 & 0.48524 \\
2.0 & 0.14750 & 3.0 & 0.25965 \\
3.0 & 0.10731 & 5.0 & 0.01357 \\
4.0 & 0.06624 & 6.0 & 0.03721 \\
5.0 & 0.03721 & & \\
6.0 & 0.01357 & & \\
\end{array}
\end{array}
\]

The Corresponding Thermoelastic Problem

It will be now shown how the above axially symmetric problem of the Quasi-static Theory of Poro-elasticity can be interpreted as a problem of the (coupled) Quasi-static Theory of Thermoelasticity discussed in the previous section 3.2.

We consider the axially symmetric stress distribution in a semi-infinite medium due to a concentrated force on the boundary. If it is assumed that the solid is purely elastic in the sense that the elastic field and the temperature field are quite independent of each other, then
\[ \tau = 1.0, \quad \mu_1 = 0.5 \]

FIG. 4.2

\[ -\frac{\partial \delta_1}{\partial \theta} \frac{f \ell p}{1 - p}(r=0) \]
the applied surface force is not expected to give rise to a temperature field. But, as has been pointed out earlier the two fields are interlinked in the real solid and hence the prescribed surface mechanical force will generate a temperature field in addition to the elastic stress field. Moreover, in the latter (coupled) case, the stress field will be different from the former (uncoupled) case. To give an idea of the magnitude of the thermal field, we wish to evaluate the temperature distribution on the axis of symmetry passing through the point on the surface at which the concentrated force is applied.

The thermoelastic problem enunciated above is exactly similar to the axially symmetric problem of poroelasticity, provided the correspondence relations are used. Replacing the fluid pressure \( p \) and the total normal poroelastic stress \( \sigma + \sigma' \) by the temperature \( \tau \) and the normal thermoelastic stress \( \sigma_z \) respectively in the results (74) and (75), we obtain the corresponding thermoelastic interpretations as

\[
- \frac{\pi \rho_0 L_0}{\alpha_0 \delta_1} \left( \frac{f_0 L}{P} \right) \tau_0 \zeta_0 = k_3 \left( M_1, \tau \right) - k_3 \left( z_1, \tau \right)
\]

(78)

\[
- \frac{4 \pi L_0^2}{\delta_1 \rho_0} \left( \frac{f_0 L}{P} \right) \zeta_0 = \frac{6L_0}{\delta_1 z_1^2} - \left[ \left( M_1 + 2 \frac{L}{P_0} \right) k_4 \left( z_1, \tau \right) - k_3 \left( M_0, \tau \right) + 2 \sqrt{\tau} \left( \frac{L}{P_0} \right) k_4 \left( z_1, \tau \right) \right]
\]

(79)

where \( \alpha_0, \beta_0, L_0, \delta_1, f_0 \) are the values corresponding to \( \alpha, \beta, L, \delta_1, f \). The exact correspondence
relations for these and some other quantities are given below as obtained from equations (71) and (72) with the help of (22), (23) and (25) of the previous section (section 4.1). From (71) we obtain the following transformation formulae:

\[
\begin{align*}
\text{Poroelasticity} & \rightarrow \text{Thermoelasticity} \\
K_1 & \rightarrow \frac{1}{\frac{2}{3} + \mu} = k_{10} \\
K_2 & \rightarrow \frac{\lambda \gamma}{\frac{2}{3} + \mu} = k_{20} \\
\ell & \rightarrow \frac{\gamma \ell}{1 + \frac{2}{\mu}} = \ell_0 \\
\alpha & \rightarrow \frac{-\frac{\alpha_0}{\beta_0}}{\frac{1}{2} (1 - \kappa_{10}) \ell_1 + (1 - \kappa_{10}) \left( \frac{3}{2} + 1 + \frac{2}{\mu} \right)} \\
\frac{\delta}{\rho} & \rightarrow \frac{-\delta_1}{2 (1 - \kappa_{10}) \ell_1 + (1 - \kappa_{10}) \left( \frac{3}{2} + 1 + \frac{2}{\mu} \right)} = \frac{\delta_0}{\rho_0}
\end{align*}
\]

where \( \ell_1 \) is an non-dimensional thermoelastic quantity defined by

\[
\ell_1 = \frac{T_0 \beta^2}{c_0 \left( \frac{2}{3} + \mu \right)}
\]

and will be called the thermoelastic coupling coefficient. Other constants \( L_0, \ell_{10}, \ldots \) can be easily calculated from (72) by replacing \( \alpha_0, \beta, \mu, \ldots \) by
The thermoelastic constants $\lambda, \mu, \beta, \kappa, \nu$, are defined as in the section 3.2 and need not be confused with the poroelastic constants.

A Numerical Example in the Thermoelastic Problem

To give an approximate idea about the magnitude of the temperature and axial normal stress distribution in the above thermoelastic problem we consider a thermoelastic material whose physical constants are those given in section 3.3. We also choose the corresponding poroelastic material in such a way that the non-dimensional parameter $\psi$, defined by the equation (23) of the previous section 4.1, has the value $\psi = 1$.

Then we obtain from (30) that

$$K_0 = K_\infty = 0.24814, \quad \epsilon_1 = 0.0012$$

and consequently

$$\frac{K_0}{\beta_0} = -0.0553.$$  

Also from (72)

$$\frac{L_s}{\beta_0} = 0.9920, \quad \mu_0 = 0.99838, \quad \frac{\delta_{10}}{L_0} = -0.2 \times 10^{-4}.$$  

With these numerical values the results (78) and (79) can be written as

$$-\pi \left( \frac{\lambda \psi_2^\tau}{P} \right)_{r=0} = 0.85511 \times 10^{-4} \left[ K_3 \left( \frac{\lambda \psi_2^\tau}{P} \right) - K_3 \left( \frac{\lambda \psi_2^\tau}{P} \right) \right].$$ (82)
\[-0.9920 \times \left( \frac{4 \pi \epsilon^2 \sigma^2}{p} \right)_{r=0} = \frac{6}{z_1^2} + 0.2 \times 10^{-4} \left[ 2 \cdot 99678 \cdot \mu_3(z_1, r) \right.\]

\[-K_3(J_{\text{int}}, r) + 1.99840 \int \mu_4(z_1, r) \right]

(83)

where \( \mu_3 \) has of course the value given earlier. The last equation shows that the axial normal stress distribution varies approximately as the inverse of the square of the distance. Also from the equation (82) it is evident that the temperature variation is very small. This is expected because of two reasons. Firstly, no surface temperature has been prescribed (only the mechanical concentrated force is applied), and secondly the coupling coefficient \( \epsilon_1 \) has a very small numerical value in this particular example.
4.3. Deformation of Spherical Bodies

The theory of deformation of fluid-saturated elastic material has been formulated for a spherically isotropic medium. The fundamental equations for the isotropic medium has been deduced as the particular case. The displacement in a spherical shell of isotropic material under internal and external pressures has been found as an illustration.

Fundamental Equations

In the deformation of a porous elastic material containing a fluid the stress tensor in spherical coordinates \((\tau, \theta, \phi)\) is [14]

\[
\begin{pmatrix}
\sigma_\tau + \sigma & \sigma_\theta & \sigma_\phi \\
\sigma_\theta & \sigma_\tau + \sigma & \sigma_\phi \\
\sigma_\phi & \sigma_\phi & \sigma_\phi + \sigma
\end{pmatrix}
\]

(1)

where \(\sigma\) is related to the porosity \(f\) and the fluid pressure \(p\) by

\[
\sigma = -fp
\]

(2)

In the absence of body forces, the equations of equilibrium are

\[
\frac{\partial}{\partial \tau} (\sigma_\tau + \sigma) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sigma_\theta + \sigma) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \phi} (\sigma_\phi + \sigma) + \frac{1}{r} \left(2 \sigma_\tau - \sigma_\theta - \sigma_\phi + \sigma_\phi \cos \theta \right) = 0
\]

\[
\frac{\partial}{\partial \theta} (\sigma_\theta + \sigma) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (\sigma_\phi + \sigma) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \phi} (\sigma_\phi + \sigma) + \frac{1}{r} \left(3 \sigma_\phi - \sigma_\phi \cos \theta \right) = 0
\]

If the medium posses anisotropy of the spherically isotropic type and the radial direction happens to be the axis of symmetry, then the stress-strain relations are

\[ \sigma_\theta = 2N\sigma_\theta + A(e_\theta + e_\phi) + F e_r r + M t \]
\[ \sigma_\phi = 2N\sigma_\phi + A(e_\phi + e_\theta) + F e_r r + M t \]
\[ \sigma_r = C e_r r + F(e_\theta + e_\phi) + \alpha t \]
\[ \sigma = M(e_\theta + e_\phi) + \alpha e_r r + R t \]

where \( N, A, F, M, C, \alpha, R, \) and \( L \) are eight elastic constants. In terms of displacements the strain components are

\[ e_r = \frac{\partial v_r}{\partial r} \]
\[ e_\theta = \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \]
\[ e_\phi = \frac{1}{r} \frac{\partial v_\phi}{\partial \phi} + \frac{1}{\sin \theta} \frac{\partial v_\theta}{\partial \theta} \]
\[ e_\phi = \frac{1}{\sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_\theta}{r} \]
\[ e_{\phi \phi} = \frac{1}{r} \frac{\partial v_\phi}{\partial \phi} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{1}{\sin \theta} \frac{\partial v_\phi}{\partial \theta} \]

\[ e = e_r + e_\theta + e_\phi \]
\[ \varepsilon = \frac{2v_r}{r} + \frac{2v_\theta}{r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{1}{\sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi}{r} \frac{\partial v_\phi}{\partial \theta} \]
where \((u_r, u_\theta, u_\varphi)\) and \((u_r, u_\theta, u_\varphi)\) denote the components of displacements of the solid and the fluid respectively. The equations of flow for the fluid in the porous solid (by Darcy's law) are

\[
\frac{1}{r}\frac{\partial}{\partial \theta} \left( r \frac{\partial u_\theta}{\partial \theta} \right) = \mu \frac{\partial^2 u_\theta}{\partial t^2} (u_\theta - u_\varphi) \\
\frac{1}{\sin \theta} \frac{\partial}{\partial \varphi} \left( \sin \theta \frac{\partial u_\varphi}{\partial \varphi} \right) = \mu_1 \frac{\partial^2 u_\varphi}{\partial t^2} (u_\varphi - u_\theta) \\
\frac{\partial u_r}{\partial r} = \mu_2 \frac{\partial^2 u_r}{\partial t^2} (u_r - u_\theta)
\]

where \(t\) denotes time. The constant \(\mu_1\) is related to the coefficient of viscosity of the fluid \(\mu\), the porosity \(\phi\) and Darcy's coefficient of permeability \(k\) in the directions of \(\theta\) or \(\varphi\) in the following way.

\[
\mu = \mu_1 = \mu_2 = \mu f^2.
\]

Similarly, if \(\mu_2\) denotes the coefficient of permeability in the \(\varphi\) direction we have

\[
\mu_2 = \mu f^2.
\]

Radial Deformation

When the body is deformed only in the radial direction the displacement components are

\[
u_r = u(r,t), \quad u_\theta = u_\varphi = 0, \quad u_\theta = u_\varphi = 0.
\]

and all the functions involved are independent of \(\theta\) and \(\varphi\). Thus from (4) and (5)
\[ e_{11} = \frac{2u_r}{\partial r} , \quad e_{22} = e_{\theta \theta} = \frac{u_t}{r} , \quad e_{r \theta} = e_{\theta r} = e_{\phi \phi} = 0 \]
\[ e = e_{rr} + 2e_{\theta \theta} , \quad e = \frac{2u_r}{\partial r} + \frac{2u_t}{\partial \theta} \]

\[ \sigma_r = (c-F) e_{rr} + Fe + q \epsilon \]
\[ \sigma_\theta = \sigma_\phi = (F - A - N) e_{rr} + (A + N) e + M \epsilon \]
\[ \sigma = (R - M) e_{rr} + Me + R \epsilon \]

\[ \sigma_{r \theta} = \sigma_{\theta r} = \sigma_{\phi \phi} = 0. \]  \hspace{1cm} (11)

The last two relations of (3) are identically satisfied, while the first relation becomes
\[ \frac{\partial}{\partial r} (\sigma_r + \sigma) + \frac{2}{\partial \theta} (\sigma_\theta - \sigma_\phi) = 0. \]

If the stress components are eliminated, the above relation with the help of (11) reduces to
\[ \frac{\partial e}{\partial r} + \frac{2\lambda}{\theta} e + \gamma_1 = 0 \]  \hspace{1cm} (12)

where
\[ (q + R) \gamma_1 = (c - F + Q - M) \frac{\partial e_{rr}}{\partial r} + 2(c - 2F + A) \frac{e_{rr}}{r} \]
\[ + (F + M) \frac{\partial e}{\partial \theta} + 2(F - A - N) \frac{e}{r}. \]  \hspace{1cm} (13)
\[ \lambda = \frac{Q - M}{Q + R}. \]
The relation (12) is written in the form.

\[ \frac{\partial}{\partial t} (r^{2\lambda} \varepsilon) = - \gamma_1 r^{2\lambda}. \]  

(14)

The first two equations of (6) are identically satisfied. Applying the operator \( \frac{\partial}{\partial t} + \frac{\partial}{\partial r} \) on both sides of the third equation of (6), it is obtained as

\[ \nabla^2 r = \lambda_2 \frac{\partial}{\partial t} (r \varepsilon - \varepsilon) , \quad \nabla^2 \varepsilon = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r}. \]  

(15)

Substitution of the value of \( \sigma \) from (11) in the above equation gives

\[ \nabla^2 \varepsilon + \gamma_2 = 2 \lambda \frac{\partial \varepsilon}{\partial t} \]  

(16)

where

\[ R \Psi_2 = (q - M) \nabla^2 \varepsilon + M \varepsilon + \lambda_2 \frac{\partial \varepsilon}{\partial t} \]

\[ R \gamma_2 = \lambda_2. \]  

(17)

Multiplying both sides of (16) by \( r^{2\lambda} \) and then differentiating with respect to \( r \varepsilon \) and using the relation (14), one gets

\[ \frac{\partial}{\partial t} (r^{2\lambda} \nabla^2 \varepsilon) = - \gamma_3 \]  

(18)

where

\[ \gamma_3 = \frac{\partial}{\partial t} (r^{2\lambda} \gamma_2) + \lambda_2 \gamma_2 r^{2\lambda} \frac{\partial \gamma_2}{\partial t}. \]  

(19)
From (12)

\[-\frac{\partial \epsilon}{\partial y^*} = \gamma_1 + \frac{2\lambda}{\gamma} \epsilon \]  

(20)

Differentiation with respect to $y^*$ gives

\[-\frac{\partial^2 \epsilon}{\partial y^*^2} = \frac{\partial \gamma_1}{\partial y^*} - 2\lambda \frac{\partial \gamma_1}{\partial y^*} - 2\lambda (2\lambda + 1) \frac{\epsilon}{\gamma^2} \]

Hence

\[-\gamma^2 \epsilon = \frac{\partial \gamma_1}{\partial y^*} - 2(\lambda - 1) \frac{\gamma_1}{\gamma^2} - 2\lambda (2\lambda - 1) \frac{\epsilon}{\gamma^2} \]  

(21)

Eliminating $\gamma^2 \epsilon$ between (18) and (21) one obtains

\[2\lambda (2\lambda - 1) \frac{\partial}{\partial y^*} \left[ \epsilon \gamma^{2(\lambda - 1)} \right] = \gamma_4 \]

(22)

where

\[\gamma_4 + \gamma_3 = \frac{\partial}{\partial y^*} \left[ \frac{\partial \gamma_1}{\partial y^*} \gamma^{2\lambda - 2(\lambda - 1)} \gamma_1 \gamma^{2\lambda - 1} \right] \]  

(23)

With the help of (20), the relation (22) can be written as

\[4\lambda (2\lambda - 1) \epsilon \gamma^{2\lambda - 3} = -\gamma_5 \]

(24)

where

\[\gamma_5 = \gamma_4 + 2\lambda (2\lambda - 1) \gamma_1 \gamma^{2(\lambda - 1)} \]  

(25)
Finally from (14) and (24)

\[ 4A(2\lambda-1) \psi_1 r^{2\lambda} = \frac{2}{2\psi_r} (r^2 \psi_5) \]  

(26)

The differential equation (26), when solved consistent with prescribed boundary and initial conditions, gives the displacement of the solid.

**Deduction for Isotropic Material**

For isotropic material,

\[ F = A, \quad C = A + 2N, \quad M = 0 \]

(27)

and consequently \( \lambda = 0 \)

The equation (26) is satisfied if \( \psi_r = 0 \).

(28)

There is no need of adding the constant of integration as is evident from (24). Equations (13), (17), (19), (23) and (25) reduce respectively to

\[ (Q + R) \psi_1 = (A + 2N + Q) \frac{2\psi}{2\psi_r} \]

\[ R \psi_2 = Q \frac{\psi_2}{\psi_r} + \lambda_3 \frac{\psi_3}{\psi_t} \]

\[ \psi_3 = \frac{\partial \psi_2}{\partial r} + \frac{\psi_4}{\psi_t} \]

\[ \psi_4 + \psi_3 = \frac{1}{3} \left( \frac{\partial \psi_2}{\partial r} + \frac{\partial \psi_4}{\partial r} \right) \]

\[ \psi_5 = \psi_4 \]  

(29)
From (28) and the last four relations of (29)
\[ R \frac{\partial}{\partial r} \left( \frac{3}{2} \dot{\gamma}^2 + \frac{2}{3} \dot{\gamma}^2 \right) - \alpha \frac{\partial}{\partial r} (\dot{\gamma}^2) = \lambda_2 \frac{\partial}{\partial t} \left( \dot{\gamma}^2 + \frac{2}{3} \dot{\gamma}^2 \right). \]  

(30)

Substituting the value of \( \dot{\gamma} \) from the first relation of (29) in (30) one gets
\[ \frac{\partial}{\partial r} (\dot{\gamma}^2) = \alpha \frac{\partial \epsilon}{\partial r, \alpha t}. \]  

(31)

where
\[ \alpha \left[ R (2 + 2N)^{-1/2} \right] = \lambda_2 \left( A + 2N + 2 \dot{\gamma} + R \right). \]  

(32)

It will be assumed that \( \alpha > 0 \).  

(33)

Equation (31) gives
\[ \dot{\gamma}^2 = \alpha \frac{\partial \epsilon}{\partial t} + a(t) \]  

(34)

where \( a(t) \) is a function of \( t \) and independent of \( r \).

With the help of the first relation of (29) and noting that \( \lambda = 0 \) in the isotropic case, the relation (12) is satisfied if
\[ (A + R) \dot{\epsilon} = -(A + 2N + 2) \dot{\epsilon}. \]  

(35)

Shell Under Internal and External Pressures

As an illustration of the foregoing theory, a thick spherical shell bounded by the internal surface \( r = r_1 \) and the external surface \( r = r_2 \) will be considered.
Let the shell be deformed by the sudden application of pressures at both the internal and external boundaries, through smooth porous sheaths. The system is assumed to be at rest and unstrained initially. The boundary and initial conditions are as follows:

\[
\sigma^+ = -N \beta_1 \text{ when } r = r_1, \quad t > 0
\]
\[
\sigma^- = -N \beta_2 \text{ when } r = r_2, \quad t > 0
\]
\[
\sigma_0 = 0 \text{ when } r = r_1 \text{ and } r = r_2, \quad t > 0
\]
\[
u = 0 \text{ when } r_1 \leq r \leq r_2, \quad t = 0.
\]  

The conditions (36) and (37) are prescribed solid and fluid stresses at the boundaries, while the condition (38) expresses the initial equilibrium of the whole system. The quantities \( \beta_1 \) and \( \beta_2 \) will be assumed as constants. Let the bar upon the function denote its Laplace Transform. Thus

\[
\tilde{u}(r, \xi) = \int_0^\infty u(r, t) \xi e^{-\xi t} dt.
\]  

If the condition (38) is used, Laplace Transform of both sides of (34) gives

\[
\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} + \xi^2 \right) \tilde{e} = \tilde{a}(\xi)
\]  

where

\[
\xi = \alpha \xi.
\]
A particular integral of (40) is
\[ \xi = \frac{a}{s^2}. \] (42)

The complementary function for (40) is obtained from the equation
\[ \left( \frac{d^2}{ds^2} + \frac{2}{s} - \frac{\lambda^2}{s^2} \right) \xi = 0. \] (43)

The substitution \( \xi = r^{-\frac{1}{2}} z \) (44)
reduce the above equation to
\[ \frac{d^2 z}{dr^2} + \frac{1}{r} \frac{dz}{dr} -(r^2 + \frac{\lambda^2}{4r^2}) z = 0, \] (45)

The solution of (45) is
\[ z = c_1 I_{\frac{\lambda}{2}} (\gamma r) + c_2 K_{\frac{\lambda}{2}} (\gamma r) \] (46)
where \( I_{\frac{\lambda}{2}} \) and \( K_{\frac{\lambda}{2}} \) are Modified Bessel functions of the first and the second kind respectively. These can be expressed in terms of elementary functions as
\[ I_{\frac{\lambda}{2}} (\gamma r) = \frac{2}{\sqrt{\pi r}} \sinh (\gamma r), \quad K_{\frac{\lambda}{2}} (\gamma r) = \frac{\pi}{\sqrt{2 \pi r}} e^{-\gamma r}. \] (47)

The quantities \( c_1 \) and \( c_2 \) are independent of \( r \), but may be functions of \( \gamma \). From (42), (44), (46) and (47) the general solution of (40) may be written as
\[ \bar{\varepsilon} = c_1 \sqrt{\frac{x}{r}} \sinh(\frac{\pi r}{x}) + c_2 \sqrt{\frac{x}{r}} e^{-\frac{\pi r}{x}} + \frac{a^2}{x^2} \quad (48) \]

We also have
\[ \bar{\varepsilon} = \frac{2}{3} \varepsilon \left( r^2 \bar{u} \right) \quad (49) \]

The elimination of \( \bar{\varepsilon} \) between (48) and (49) and then integration with respect to \( r \), yield
\[ \bar{u} = \frac{1}{3} \frac{ex}{x^2} + \frac{c_3}{x^2} \psi + \frac{\psi}{x^2} \left( \frac{2}{2+\gamma} \sinh \frac{\pi r}{x} - 2x \sinh \frac{\pi r}{x} \right) + 2 \psi \]
\[ + \frac{2}{3} \psi \left( \frac{2}{3} \sinh \frac{\pi r}{x} \right) - \frac{\sqrt{r}}{2x^2} \left( \frac{c_1}{2+\gamma} \sinh \frac{\pi r}{x} + 2x \sinh \frac{\pi r}{x} + 2 \right) e^{-\frac{\pi r}{x}} \quad (50) \]

where \( c_3 \) is defined like \( c_1 \) or \( c_2 \).

Differentiation of (50) with respect to \( r \) gives
\[ \bar{e}_{rr} = \frac{d}{dr} \left( \frac{2x}{2+\gamma} \right) \psi + \frac{2x}{2+\gamma} \left( \frac{2}{2+\gamma} \sinh \frac{\pi r}{x} + 2x \sinh \frac{\pi r}{x} \right) \]
\[ - \frac{2}{3} \left( 1 + \frac{2}{x^2} \right) \psi \]
\[ + \frac{2}{3} \left( 1 + \frac{2}{x^2} \right) \psi e^{-\frac{\pi r}{x}} \quad (51) \]

In the case of isotropy, the relations (11) with the help of (35) reduce to
\[ \sigma_r / N = 2 \bar{e}_{rr} + \beta_1 \bar{e} \quad , \quad \sigma_\theta / N = \sigma_\phi / N = - \bar{e}_{rr} + \beta_3 \bar{e} \quad (52) \]
\[ \sigma / q = - \beta_2 \bar{e} \quad , \quad \sigma / q = - \beta_2 \bar{e} \]
where
\[
\beta_1 = \frac{AR - A(2N + a)}{N(A + R)}
\]
\[
\beta_2 = \frac{R(A + 2N) - a^2}{a(A + R)}
\]
\[
\beta_3 = \frac{R(A + N) - a(N + a)}{N(A + R)}
\]

Laplace Transforms of (36) and (37) give
\[
\bar{\alpha} = -\frac{\bar{N}}{4}, \quad \alpha = \gamma_1
\]
\[
= -\frac{\bar{N}_2}{4}, \quad \alpha = \gamma_2.
\]
\[
\gamma = 0, \quad \alpha = \gamma_1, \quad \alpha = \gamma_2.
\]

Finally the Laplace Transform of the first and third equations of (52) with the boundary conditions (54) and (55) and the relations (43) and (51) give four equations to determine four unknowns \(\bar{\alpha}, c_1, c_2, c_3\) and \(\gamma\). These are
\[
\frac{\bar{\alpha}}{\bar{\psi}_2} + c_1 \bar{S}_1 + c_2 \bar{E}_i = 0, \quad i = 1, 2
\]
\[
-\frac{4c_3}{\bar{\psi}_1^3} + \frac{\bar{\alpha}}{\bar{\psi}_1^2} + c_1' (\bar{g}_i \bar{S}_1 - \bar{h}_i \bar{T}_i) + c_2' (\bar{g}_i + \bar{h}_i) \bar{E}_i = \frac{\bar{b}_i}{\bar{\psi}_1} = 0
\]
where
\[
c_1' = \sqrt{\frac{2}{\bar{\psi}_1}} c_1, \quad c_2' = \sqrt{\frac{1}{\bar{\psi}_1^2}} c_2, \quad \bar{\psi} = \frac{2}{3} + \beta_1
\]
\[
\bar{S}_1 = \sinh(\bar{\psi} \gamma), \quad \bar{T}_i = \cosh(\bar{\psi} \gamma), \quad \bar{E}_i = \bar{E}_i \gamma
\]
\[
\bar{g}_i = \beta_1 + 2(1 + \frac{4}{\bar{\psi}^2}), \quad \bar{h}_i = \frac{4}{\bar{\psi}^2} (1 + \frac{2}{\bar{\psi}^2}), \quad i = 1, 2.
\]
The coefficients being known from (56), the Inverse Laplace Transform of (50) gives
\[ u = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{L}^{-1} \left( \frac{\gamma t}{e^{-\gamma t} - 1} \right) dt. \] (58)

**Thin Shell**

When the thickness of the shell is small compared with the internal radius, the external radius may be taken as
\[ r_2 = r_1 (1 + m) \] (59)
where \( m \) is a small quantity. Neglecting the squares and higher powers of \( m \), one obtains from (57),
\[ S_2 = S_1 + m \gamma_1 T_1, \quad T_2 = T_1 + m \gamma_1 S_1, \quad E_2 = E_1 (1 - \gamma_1), \]
\[ g_2 = g_1 - \frac{16m}{3\gamma_1}, \quad m = \frac{4m}{3\gamma_1} \left( 1 - \frac{6}{\pi^2 \gamma_1} \right). \] (60)

Then relations (56) reduce to
\[ \frac{\alpha}{s^2} + c_1 s + c_2 E_1 = 0, \]
\[ \frac{\beta}{s^2} + c_1 (s + m \gamma_1 T_1) + c_2 (1 - m \gamma_1) E_1 = 0, \]
\[ -\frac{4c_3}{\gamma_1^2} + c_1 \left( g_2 s_1 - \gamma_1 T_1 \right) + c_2 \left( g_1 + g_1 \right) E_1 + \frac{\beta_1}{\epsilon} = 0, \] (61)
\[ -\frac{4c_3}{\gamma_1^2} \left( 1 - 2m \right) + \frac{4\alpha^2}{s^2} + c_1 \left[ (g_2 s_1 - \gamma_1 T_1) + m \gamma_1 \left( g_1 T_1 - g_1 s_1 \right) \right] + \frac{4c_3}{\gamma_1^2} \left[ (g_1 + g_1) \left( 1 - \gamma_1 \right) - \frac{16m}{3\gamma_1} \right] \]
\[ - \frac{4m}{3\gamma_1} \left( 1 + \frac{6}{\pi^2 \gamma_1} \right) E_1 \frac{\alpha}{s} + \frac{\beta_1}{\epsilon} = 0. \]
From the first two equations of (61), noting that
\[ T_1 + S_1 = \frac{1}{E_1}, \]
we get
\[ c_1^T_1 \neq c_2^T_1 E_1, \quad \frac{\sqrt{\xi}}{S_2} + \frac{c_1}{E_1} = 0. \]  \hspace{1cm} (62)

The third equation of (61) gives
\[ -\frac{4c_2}{\eta^3} + \frac{c_1}{E_1} (\varphi - \xi) + \frac{b_1}{\xi} = 0. \]  \hspace{1cm} (63)

Subtracting the third equation of (61) from the fourth, and using (62), one obtains
\[ \frac{12c_2}{\eta^3} - \frac{c_1}{E_1} \left( \eta f_1 - \frac{16}{5^2 \eta^2} \right) + \frac{b_2 - b_0}{\eta \xi} = 0. \]  \hspace{1cm} (64)

From equations (62)-(64), there follow the results
\[ c_1 = -E_1 \frac{\varphi}{\Delta}, \quad \varepsilon_1 = -T_1 \frac{\varphi}{\Delta}, \quad \varepsilon_2 = \frac{\varphi}{\Delta}, \quad \varepsilon_3 = \frac{\varphi}{\Delta}, \]  \hspace{1cm} (65)
\[ \frac{\varphi^3}{\Delta} = \frac{\eta^2}{4^2} - \frac{\varphi}{\Delta} (\varphi - \xi), \quad \Delta = \frac{\eta^2}{4^2} \left( \frac{\varphi}{\Delta} - \frac{\varphi}{\Delta} \right) + \frac{16}{3^2 \eta^2} \]
\[ b_2 = \frac{b_2 - b_0}{\eta \xi}, \quad \Delta = \frac{\eta^2}{4^2} \left( \frac{\varphi}{\Delta} - \frac{\varphi}{\Delta} \right) + \frac{16}{3^2 \eta^2} \]

Finally, using (57) in (65) and noting that \( \xi^2 = \varphi \),
we get
\[ c_1 = -\frac{1}{3^2} \alpha \eta^2 \varphi \xi^2 \xi^2, \quad \varepsilon_2 = -\frac{1}{3^2} \alpha \eta^2 \varphi ^2 \xi^2 \xi^2 \xi^2, \quad \varepsilon_3 = \frac{\alpha \eta^2}{4^2} \left( \frac{\varphi}{\Delta} - \frac{\varphi}{\Delta} \right), \]  \hspace{1cm} (66)
\[ \frac{\alpha}{\xi^2} = \frac{1}{3^2} \alpha \eta^2 \varphi \xi^2 \xi^2 \xi^2. \]
Substitution of the values of the coefficients from (66) in (50) gives

\[
\tilde{u} = \frac{1}{q_6} \alpha \eta^2 \phi_3 (\eta - \frac{\eta_2}{\eta_1}) + \frac{1}{q_4} \frac{\eta^3}{\xi^2} (\phi_1 - \frac{1}{2} \phi_3) \\
+ \frac{1}{16 \xi} \frac{\eta^2 + 2 \eta}{\eta} \sinh s (\eta - \eta_1) \\
+ \frac{1}{32} \alpha \eta^2 \phi_3 \left( \frac{1}{5} - \frac{2}{3 \xi^2} \right) \sinh s (\eta - \eta_1).
\]  

(67)

Since \( r \simeq r_1 (1 + m) \) we can put in (67),

\[
\cosh s (\eta - \eta_1) = 1, \quad \sinh s (\eta - \eta_1) = s (\eta - \eta_1).
\]  

(68)

Substituting the values from (68) in (67) and then taking the Inverse Laplace Transform, the displacement is obtained as

\[
u = \frac{1}{q_6} \alpha \eta^2 \phi_3 (3 \eta_1 - 2 \eta - \frac{\eta_2}{\eta_1}) \delta (t) \\
+ \frac{1}{q_4} \frac{\eta^3}{\xi^2} + 16 \frac{\phi_3 \eta_1}{\eta} - \frac{1}{16} \alpha \eta^2 \phi_3 \left( \frac{1}{5} - \frac{2}{3 \xi^2} \right) \delta (t),
\]  

(69)

where \( \delta (t) \) is Dirac delta function whose Laplace Transform is unity, and \( \delta (t) \) denotes Heaviside function. The part of the displacement represented by the coefficient of \( \delta (t) \) in the right side of (69) is the displacement produced just after the application of the load and this part relaxes immediately. The remaining part represents the steady state solution.
4.4. **Deformation of Transversely Isotropic Material**

The theory of deformation of an elastic material containing a fluid has been formulated for a transversely isotropic medium. The problem is reduced to the determination of two different stress functions satisfying the same differential equation. The stress distribution in a semi-infinite body due to symmetrical surface loads has been considered as an illustration. The results have been expressed in terms of complex integrals.

**Fundamental Equations**

In the deformation of a porous elastic material containing a fluid the stress tensor is [18]

\[
\begin{pmatrix}
\sigma_{xx} + \sigma & \sigma_{xy} & \sigma_{xz} \\
\sigma_{yx} & \sigma_{yy} + \sigma & \sigma_{yz} \\
\sigma_{zx} & \sigma_{zy} & \sigma_{zz} + \sigma
\end{pmatrix}
\]  

(1)

where \( \sigma \) is related as usual to the porosity \( f \) and the fluid pressure \( p \) by

\[
\sigma = -fp
\]

(2)

In the absence of the body forces the equations of equilibrium are

\[
\frac{\partial}{\partial x}(\sigma_{xx} + \sigma) + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0,
\]

If the medium possesses anisotropy of the transversely isotropic type about the \( z \) axis, the stress strain relations are

\[
\begin{align*}
\sigma_x &= 2N \epsilon_{xx} + A (\epsilon_{xx} + \epsilon_{yy}) + F \epsilon_{zz} + M \epsilon \\
\sigma_y &= 2N \epsilon_{yy} + A (\epsilon_{xx} + \epsilon_{yy}) + F \epsilon_{zz} + M \epsilon \\
\sigma_z &= C \epsilon_{zz} + F (\epsilon_{xx} + \epsilon_{yy}) + Q \epsilon \\
\sigma_{yz} &= L \epsilon_{yz}, \\
\sigma_{zx} &= L \epsilon_{zx}, \\
\sigma_{xy} &= L \epsilon_{xy},
\end{align*}
\]  

(3)

where \( N, A, F, M, C, Q, R \) and \( L \) are eight elastic constants.

We also have

\[
\begin{align*}
\epsilon_{xx} &= \frac{\partial u}{\partial x}, & \epsilon_{yy} &= \frac{\partial v}{\partial y}, & \epsilon_{zz} &= \frac{\partial w}{\partial z}, \\
\varepsilon &= \frac{\partial u}{\partial z} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x},
\end{align*}
\]  

(5)

where \( (u, v, w) \) and \( (U, V, W) \) are components of displacements of the solid and the fluid respectively. The equations of flow for the fluid in the porous solid (by Darcy's law) are
\[
\frac{\partial u}{\partial x} = k_1 \frac{\partial^2 u}{\partial t^2} (u - u), \quad \frac{\partial v}{\partial y} = k_1 \frac{\partial^2 v}{\partial t^2} (v - v), \quad \frac{\partial w}{\partial z} = k_2 \frac{\partial^2 w}{\partial t^2} (w - w),
\]
where \( t \) denotes time. The constant \( k \) is related to the coefficient of viscosity of the fluid \( \mu \), the porosity \( \phi \) and Darcy's coefficient of permeability \( k_1 \) in the directions of \( x \) and \( y \) as
\[
k_1 = \frac{\mu t^2}{\mu_1}.
\]

Similarly if \( k_2 \) denotes the coefficient of permeability in the \( z \) direction,
\[
k_2 = \frac{\mu t^2}{\mu_2}.
\]

### Displacement Functions

Let the displacement functions \( \Phi(x, y, z, t) \) and \( \Psi(x, y, z, t) \) satisfy the following relations:
\[
u = \frac{\partial \Phi}{\partial x}, \quad w = \frac{\partial \Phi}{\partial y}, \quad \omega = k_1 \frac{\partial \Phi}{\partial z} + k_2 \frac{\partial \Psi}{\partial z} + \gamma,
\]
where
\[
k_1 = \frac{p + \mu}{l^2 (\alpha + \gamma + l)}, \quad k_2 = \frac{l}{l^2 (\alpha + \gamma + l)},
\]
\[
p = A + 2N, \quad \alpha = \frac{\gamma \partial^2}{\partial x^2} + \frac{\gamma \partial^2}{\partial y^2}.
\]

With the help of (4), (5) and (9), the first two relations of (3) reduce to
\[ \frac{\partial}{\partial x} \left[ (F + L + \Phi) \frac{\partial \Phi}{\partial x} + (M + R) \Phi \right] = 0, \]
\[ \frac{\partial}{\partial y} \left[ (F + L + \Phi) \frac{\partial \Phi}{\partial y} + (M + R) \Phi \right] = 0. \]

Both these relations are satisfied if
\[ \Phi = -k_3 \frac{\partial \psi}{\partial z}, \]
with
\[ k_3 = \frac{F + L + \Phi}{M + R}. \]

The third relation of (3) then gives
\[ L \left[ \nabla^2 \Phi + \left\{ k_1 (c + \Phi) + k_2 (L - (F + L + M)) \right\} \nabla^2 \frac{\partial \Phi}{\partial z^2} + k_2 (c + \Phi) \frac{\partial \Phi}{\partial z} + (c + \Phi) - k_3 (q + R) \right] \frac{\partial \psi}{\partial z} = 0. \]

The above equation may be written in the form
\[ k_1 \left( \nabla^2 + \frac{1}{\gamma_1} \frac{\partial^2}{\partial \alpha^2} \right) \left( \nabla^2 + \frac{1}{\gamma_2} \frac{\partial^2}{\partial \beta^2} \right) \Phi + \left( \nabla^2 + \frac{1}{\gamma_3} \frac{\partial^2}{\partial \gamma^2} \right) \psi = 0, \]
where \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) are determined from the following equations
\[ k_1 \left( \frac{1}{\gamma_1} + \frac{1}{\gamma_2} \right) = \left\{ k_1 (c + \Phi) + k_2 (L - (F + L + M)) \right\} / L, \]
\[ k_1 L = k_2 (c + \Phi) \gamma_1 \gamma_2^2, \]
\[ \left\{ c + \Phi - k_3 (q + R) \right\} \gamma_3^2 = L. \]
The three equations in (6) may be written in the combined form as
\[
\frac{1}{x_1} \nabla_1^2 \sigma + \frac{1}{x_2} \frac{\partial^2 \sigma}{\partial x^2} = \frac{2}{\delta t} (\varepsilon - \varepsilon)
\]  
(16)

where
\[
\sigma = \frac{3u}{\partial x} + \frac{3v}{\partial y} + \frac{3w}{\partial z}.
\]  
(17)

Note that \(\sigma\) can be expressed in terms of \(\varphi\) and \(\psi\) by (4'), (5), (9) and (11) as
\[
\sigma = \frac{2}{\delta z} \left[ (q - \Delta K - \nabla_2^2 \varphi + K_2 \frac{\partial^2 \varphi}{\partial y^2} + (q - \Delta K) \psi \right].
\]  
(18)

Now if \(\sigma\), \(\varepsilon\), and \(\varepsilon\) are eliminated in terms of \(\varphi\) and \(\psi\) the equation (18) reduces to
\[
(\nabla_1^2 + \frac{1}{x_2} \frac{\partial^2}{\partial x^2}) \left[ (K_1 + - \nabla_2^2 \varphi + K_2 \frac{\partial^2 \varphi}{\partial y^2} + (q - \Delta K) \psi \right] = \frac{1}{\delta t} \left[ (1 - \Delta K) \nabla_2^2 \varphi - K_2 \frac{\partial^2 \varphi}{\partial y^2} - (1 + \Delta K) \psi \right].
\]  
(19)

The above relation may be written in the form
\[
(\nabla_1^2 + \frac{1}{x_0} \frac{\partial^2}{\partial x^2}) \left( \nabla_1^2 + \frac{1}{x_1} \frac{\partial^2}{\partial x^2} + \frac{1}{x_2} \frac{\partial^2}{\partial y^2} \right) \varphi + \varrho_1 \left( \nabla_1^2 + \frac{1}{x_0} \frac{\partial^2}{\partial x^2} + \frac{1}{x_2} \frac{\partial^2}{\partial y^2} \right) \frac{\partial \varphi}{\partial t} + \varrho_2 \left( \nabla_1^2 + \frac{1}{x_0} \frac{\partial^2}{\partial x^2} + \frac{1}{x_2} \frac{\partial^2}{\partial y^2} \right) \psi + \varrho_3 \frac{\partial \psi}{\partial t} = 0
\]  
(20)

where
\[
\frac{1}{\gamma_0^2} = \frac{\gamma_1}{\gamma_1}, \quad \frac{1}{\gamma_2^2} = \frac{\kappa_3}{\kappa_1 \gamma - m}, \quad \frac{1}{\gamma_3^2} = \frac{\kappa_2}{\kappa_1 - 1},
\]

\[
\gamma_1 = \frac{\gamma_1(\kappa_1 - 1)}{\kappa_1 \gamma - m}, \quad \gamma_2 = \frac{\kappa_3}{\kappa_1 \gamma - m}, \quad \gamma_3 = \frac{\gamma_1 (1 + \kappa_3)}{\kappa_1 \gamma - m}.
\]

In order to eliminate the function \( \psi \) between (14) and (20), we first differentiate the left hand side of (14) with respect to \( t \) and substitute in it the value of \( \frac{\partial \psi}{\partial t} \) as given by (20), and then in the expression thus obtained we put the value of \( \psi \) as given by (14). This gives the result

\[
(\nabla_1^2 + \frac{1}{\gamma_0^2} \frac{\partial^2}{\partial t^2}) \left\{ \left( \nabla_1^2 + \frac{1}{\gamma_2^2} \frac{\partial^2}{\partial z^2} \right) \left( \nabla_2^2 + \frac{1}{\gamma_3^2} \frac{\partial^2}{\partial z^2} \right) \right\} \varphi
\]

\[
- \kappa_1 \gamma_2 \left( \nabla_1^2 + \frac{1}{\gamma_2^2} \frac{\partial^2}{\partial z^2} \right) \left( \nabla_2^2 + \frac{1}{\gamma_3^2} \frac{\partial^2}{\partial z^2} \right) \varphi.
\]

\[
\left( \kappa_1 \gamma_3 \left( \nabla_1^2 + \frac{1}{\gamma_2^2} \frac{\partial^2}{\partial z^2} \right) \left( \nabla_2^2 + \frac{1}{\gamma_3^2} \frac{\partial^2}{\partial z^2} \right) \right) \varphi.
\]

Equation (22) may be written in the form

\[
\nabla_1^2 \nabla_2^2 \nabla_3^2 \varphi = \gamma_0 \nabla_{14}^2 \frac{\partial \varphi}{\partial t}.
\]

where

\[
\nabla_{14}^2 = \nabla_1^2 + \frac{1}{\delta_1 \delta_2} \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{\delta_1 \delta_2} \frac{\partial^2}{\partial z^2},
\]

\[
(\delta_1 = 0, 1, 2, 3, 4)
\]

\[
\delta_0 = \gamma_0.
\]
and \( \delta_i(\sigma = 1, 2, 3, 4) \) can be calculated from the equations
\[
(1 - \kappa_1) \left( \frac{\delta_1}{\delta_1} + \frac{\delta_2}{\delta_2} \right) = \frac{1}{\gamma_1} + \frac{1}{\gamma_2} - \kappa_1 \gamma_2 \left( \frac{1}{\gamma_1} + \frac{1}{\gamma_2} \right),
\]
\[
\frac{1 - \kappa_2}{\delta_1 \delta_2} = \frac{1}{\gamma_3} + \frac{1}{\gamma_4} - \kappa_2 \gamma_4 \frac{1}{\gamma_3}, \quad \alpha = \frac{\kappa_1 \gamma_3 - \kappa_2}{1 - \kappa_2},
\]
\[
(\kappa_2 \gamma_3 - \kappa_1) \left( \frac{\delta_2}{\delta_2} + \frac{\delta_3}{\delta_3} \right) = \kappa_2 \gamma_3 \left( \frac{1}{\gamma_3} + \frac{1}{\gamma_4} \right) - \gamma_1 \left( \frac{1}{\gamma_3} + \frac{1}{\gamma_4} \right),
\]
\[
\frac{\kappa_1 \gamma_3 - \gamma_1}{\delta_2 \delta_3} = \frac{\kappa_1 \gamma_3}{\gamma_3} - \frac{\gamma_1}{\gamma_3}.
\]

Again, the elimination of \( \varphi \) between (14) and (28) gives the differential equation for \( \psi \), viz.
\[
\nabla_{10} \nabla_{11} \nabla_{12} \psi = \alpha \nabla_{10} \nabla_{14} \frac{\partial \psi}{\partial t}. \tag{26}
\]

The functions \( \varphi \) and \( \psi \) are determined from (23) and (26) consistent with (13) and the given boundary and initial conditions. Then stresses and displacements can be calculated in terms of these functions.

The results for the isotropic material will be obtained by putting
\[
F = A, \quad C = P, \quad M = \delta, \quad L = N, \quad \lambda_1 = \lambda_2, \tag{27}
\]
which imply
\[ \gamma_i = \delta_i = 1, \quad (i = 0, 1, 2, 3, 4), \quad \alpha = \frac{\lambda_1 (P+q+R)}{PR-q^2} \]

\[ \begin{align*}
\kappa_1 &= \frac{P+q}{A+N+q} ,
\kappa_2 &= \frac{N}{A+N+q} ,
\kappa_3 &= \frac{A+N+q}{q+R} .
\end{align*} \]

\[ \begin{align*}
\nu_1 &= \frac{\lambda_1}{A+N+q} ,
\nu_2 &= \frac{(A+N+q) \{ Q^2 - R(A+N)^2 \}}{AN(q+R)} ,
\nu_3 &= \frac{\lambda_1 (A+N+q)(A+N+2q+R)}{AN(q+R)} .
\end{align*} \]  

Equation (23) simplifies, in this case, to

\[ \nabla^6 \phi = \beta \nabla^4 \frac{\partial \phi}{\partial t} \]  

where

\[ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]

\[ \beta = \frac{\lambda_1 (P+q+R)}{PR-q^2} . \]  

Stress Components

The stress components can now be expressed in terms of the functions \( \phi \) and \( \psi \). Thus, from (4) with the help of (5), (6) and (11), one gets,

\[ \sigma_x = \frac{3}{2z} \left[ (k_1 F - P) \frac{\partial \phi}{\partial x} + (k_1 F - A) \frac{\partial \psi}{\partial y} + r_2 F \frac{\partial \phi}{\partial z} + (F - \kappa_3 M) \psi \right] . \]

\[ \sigma_y = \frac{3}{2z} \left[ (k_1 F - A) \frac{\partial \phi}{\partial z} + (k_1 F - P) \frac{\partial \psi}{\partial y} + r_2 F \frac{\partial \phi}{\partial z} + (F - \kappa_3 M) \psi \right] . \]

\[ \sigma_{xy} = -N \frac{\partial \phi}{\partial x \partial y} . \]  

\[ \sigma_{xz} = \sigma_{yz} = 0 . \]
The component $\sigma$ is given by the relation (18).

**Half-Space Under Given Normal Loads**

As an example we consider the half space bounded by the plane $z = 0$. The positive direction of the $z$ axis is taken into the half-space. Let a distributed load of constant magnitude $\gamma$ per unit area be applied on the surface through a porous, smooth rectangular slab whose sides are given by $x = \pm a$, $y = \pm b$. It will be assumed that the system is at rest and unstrained initially, and the load is applied suddenly. The boundary and initial conditions are then as follows:

\[
\sigma_x = -\gamma, \quad \sigma_y = -\gamma, \quad z = 0, \quad -a < x < a, \quad -b < y < b, \quad t > 0 \quad (33)
\]

\[
\sigma_{yz} = \sigma_{zx} = 0, \quad z = 0, \quad |x| > a, \quad |y| > b, \quad t > 0 \quad (34)
\]

\[
\sigma_z = 0, \quad z = 0, \quad |x| > 0, \quad |y| > 0, \quad t > 0 \quad (35)
\]
Conditions (33) - (35) are the prescribed solid and fluid stresses on the boundary. The condition (36) expresses the initial equilibrium of the whole system while the condition (37) shows that the loading process has no effect at a great distance.

**Transform Solution**

Let the prime denote the Laplace transform of the function. Thus,

\[ \varphi'(x,y,z,t) = \int_0^\infty \varphi(x,y,z,t) e^{-zt} \, dt, \]

\[ \psi'(x,y,z,t) = \int_0^\infty \psi(x,y,z,t) e^{-zt} \, dt, \]

(38)

Since the loading is symmetrical with respect to the axes of \( x \) and \( y \), the functions \( \varphi' \) and \( \psi' \) are assumed in the form of Fourier's double cosine integrals

\[ \varphi' = \int_0^{\infty} \int_0^{\infty} \varphi'(z,\xi) \cos(\xi x) \cos(\eta y) \, d\xi \, d\eta, \]

\[ \psi' = \int_0^{\infty} \int_0^{\infty} \psi'(z,\xi) \cos(\xi x) \cos(\eta y) \, d\xi \, d\eta, \]

(39)
where $\bar{\varphi}'$ and $\varphi'$ function of $z$ and $\tau$. All the integrals are supposed to be convergent. Using the condition (36) and the definition (38), the equation (23) is written as

$$\int_{\nu_1}^{\nu_2} \int_{\nu_3}^{\nu_4} \varphi' (x, y, z, \tau) - \alpha \tau \int_{\nu_5}^{\nu_6} \int_{\nu_7}^{\nu_8} \varphi' (x, y, z, \tau) = 0.$$  \hspace{1cm} (40)

The above relation with the help of $\varphi'$ given by (39) is satisfied if

$$\left[ (\frac{\partial^2}{\partial z^2} - v^2) (\frac{\partial^2}{\partial \tau^2} - v^2) - \alpha \tau (\frac{\partial^2}{\partial z^2} - v^2) (\frac{\partial^2}{\partial \tau^2} - v^2) \right] \varphi' (z, \tau) = 0,$$  \hspace{1cm} (41)

where $\gamma = \xi + \eta^2$, $\delta \equiv \frac{3}{\beta^2}$.

The auxiliary equation corresponding to the ordinary differential equation (41) in $x$ is a cubic in $m^2$, viz.

$$\left( \frac{m^2 - v^2}{\delta^2} \right) \left( \frac{m^2 - v^2}{\delta^2} \right) - \alpha \tau \left( \frac{m^2 - v^2}{\delta^2} \right) \left( \frac{m^2 - v^2}{\delta^2} \right) = 0.$$  \hspace{1cm} (42)

The roots of (42) can be found by the method of Cardan. The solution of (41) depends on the nature of roots (real or complex conjugates). It will be assumed that the cubic in $m^2$ has the real positive roots

$$m = m_i^\nu, \quad (i = 1, 2, 3)$$  \hspace{1cm} (43)
Then the solution of (41) consistent with the boundary condition (37) is given by

$$\nabla' = \sum_{i=1}^{3} B_i e^{-m_i z}$$

(44)

where $B_i (i = 1, 2, 3)$ is independent of $u$ but may be functions of $u$ and $z$. Similarly from (26) and (37) we obtain

$$\nabla' = \sum_{i=1}^{3} E_i e^{-m_i z}$$

(45)

The values of $\nabla'$ and $\nabla'$ as given by (44) and (45) will satisfy (14), provided,

$$\sum_{i=1}^{3} \left[ k_i B_i \left( \frac{m_i^2}{\gamma_i^2} - \gamma_i^2 \right) \left( \frac{m_i^2}{\gamma_i^2} - \gamma_i^2 \right) + E_i \left( \frac{m_i^2}{\gamma_i^2} - \gamma_i^2 \right) \right] e^{-m_i z} = 0.$$  

(46)

Since the relation (46) holds for all values of $z$, the coefficients of $\exp(-m_i z)$ must separately vanish. Hence

$$k_i B_i \left( \frac{m_i^2}{\gamma_i^2} - \gamma_i^2 \right) \left( \frac{m_i^2}{\gamma_i^2} - \gamma_i^2 \right) + E_i \left( \frac{m_i^2}{\gamma_i^2} - \gamma_i^2 \right) = 0,$$

$$i = 1, 2, 3.$$  

(47)

Both the conditions of (34) are satisfied by taking

$$\sum_{i=1}^{3} \left[ E_i \left\{ (\kappa_i - \gamma_i^2) m_i^2 - \kappa_i \gamma_i^2 \right\} + E_i \right] = 0.$$  

(48)
The condition (35) gives

\[ \sum_{i=1}^{3} \left( B_i \left( \frac{m_i^c}{y_i^2} - \xi^2 \right) + n_i E_i \right) = 0. \]  

(49)

Now the Laplace transform of (33) is

\[ \sigma_1' = -\frac{n}{E} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\sin a x \sin \eta_1}{z} \cos x \cos \eta_1 \, dx \, d\eta_1 \quad z = 0. \]  

(50)

By Fourier Integral Theorem, the above condition can be expressed as

\[ \sigma_1' = -\frac{n}{E} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\sin a x \sin \eta_1}{z} \cos x \cos \eta_1 \, dx \, d\eta_1. \]

From the first relation of (32) it is seen that (50) will be satisfied if

\[ \sum_{i=1}^{3} \left[ B_i \left( \frac{m_i^c}{y_i^2} - \xi^2 \right) - k_i C \frac{m_i^c}{y_i^2} - (c - k_i \xi) E_i \frac{m_i}{y_i^2} \right] = -\frac{n}{E} \int_{-\infty}^{\infty} \frac{\sin a x \sin \eta_1}{z} \cos x \cos \eta_1 \, dx \, d\eta_1. \]  

(51)

We write (47) in the form

\[ E_i = -k_i A_i B_i, \quad i = 1, 2, 3 \]  

(52)

where \( A_i \) is given by

\[ \left( \frac{m_i^c}{y_i^2} - \xi^2 \right) A_i = \left( \frac{m_i^c}{y_i^2} - \xi^2 \right) \left( \frac{m_i^c}{y_i^2} - \xi^2 \right). \]  

(53)
Then from (48), (49) and (51) it is obtained that
\[ \sum_{i=1}^{3} \varepsilon_i B_i = 0, \quad \sum_{i=1}^{3} \varepsilon_i' B_i = 0, \quad \sum_{i=1}^{3} \varepsilon_i'' B_i = -X, \]  
(54)

where
\[ \varepsilon_i = (\varepsilon_i - 1) m_i^2 - \varepsilon_i', \quad \varepsilon_i' = \frac{m_i^2}{\lambda_i} - \varepsilon_i - \varepsilon_i'' \lambda_i, \]
(55)
\[ \varepsilon_i'' = m_i \left[ \gamma (\varepsilon_i - 1) - \frac{1}{2} \varepsilon_i' \right] + \varepsilon_i' (\varepsilon_i - \varepsilon_i'' \lambda_i), \]
(56)
\[ X = \frac{4q}{\pi \gamma} \sin \frac{\pi \theta}{6 \gamma}, \]
(57)
\[ \sum_{i=1}^{3} \left[ \varepsilon_i'' \left( \varepsilon_i'_{i+1} - \varepsilon_i'_{i+2} \right) \right] = -\frac{X}{p}. \]
(58)

Equations in (54) can be solved for \( B_i \) to give
\[ B_i = \frac{1}{p} \left( \varepsilon_i \varepsilon_{i+1}^1 - \varepsilon_i \varepsilon_{i+2}^1 \right), \]
(59)

where \( p \) is calculated from the equation
\[ \sum_{i=1}^{3} \left[ \varepsilon_i'' \left( \varepsilon_i'_{i+1} - \varepsilon_i'_{i+2} \right) \right] = -\frac{X}{p}. \]
(60)

The transformed normal stress \( \sigma_z^2 \) is then obtained from (32) as
The Inverse Laplace transform of (61) gives

\[ \sigma_2' = \frac{1}{2\pi i} \int_{t-i\infty}^{t+i\infty} \sigma_2(t) e^{+t} dt. \]

Special Type of Transverse Isotropy

Let it be assumed that the elastic constants satisfy the relations

\[ \delta_1 = \delta_3, \quad \delta_2 = \delta_4, \quad \gamma_2 = \gamma_3. \]  

(62)

In spite of these restrictions imposed, the deviation from isotropy is quite appreciable.

However, the equation (42) in this case gives the roots

\[ m_i = \delta_i \gamma_i, \quad (i = 1, 2), \]

\[ m_3 = \delta_0 \sqrt{\gamma_2^2 + \alpha^2}. \]  

(63)

Consequently from (63)

\[ \lambda_i = \frac{\delta_i \gamma_i^2}{\gamma_i^2 - 1}, \quad \lambda_i = \frac{\delta_i \gamma_i^2}{\gamma_i^2 - 1}, \quad i = 1, 2, \]

\[ \lambda_3 = \left( \frac{\delta_3 \gamma_2^2}{\gamma_2^2 - 1} \right) \gamma_2^2 + \frac{\delta_3 \gamma_2^2}{\gamma_2^2} \alpha \gamma_i. \]  

(64)
Also from (55), (56) and (57),

\[ \epsilon_i = q_i s^2, \quad q_i = \delta_i \left( \frac{v_i}{\epsilon_{i+1}} - 1 \right), \quad i = 1, 2, \]

\[ \epsilon_3 = \delta_0 \left( \frac{v_3}{\epsilon_{3+1}} - 1 \right) \left( s^2 + \kappa \right), \]

\[ \epsilon_i = \delta_i \frac{s^2}{\epsilon_i^2} - 1 - \kappa_i \lambda_i, \quad i = 1, 2, \]

\[ \epsilon_3 = \left[ \frac{\delta_0}{\epsilon_3} - 1 - \kappa_3 \lambda_3 \right] s^2 + \left( \frac{\delta_0}{\epsilon_3} - \kappa_3 \right) \kappa \left( \frac{s^2}{\epsilon_3^2} - 1 \right). \]

\[ \epsilon_i^n = \lambda_i s^2, \quad \lambda_i = \delta \left\{ \left( \lambda_i C - F - \lambda_i C \delta_i^2 \right) + \lambda_i \left( C - \lambda_i \right) \lambda_i' \right\}, \quad (i = 1, 2) \]

\[ \epsilon_3^n = \delta_0 \left( s^2 + \kappa \right) \frac{1}{2} \left[ \left\{ \lambda_3 C - F - \lambda_3 C \delta_3^2 \right\} + \lambda_3 \left( C - \lambda_3 \right) \left( \frac{\delta_3}{\epsilon_3} - 1 \right) \right] s^2 + \left\{ \lambda_i \left( C - \lambda_i \right) \frac{\delta_i}{\epsilon_i^2} - \lambda_i C \delta_i^2 \right\} \kappa \right\} \kappa. \]

Finally equation (60) gives,

\[ \phi = -x / \left( \frac{\epsilon^4 \Delta}{} \right) \]

where

\[ \Delta = s \left( \alpha_1 s^2 + \alpha_2 \kappa \right) + \delta_0 \left( s^2 + \kappa \right) \frac{1}{2} \left( \delta_3 \delta_3^2 + \lambda_3 \kappa \right). \]
and \( s_i \) \((i = 1, 2, 3, 4)\) are defined as follows.

\[
\begin{align*}
\sigma_0 &= \delta_0 \left( g_1 h_2 - h_1 g_2 \right), \\
\lambda_1 &= \delta_0 \left( \frac{g_2}{\gamma_1} - 1 \right) \left( h_1 l_2 - h_2 l_1 \right) + \left\{ \frac{\delta_0}{\gamma_1} \frac{g_2}{\gamma_1} - 1 - k_1 v_2 \left( \frac{\delta_0}{\gamma_1} - 1 \right) \right\} \left( l_1 g_2 - l_2 g_1 \right), \\
\lambda_2 &= \delta_0 \left( \frac{g_2}{\gamma_1} - 1 \right) \left( h_1 l_2 - h_2 l_1 \right) + \left\{ \frac{\delta_0}{\gamma_1} - k_1 v_2 \frac{\delta_0}{\gamma_1} \right\} \left( l_1 g_2 - l_2 g_1 \right), \\
\lambda_3 &= \left( c - \frac{\delta_0}{\gamma_1} \right) \lambda_3 - k_1 \left( c - \frac{\delta_0}{\gamma_1} \right) \lambda_3 - k_1 \lambda_3 \lambda_2, \\
\lambda_4 &= k_1 \left( c - \frac{\delta_0}{\gamma_1} \right) - k_1 \lambda_3 \lambda_2.
\end{align*}
\]

Hence equation (61) reduces to

\[
\begin{align*}
\mathcal{E}_2 = - &\frac{\gamma_1}{\eta_1 \gamma_2 \eta_3} \int_0^\infty \int_0^\infty \left[ l_1 \left( \frac{g_2}{\gamma_1} + h_1 \frac{\gamma_1}{\gamma_2} \right) + e^{\delta_0 z} \\
&- l_2 \left( \frac{g_2}{\gamma_1} + h_1 \frac{\gamma_1}{\gamma_2} \right) \right] e^{-\delta_0 z} \\
+ &\sigma_0 \left( \frac{g_2}{\gamma_1} + h_1 \frac{\gamma_1}{\gamma_2} \right) \left( \frac{g_2}{\gamma_1} + h_1 \frac{\gamma_1}{\gamma_2} \right) \exp \left\{ - \sigma_0 \left( \frac{g_2}{\gamma_1} + h_1 \frac{\gamma_1}{\gamma_2} \right) \right\}
\end{align*}
\]

(70)

where

\[
\begin{align*}
\mathcal{G}_i &= \frac{g_i}{\gamma_1} \left( \frac{g_2}{\gamma_1} - 1 - k_1 v_2 \left( \frac{\delta_0}{\gamma_1} - 1 \right) \right) - k_1 \delta_0 \left( \frac{g_2}{\gamma_1} - 1 \right), \\
\mathcal{H}_i &= \frac{g_i}{\gamma_1} \left( \frac{g_2}{\gamma_1} - k_1 v_2 \delta_0 \right) - k_1 \delta_0 \left( \frac{g_2}{\gamma_1} - 1 \right),
\end{align*}
\]

(71)

\( i = 1, 2 \).
The Inverse Laplace transform of (61) or (70) may be obtained for small values of time by the method of expansion as given in chapter III. Then the double integrals with respect to the space coordinates can be evaluated by the numerical methods.

4.5. Deformation of a Porous Visco-Elastic Body Containing a Fluid Under Steady Pressures

The plain strain deformation of a porous visco-elastic body containing a fluid in the shape of a circular cylinder has been found when steady pressures are applied on the boundary through a perfectly previous sheath. The boundary displacement has been evaluated by an approximate method.

Introduction

The previous sections of this chapter deal with problems in the theory of a porous material containing a viscous fluid when the solid skeleton behaves as a perfectly elastic body. The present section is concerned with a problem when the solid framework of the porous body is assumed to respond as linear visco-elastic material. The generalisation of the idea of elasticity to linear visco-elasticity is usually carried out by a mathematical device which consists in replacing the elastic constants by operators which are functions of the time derivatives. It is to be remembered however that the operators may not always...
be simply linear differential ones. The interpretation of these operators are then advanced with the help of a model made up of elastic springs and dash-pots by their different combinations. It is gratifying however to note that there exists extensive literature \([25,26,27,28]\) where the general propositions of linearly viscoelastic non-porous body have been rigorously deduced by using the principles of non-reversible thermodynamics. These propositions are then generalised from non-porous to porous body in the same way as from non-porous to porous elastic body \([22]\). The theory of such a fluid-saturated visco-elastic porous body is applicable to a variety of problems, for example, those of creep at high temperature in a porous wall cooling, stresses in a dam, flow of oil or water in petroleum reservoirs, settlement and consolidation of clay in foundations, seepage through a plastic porous medium and so on. In the present section the above theory has been applied to determine the deformation in a long circular cylindrical column when a steady pressure is applied on the surface through a perfectly pervious sheath so that the fluid pressure on the boundary surface is always zero \([23]\).

Fundamental Equations

As before the stress tensor in cylindrical coordinates is defined as

\[
\begin{pmatrix}
\sigma_r + \sigma & \sigma_\theta & \sigma_r \gamma \\
\sigma_\theta r & \sigma_\theta + \sigma & \sigma_\theta \gamma \\
\sigma_r r & \sigma_\theta \gamma & \sigma_\gamma + \sigma
\end{pmatrix}
\]  

(1)
The strain components for the solid are defined as
\( \varepsilon_{yy} = \frac{\partial u_y}{\partial y} \) etc. where \((u_x, u_y, u_z)\) represent the displacement vector of the solid. The only significant strain component for the fluid is
\[
\varepsilon = \frac{1}{\rho} \frac{\partial (u r_y)}{\partial y} + \frac{1}{\rho} \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \tag{2}
\]
with \( \varepsilon_{yy} = \frac{\partial u_y}{\partial y} \) etc. and \((u_r, u_\theta, u_z)\) representing the displacement vector for the fluid. The operational stress-strain relations for the visco-elastic isotropic body are \([22]\)
\[
\begin{align*}
\sigma_r &= 2 N e_{rr} + A e + \epsilon e \\
\sigma_\theta &= 2 N e_{\theta\theta} + A e + \epsilon e \\
\sigma_z &= 2 N e_{zz} + A e + \epsilon e \\
\sigma_r &= N e_{rr} \\
\sigma_\theta &= N e_{\theta\theta} \\
\sigma_z &= N e_{zz} \\
\sigma &= \epsilon e + R e
\end{align*} \tag{3}
\]
with
\[
\epsilon = \frac{1}{\rho} \frac{\partial (u r_y)}{\partial y} + \frac{1}{\rho} \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \tag{4}
\]
If we consider the simplified case where there are only two relaxation constants, \(A\) and \(B\), and the solid is elastic for rapid deformation while it contains a viscous term for slow deformation, then the operators \(N, A, B\) and \(R\) are defined as
The variable \( t \) denotes time. The constants \( N_1, N_2, A_1, A_2, R_1, \) and \( R_2 \) are characteristic of the material. If the strain is symmetrical about the \( z \) axis
\[
\omega_z = U_\theta = 0 \tag{6}
\]
and the remaining displacement components are independent of \( \theta \). If it is further assumed that the strain is the same in all planes perpendicular to the \( z \) axis the non-vanishing displacement components are independent of \( z \). Thus
\[
\omega_x = u(x, t), \quad U_x = U(y, t), \quad U_z = U_2 = 0. \tag{7}
\]

[As in elasticity a uniform extension may be assumed in the \( z \) direction, in which case \( U_2 = z \lambda(t) \), \( U_z = z \lambda(t) \).]

The strain components reduce to
If the body force be neglected the nonvanishing equation of equilibrium is
\[ \frac{\partial (\sigma_{xx} + \sigma_{yy})}{\partial x} + \frac{1}{\mu} (\sigma_{xx} - \sigma_{yy}) = 0. \] (9)

The nonvanishing flow resistance equation given by Darcy's law is
\[ \frac{\partial \sigma_{xx}}{\partial x} = \lambda \cdot D (U - u), \] (10)
where \( \lambda \) is a permeability coefficient of the material. Thus the fundamental equations are (3), (8), (9) and (10) which are to be solved by using given boundary and initial conditions.

**Reduction of the Fundamental Equations**

Substitution for \( \sigma_{xx}, \sigma_{yy} \) and \( \sigma \) in terms of \( u \) and \( v \) reduces (9) and (10) to
\[ (2N + A + B) \nabla u + (Q + R) \nabla^2 u = 0, \] (11)
\[ Q \nabla^2 u + R \nabla^2 u = \lambda D (U - u), \] (12)
Let $\bar{u}$ and $\bar{v}$ denote the Laplace transform of $u$ and $v$ respectively, so that

$$
\bar{u} = \int_0^\infty u e^{-pt} \, dt, \quad \bar{v} = \int_0^\infty v e^{-pt} \, dt.
$$

(14)

Let the system be stressed by sudden application of forces so that initially all the unknown quantities are zero. Consequently,

$$
\int_0^\infty N_u e^{-pt} \, dt = N_3 \bar{u},
$$

(15)

where

$$
N_3 = \frac{N_1}{p + \lambda} + N_2.
$$

(16)

Similar results hold for other operators. It is seen from relations (3), (10), (11) and (12) that the forms of these remain the same when $u$ and $v$ are replaced by their Laplace Transforms $\bar{u}$ and $\bar{v}$ provided $\bar{u}$ is replaced by $\bar{b}$. in $N, A, \varepsilon$, etc. The relations (11) and (12) transform to

$$
(2 N_3 + A_3 + \Theta_3) v \bar{u} + (A_3 + R_3) v^2 \bar{u} = 0,
$$

$$
\Theta_3 \bar{v} \bar{u} + R_3 \bar{v} = \lambda \beta (\bar{u} - \bar{v}),
$$
which give

\[ \nabla_1^4 \bar{u} + L^2 \nabla_1^2 \bar{u} = 0, \]  

\[ \bar{u} = \bar{u} + M^2 \nabla_1^2 \bar{u}, \]  

where

\[ L^2 = 4 \left( 2N_2 + 2A_2 + A_3 + R_3 \right) \left[ \left( \frac{A_3}{2N_3} - \frac{R_3}{2N_3 + A_3} \right)^2 \right], \]  

\[ M^2 = \frac{A_3}{2N_3} - \frac{R_3}{2N_3 + A_3} \left[ \left( \frac{A_3}{2N_3} - \frac{R_3}{2N_3 + A_3} \right)^2 \right]. \]

The relation (17) shows that \( \bar{u} \) can be written as

\[ \bar{u} = \phi_1 + \phi_2, \]  

and \( \phi_1, \phi_2 \) satisfy the equations

\[ \nabla_1^2 \phi_1 = 0, \quad \nabla_1^2 \phi_2 + L^2 \phi_2 = 0. \]

From (18)

\[ \bar{u} = \phi_1 + \phi_2 - L^2 M^2 \phi_2. \]

Solution of the Problem

For a complete cylinder the solution of (22) may be taken as

\[ \phi_1 = \alpha_1 r, \quad \phi_2 = \alpha_2 J_1 (lr), \]

\[ \alpha_1, \alpha_2, r, l, \]
and \( \alpha_1 \) and \( \alpha_2 \) being integration constants, and \( J_1 \) denoting Bessel function of order one. From (8), (21) and (23) one gets

\[
\bar{\alpha} = 2\alpha_1 + \alpha_2 L J_0(Lr) , \\
\bar{\epsilon} = 2\alpha_1 + (1 - M^2 L^2) \alpha_2 L J_0(Lr) ,
\]

using the recurrence formula

\[
J_n'(r) = J_{n-1}(r) - \frac{n}{r} J_n(r) .
\]

From (3) and (25),

\[
\overline{\sigma_\nu} = 2 \alpha_1 (N_3 + A_3 + A_3) - L \alpha_2 \left[ \frac{2 N_3}{L^2 r} J_1(Lr) + \lambda \beta M^2 J_0(Lr) \right] ,
\]

\[
\overline{\sigma} = 2 \alpha_1 (A_3 + R_3) + \lambda \beta M^2 \alpha_2 \epsilon J_0(Lr) .
\]

Let

\[
\overline{\sigma_\nu} = -\frac{\rho_0}{\nu} \quad \text{(a constant) on } \nu = a, \quad 0 < t < \infty ,
\]

\[
\overline{\sigma} = 0 \quad \text{on } \nu = a, \quad 0 < t < \infty .
\]

Then

\[
(\overline{\sigma_\nu})_{\nu = a} = -\frac{\rho_0}{\rho} , \quad (\overline{\sigma})_{\nu = a} = 0 .
\]

From (26) and (27),
\[ a_1 = -\left(\frac{\omega \rho \sigma}{2}\right) M^2 L J_0(La) \left\{ \frac{2 N_3(a_3 + R_3)}{2 N_3 (a_3 + R_3) J_1(La)} \right\} \]
\[ + \frac{\omega \rho M^2 L (N_3 + A_3 + 2 A_3 + R_3)}{2 \left( N_3 + A_3 + 2 A_3 + R_3 \right) J_0(La)} \]  
\[ \alpha_2 = \frac{(\omega \rho \sigma)}{(a_3 + R_3)} \left\{ \frac{2 N_3(a_3 + R_3)}{2 N_3 (a_3 + R_3) J_1(La)} \right\} \]
\[ + \frac{\omega \rho M^2 L (N_3 + A_3 + 2 A_3 + R_3)}{2 \left( N_3 + A_3 + 2 A_3 + R_3 \right) J_0(La)} \]

Thus \( a_1 \) and \( \alpha_2 \) being determined, the inverse Laplace transform of (21) gives

\[ u = \int \left\{ a_1 r + \alpha_2 J_1(La) \right\} e^{t \sigma} dt \]

Approximate Evaluation of the Boundary Displacement

On the boundary \( r = a \), we have

\[ (\mathcal{U})_{r=a} = -\left(\frac{\omega \rho (a_3 + R_3)}{2A} \right)^2 \left\{ \frac{\omega \rho M^2 L J_0(La) - 2 \left( A_3 + R_3 \right) J_1(La)}{2 N_3 \left( A_3 + R_3 \right) J_0(La) + 2 N_3 \left( A_3 + R_3 \right) J_1(La)} \right\} \]

The ratio of Bessel functions of two consecutive orders can be expanded by a continued fraction in the form

\[ \frac{J_{n+1}(z)}{J_n(z)} = \frac{(z/\nu)^2}{n+2} - \frac{(z/\nu)^3}{n+3} \]

[24]
As a first approximation, retaining only the first convergent, one gets

\[
\left\{ \frac{J_1(la)}{J_0(la)} \right\} \approx \frac{1}{l} a.
\]

(33)

Now (31) reduces to

\[
(u)_{r=a} = \frac{a \beta_0}{2\pi} \frac{R_3}{R_3(N_3 + A_3) - \alpha_3^2}.
\]

Substitution of the values of \( R_3, N_3, A_3 \) and \( \alpha_3 \) gives

\[
(u)_{r=a} = -\frac{1}{2} a \beta_0 \left( \frac{a_1 b^2 + a_2 b^2 + a_3 b + a_4}{b (A_1 b^2 + A_2 b^2 + A_3 b + A_4)} \right)
\]

(34)

where

\[
a_1 = R_1 + R_2, \quad a_2 = \sqrt{R_2 + (\gamma + \lambda)(R_1 + R_2)},
\]

\[
a_3 = \sqrt{A_1 + A_2 + N_1 + N_2} - \frac{A_1 + A_2}{A_1 + A_2 + N_1 + N_2}, \quad a_4 = \sqrt{R_2},
\]

\[
\begin{align*}
\lambda_1 &= \frac{1}{2} \left( R_1 + R_2 \right) \left( A_1 + A_2 + N_1 + N_2 \right) - (A_1 + A_2)^2, \\
\lambda_2 &= \frac{1}{2} \left( R_1 + R_2 \right) \left( A_1 + A_2 + N_1 + N_2 \right) - (A_1 + A_2)^2 + \frac{1}{2} \left( R_1 + R_2 \right) \left( A_1 + A_2 + N_1 + N_2 \right) - (A_1 + A_2)^2, \\
\lambda_3 &= \frac{1}{2} \left( R_2 \left( A_1 + A_2 + N_1 + N_2 \right) - \theta_2 \right)^2 + \frac{\nu}{2} \left( N_0 R_2 + (A_2 + \theta_2)(R_1 + R_2) \right)^2 - 2 \theta_2 \left( A_1 + A_2 + N_1 + N_2 \right) - \theta_2 \right)^2, \\
\lambda_4 &= \frac{1}{2} \left( R_1 + (A_2 + N_2) - \theta_2 \right)^2.
\end{align*}
\]
If \(-\lambda_1, -\lambda_2, -\lambda_3\) are the roots of the equation

\[ a_1 p^3 + a_2 p^2 + a_3 p + a_4 = 0 \]  

the relation (34) may be put in the partial fraction form

\[ (\bar{u})_{x=\lambda} = \frac{a_0 a_4}{a_1} \left[ \frac{B_0}{p} + \frac{B_1}{p + \lambda_1} + \frac{B_2}{p + \lambda_2} + \frac{B_3}{p + \lambda_3} \right] \]  

where

\[ B_0 = \frac{a_0}{\lambda_1 \lambda_2 \lambda_3} \]
\[ B_1 = \frac{a_1 \lambda_1^2 - a_2 \lambda_1^2 + a_3 \lambda_1 - a_4}{\lambda_1 (\lambda_1 - \lambda_2) (\lambda_1 - \lambda_3)} \]
\[ B_2 = \frac{a_1 \lambda_2^2 - a_2 \lambda_2^2 + a_3 \lambda_2 - a_4}{\lambda_2 (\lambda_1 - \lambda_2) (\lambda_2 - \lambda_3)} \]
\[ B_3 = \frac{a_1 \lambda_3^2 - a_2 \lambda_3^2 + a_3 \lambda_3 - a_4}{\lambda_3 (\lambda_1 - \lambda_3) (\lambda_2 - \lambda_3)} \]

The inverse Laplace transform of (36) gives

\[ (u)_{x=\lambda} = -\frac{a_0 a_4}{a_1} \left[ B_0 + B_1 e^{\lambda_1 t} + B_2 e^{\lambda_2 t} + B_3 e^{\lambda_3 t} \right] \]  

The procedure will be the same if, in the continued fraction (32), more than the first convergent is taken. It may be mentioned in this connection that the analysis may easily be extended to the case when the material is cylindrically aeolotropic.
4.6. Dynamic Stresses in a Porous Elastic Plane Under an Impulsive Load

The dynamic theory of deformation of an isotropic elastic material containing a fluid has been developed for the axisymmetric loading condition in a semi-infinite medium under an impulsive load. The stresses and displacements have been expressed in the complex integral forms.

Introduction

In the previous sections of the present chapter we have investigated problems which fall under the quasi-static theory of deformation of the porous elastic and visco-elastic material containing a fluid. In the corresponding dynamic theory, the equations of equilibrium is to be replaced by the equations of motion. In general, this is done by simply adding the inertia term and thus introducing the density of the material, provided the medium is a single-phased system of either a solid or a fluid. In the current discussion of the two-phased system of solid-fluid, the addition of the inertia term introduces not only the density of the solid and the density of the fluid but also a third coupling density characteristic of the medium. We shall not dilate upon the theory. This has been discussed elsewhere [15] with the help of Lagrange's equations of motion for a general dynamical system. It has also been shown that [15,30] if the medium extends to infinity in all directions, three kinds of plane waves can

be propagated through it. One of these is the transverse wave, while the other two are dilatation waves of different types. This phenomenon is in striking contrast with what happens in the purely elastic body. In the latter case there are only two kinds of plane waves, one dilatation and one transverse. The effect of the plane boundary of a semi-infinite body upon the reflection and refraction of the waves has also been discussed recently on the assumption that there is no dissipation in medium \cite{31}.

In the present section we shall investigate a particular problem of the transient type to illustrate the theory. It is the classical Lamb's problem of determining stresses in the half plane under an impulsive load on the free surface. The formulae for the non-dissipative system will be deduced therefrom. All the results are expressed in the form of complex integrals.

**Fundamental Equations**

In the axisymmetric deformation of a porous elastic solid containing a fluid the stress tensor can be separated into two parts \cite{16}. The first part given by

\[
\begin{pmatrix}
\sigma_r & 0 & \sigma_{rz} \\
0 & \sigma_\theta & 0 \\
\sigma_{zr} & 0 & \sigma_z
\end{pmatrix}
\]

(1)

represents the force component acting on the solid skeleton, while the other part given by
represents the force component on the fluid part of the solid-fluid system. The scalar \( \sigma \) is proportional to the fluid pressure \( p \) according to
\[
\sigma = -f p \tag{3}
\]
where \( f \) is the porosity of the system. The stress-strain law in the isotropic case is given by
\[
\begin{align*}
\sigma_x &= 2N e_x + A e + \epsilon_x, \\
\sigma_y &= 2N e_y + A e + \epsilon_y, \\
\sigma_z &= 2N e_z + A e + \epsilon_z, \\
\end{align*}
\tag{4}
\]
Here \( A, N, E, \) and \( R \) are four constants characterising the elastic property of the solid, and
\[
\begin{align*}
e_{xx} &= \frac{\partial w}{\partial x^2}, & e_{yy} &= \frac{\partial w}{\partial y^2}, & e_{zz} &= \frac{\partial w}{\partial z^2}, \\
e_{xy} &= \frac{\partial w}{\partial x \partial y}, & e_{yx} &= \frac{\partial w}{\partial y \partial x}, & e &= \frac{\partial w}{\partial x} + \frac{1}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right), \\
\epsilon &= \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right), \\
\end{align*}
\tag{5}
\]
\((u_x, u_y, u_z)\) and \((v_x, v_y, v_z)\) being the components of displacements of the solid and the fluid respectively. The dynamic equations for the system may be deduced from Lagrange's equations of motion in generalised coordinates with a dissipation function [15]. In terms of stress components
these are.

\[
\frac{\partial \sigma_{xx}}{\partial t} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \sigma_{yy}}{\partial y} = \frac{1}{\partial t} (\varphi \nu \nu + \varphi_{12} \nu \nu) + \frac{\partial \nu}{\partial t} (\nu \nu - \nu \nu),
\]

\[
\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yx}}{\partial x} = \frac{1}{\partial t} (\varphi \nu \nu + \varphi_{12} \nu \nu) + \frac{\partial \nu}{\partial t} (\nu \nu - \nu \nu).
\]

(6)

\[
\frac{\partial \sigma_{yy}}{\partial y} = \frac{\partial \nu}{\partial t} (\varphi \nu \nu + \varphi_{12} \nu \nu) - \frac{3}{\partial t} (\nu \nu - \nu \nu),
\]

\[
\frac{\partial \sigma_{xx}}{\partial x} = \frac{\partial \nu}{\partial t} (\varphi \nu \nu + \varphi_{12} \nu \nu) - \frac{3}{\partial t} (\nu \nu - \nu \nu).
\]

(7)

The coefficient \( \lambda \) is a constant related to Darcy's coefficient of permeability \( k \) by

\[
\lambda \kappa = \mu f^2
\]

where \( \mu \) is the fluid viscosity. The mass coefficients \( \varphi \), \( \varphi_{12} \) and \( \varphi_{22} \) are also constants having the following properties.

\[
\varphi > 0, \quad \varphi_{12} > 0, \quad \varphi_{12} < 0, \quad \varphi \varphi_{12} - \varphi_{12}^2 > 0, \quad \varphi + \varphi_{12} + 2 \varphi_{12} > 0.
\]

(9)

Appropriate physical interpretation may be given to them.

[15]. We may simply state here that, besides the density of solid and the density of the fluid, there is a third coupling density for the two-phased solid-fluid system. The mass coefficients \( \varphi \), \( \varphi_{12} \) and \( \varphi_{22} \) can be expressed in terms of these three fundamental densities.
Substitution of values of stress components in terms of displacements from (4) and (5) in (6) and (7) gives

\[
\frac{\partial}{\partial r}(p_0 + \phi \varepsilon) + N \frac{\partial \varepsilon}{\partial z} = \frac{\partial}{\partial r} \left( f_{11} u_r + f_{12} u_r \right) + \lambda \frac{\partial}{\partial \theta} (u_r - u_r),
\]

\[
\frac{\partial}{\partial z} \left( p_0 + \phi \varepsilon \right) - N \left( \frac{\partial u_r}{\partial r} + \frac{u_r}{r} \right) = \frac{\partial}{\partial r} \left( f_{11} u_z + f_{12} u_z \right)
+ \lambda \frac{\partial}{\partial \theta} (u_z - u_z),
\]

(10)

\[
\frac{\partial}{\partial r} (q_0 R + R \varepsilon) = \frac{\partial}{\partial z} \left( f_{12} u_r + f_{22} u_r \right) - \lambda \frac{\partial}{\partial \theta} (u_r - u_r),
\]

\[
\frac{\partial}{\partial z} (q_0 R + R \varepsilon) = \frac{\partial}{\partial z} \left( f_{12} u_z + f_{22} u_z \right) - \lambda \frac{\partial}{\partial \theta} (u_z - u_z),
\]

(11)

where

\[
\omega = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r},
\]

\[
P = \lambda \frac{\partial}{\partial \theta} (u_z - u_z). \tag{12}
\]

Applying the operators \( \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) \) on the two relations of (10) repectively and adding one gets

\[
\nabla^2 (p_0 + \phi \varepsilon) = \frac{\partial^2}{\partial r^2} (f_{11} \varepsilon + f_{12} \varepsilon) + \lambda \frac{\partial}{\partial \theta} (\varepsilon - \varepsilon),
\]

(13)

\[
\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.
\]

Elimination of \( (p_0 + \phi \varepsilon) \) between the two equations of (10) gives

\[
N (\nabla^2 \omega - \frac{\omega}{\partial z^2}) = \frac{\partial^2}{\partial r^2} \left( f_{11} \omega + f_{12} \omega \right) + \lambda \frac{\partial}{\partial \theta} (\omega - \omega), \tag{14}
\]
where
\[ n = \frac{\partial u_r}{\partial z} - \frac{\partial u_2}{\partial r} \]  \hspace{1cm} (15)

If we repeat the same process by which equations (13) and (14) are obtained from (10), the relation (11) gives
\[ \nabla^2 (\rho \varepsilon + \kappa \varepsilon) = \frac{3}{\varepsilon} \left( \phi_1 \varepsilon + \phi_2 \varepsilon \right) - 4 \frac{3}{\varepsilon} (\varepsilon - \varepsilon), \] \hspace{1cm} (16)
\[ \frac{3}{\varepsilon} \left( \phi_1 \omega + \phi_2 n \right) - \rho \frac{3}{\varepsilon} (\omega - n) = \rho. \] \hspace{1cm} (17)

The equations (13), (14), (16) and (17) give the four unknowns \( e, \varepsilon, \omega, n \), and the substitution of values of these in (10) and (11) gives four equations to determine the unknowns \( u_r, u_z, \phi_r, \phi_z \).

**Boundary and Initial Conditions**

In the present discussion the semi-infinite medium bounded by \( z = 0 \) will be considered with \( z \) axis taken into the medium. Let an impulsive load of constant magnitude \( \rho_0 \) per unit area be applied on the surface \( z = 0 \) through a porous smooth circular rigid slab of radius \( a \). The boundary and initial conditions are then as follows.

\[ u_r = -\rho_0 \delta(t), \quad 0 \leq r \leq a, \quad z = 0, \] \hspace{1cm} (18)
\[ = 0, \quad r > a, \quad z = 0. \]
where $\delta(t)$ is the Dirac delta function.

\begin{align}
\tau_{xz} = 0, \quad z = 0, \quad \gamma > 0, \quad t > 0, \\
\sigma = 0, \quad z = 0, \quad \gamma > 0, \quad t > 0,
\end{align}

(19)

(20)

\begin{align}
u_r = u_z = v_r = v_z = \frac{\partial u_r}{\partial t} = \frac{\partial u_z}{\partial t} = \frac{\partial v_r}{\partial t} = \frac{\partial v_z}{\partial t} = 0,
\end{align}

(21)

All the stresses and displacements tend to zero as

\begin{align}
g \rightarrow \infty, \quad \nu \rightarrow \alpha, \quad t \rightarrow 0.
\end{align}

(22)

The conditions (18) and (19) are the prescribed boundary stresses on the solid. The condition (20) shows that the liquid pressure is zero at the free surface as well as under the rigid slab, the slab being porous. Initial equilibrium of the whole system is expressed by the condition (21), while the condition (22) shows that the loading process has no effect at a great distance.

**Transform Solution**

The transform of Laplace-Hankel will be adopted to solve the problem. A function with a prime on it will be taken as its Laplace transform with respect to the time variable $t$. Thus

\begin{align}
e'(r, z, t) = \int_0^\infty e(r, z, t) e^{-st} dt.
\end{align}
We also define Hankel transforms and denote them by dots on functions in the following way.

\[ \hat{e}'(\eta, r, z, \xi) = \int_0^\infty e'(r, z, \xi) \cdot J_\nu(\eta r) \, dr, \]

\[ \hat{e}'(\eta, r, z, \xi) = \int_0^\infty e'(r, z, \xi) \cdot J_\nu(\eta r) \, dr, \]

where \( J_\nu \) is the Bessel function of order \( \nu \).

If the condition (21) is used, then equations (13) and (16) can be written as

\[ \nabla^2 (\rho e' + q e') = \frac{\hat{\rho}}{\hat{\rho}^\prime} \left( \hat{f}_{11} e' + \hat{f}_{12} \hat{e}' \right) + \frac{\hat{q}}{\hat{q}^\prime} \left( e' - \hat{e}' \right), \]

\[ \nabla^2 (q e' + q \hat{e}') = \frac{\hat{q}}{\hat{q}^\prime} \left( \hat{f}_{12} e' + \hat{f}_{12} \hat{e}' \right) - \frac{\hat{\rho}}{\hat{\rho}^\prime} \left( e' - \hat{e}' \right). \quad (23) \]

Assuming further that

\[ \rho e' = q e' = \rho \frac{\partial e'}{\partial \rho} = \rho \frac{\partial \hat{e}'}{\partial \rho} = 0, \quad \text{as } \rho \to 0, \text{ and } \rho \to \infty, \quad (24) \]

(which are required for the application of Hankel transform and may be verified when the solution has been obtained) one gets from (23)

\[ (D^2 - \eta^2)(\rho e' + q \hat{e}') = \frac{\hat{\rho}}{\hat{\rho}^\prime} \left( \hat{f}_{11} e' + \hat{f}_{12} \hat{e}' \right) + \frac{\hat{q}}{\hat{q}^\prime} \left( e' - \hat{e}' \right), \]

\[ (D^2 - \eta^2)(q e' + q \hat{e}') = \frac{\hat{q}}{\hat{q}^\prime} \left( \hat{f}_{12} e' + \hat{f}_{12} \hat{e}' \right) - \frac{\hat{\rho}}{\hat{\rho}^\prime} \left( e' - \hat{e}' \right), \quad (25) \]

\[ D \equiv \frac{\partial}{\partial z}. \]
Elimination of \( \alpha_1 \) in (25) gives
\[
\alpha_1 \rho^2 \dot{\epsilon}' = (\alpha_1 \eta^2 + \alpha_2 \xi^2) e' + (\alpha_3 \xi - \alpha_1) \xi \dot{\epsilon}'.
\] (26)

Again elimination of \( \dot{e}' \) between the first relation of (25) and (26) yields
\[
\alpha_1 \rho^4 \dot{\epsilon}'' - (2 \alpha_1 \eta^2 + \alpha_4 \xi + \alpha_2 \xi) \dot{\rho}^2 \dot{\epsilon}' + \frac{1}{\alpha_1} \alpha_1 \eta^4 + (\alpha_4 \xi + \alpha_2) \xi \eta^2
+ (\rho + \xi \xi) \eta^2 \frac{\xi^3}{3} \dot{\epsilon}' = 0,
\] (27)

where
\[
\begin{align*}
\alpha_1 &= \rho R - \eta^2 \\
\alpha_2 &= R \rho - \eta \rho_1 \\
\alpha_3 &= \rho \rho_1 - \eta \rho_2 \\
\alpha_4 &= \rho \rho_2 + \rho - 2 \alpha \rho_2
\end{align*}
\] (28)

The solution of (27) vanishing at \( \tau \rightarrow \infty \), consistent with the condition (22), can be written as
\[
\dot{\epsilon}' = g_1(\eta) e^{-j_1 \eta} + g_2(\eta) e^{-j_2 \eta},
\] (29)

where
\[
\begin{align*}
\xi_1 &= + \left( \eta^2 + \xi_1 \right)^{\frac{1}{2}} \\
\xi_2 &= + \left( \eta^2 + \xi_2 \right)^{\frac{1}{2}}
\end{align*}
\] (30)
The coefficients $\alpha_1$ and $\alpha_2$ are to be determined from given conditions. In (30) and (31) only positive values of the square roots are to be considered. From (26) and (29)

$$
\dot{z}' = \frac{1}{\delta (\alpha_2 x^2 - \xi)} \left[ (\alpha_1 x_1 - \alpha_2 x^2 - c_1 x^2) \alpha_1 e^{-x_1^2} + (\alpha_1 x_2 - \alpha_2 x^2 - c_1 x^2) \alpha_2 e^{-x_2^2} \right].
$$

Equations (14) and (17) with condition (21) and assumption (24) give

$$
\begin{align*}
\frac{1}{2} \left( f_{11} \dot{\omega}' + f_{12} \ddot{\omega}' \right) + \frac{1}{2} \left( \ddot{\omega}' - \dot{j} \ddot{j}' \right) + N \gamma^2 \dot{\omega}' &= N \dot{\omega}' \ddot{\omega}', \\
\frac{1}{2} \left( f_{12} \dot{\omega}' + f_{22} \ddot{\omega}' \right) - \lambda \left( \ddot{\omega}' - \dot{j} \ddot{j}' \right) &= 0,
\end{align*}
$$

from which

$$
\ddot{j}' = -\frac{f_{12} \frac{1}{2} - \lambda}{f_{22} \frac{1}{2} + \lambda} \dot{\omega}',
$$

$$
D = \dot{\omega}' = \lambda^2 \dot{\omega}',
$$

where

$$
\lambda = +\left( \gamma^2 + \nu^2 \right)^{\frac{1}{2}},
$$

$$
\nu^2 = \frac{(\rho_0 \frac{1}{2} + \xi \frac{1}{2}) \frac{1}{2}}{N (f_{22} \frac{1}{2} + \lambda)}.
$$
The solution of (34) vanishing as \( z \to \infty \), is
\[
\hat{\omega}' = g_3(\eta) e^{-\lambda z}
\]  
(36)

\( g_3 \) being the integration constant.

Taking the inverse Hankel transform of (29), (32) and (36), we obtain

\[
e' = \int_0^\infty \left[ a_1 e^{-\alpha_2 \xi^2} + a_2 e^{-\alpha_4 \xi^2} \right] \eta J_0 (\eta \xi) d\eta
\]  
(37)

\[
e' = \frac{1}{\frac{1}{2} (\alpha_3^2 - \alpha_4^2)} \int_0^\infty \left[ (\alpha_1 \chi_4 - \alpha_2 \xi^2 - \alpha_4 \xi^2) a_1 e^{-\alpha_2 \xi^2} + (\alpha_1 \chi_2 - \alpha_2 \xi^2 - \alpha_4 \xi^2) a_2 e^{-\alpha_4 \xi^2} \right] \eta J_0 (\eta \xi) d\eta
\]  
(38)

\[
\omega' = \int_0^\infty a_3 e^{-\lambda z} \eta J_1 (\eta \xi) d\eta
\]  
(39)

Relations (10) and (11) with the condition (21) give

\[
\frac{\partial}{\partial \tau} (p e' + q e') + N \frac{\partial u'}{\partial z} = \frac{1}{2} \left( \frac{\partial u'}{\partial \tau} + \nabla \cdot \frac{u'}{\partial \tau} \right) + \frac{1}{2} \left( \frac{\partial u'}{\partial \tau} + \nabla \cdot \frac{u'}{\partial \tau} \right)
\]

\[
\frac{\partial}{\partial z} (p e' + q e') - N \left( \frac{\partial u'}{\partial \tau} + \frac{\partial v'}{\partial \tau} \right) = \frac{1}{2} \left( \frac{\partial u'}{\partial \tau} + \nabla \cdot \frac{u'}{\partial \tau} \right)
\]

+ \frac{1}{2} \left( \frac{\partial u'}{\partial \tau} + \nabla \cdot \frac{u'}{\partial \tau} \right).
\[
\frac{\partial}{\partial r} (q e^i + R e^j) = \frac{1}{2} \left( f_{12} u_1^i + f_{22} u_2^i \right) - \frac{1}{2} \left( u_1^i - u_2^i \right),
\]
\[
\frac{\partial}{\partial z} (q e^i + R e^j) = \frac{1}{2} \left( f_{12} u_1^i + f_{22} u_2^i \right) - \frac{1}{2} \left( u_1^i - u_2^i \right).
\]

From the first and the third of these relations, eliminating \( u_1^i \), we have
\[
\begin{align*}
\dot{u}_1^i = & \left. \frac{-1}{\frac{1}{2} \left( f_{12} + f_{22} \right)} \left[ \left\{ \frac{\partial}{\partial r} \left( \xi_1 - \xi_2 \right) + \xi_2 - \xi_1 \right\} e^l - \frac{\partial \xi_1}{\partial r} \right] \\
& - N \left( f_{12} \frac{\partial}{\partial r} + \frac{\partial N}{\partial r} \right) \frac{\partial \xi_1}{\partial r},
\end{align*}
\]

(40)

Similarly the second and the fourth ones give, on elimination of \( u_2^i \),
\[
\begin{align*}
\dot{u}_2^i = & \left. \frac{-1}{\frac{1}{2} \left( f_{12} + f_{22} \right)} \left[ \left\{ \frac{\partial}{\partial r} \left( \xi_1 - \xi_2 \right) \xi + \xi_2 - \xi_1 \right\} e^l - \frac{\partial \xi_2}{\partial r} \right] \\
& - N \left( f_{12} \frac{\partial}{\partial r} + \frac{\partial N}{\partial r} \right) \left( \frac{\partial \xi_2}{\partial r} + \frac{\partial N}{\partial r} \right),
\end{align*}
\]

(41)

Substitution for \( e^i, \xi^i, \omega^i \) in (40) and (41) from (37) - (39) gives
\[
\begin{align*}
\dot{u}_1^i = & \left. \frac{1}{\frac{1}{2} \left( f_{12} + f_{22} \right)} \right[ \alpha \int_0^\infty G_1 \gamma^2 \widehat{T}_1 \left( \eta \gamma \right) e^{-i \xi \gamma} d\gamma \\
& + \beta \int_0^\infty G_2 \eta^2 \widehat{T}_2 \left( \eta \gamma \right) e^{-i \xi \gamma} d\gamma \\
& - N \left( f_{12} \frac{\partial}{\partial r} + \frac{\partial N}{\partial r} \right) \left( \frac{\partial \xi_1}{\partial r} + \frac{\partial N}{\partial r} \right),
\end{align*}
\]

(42)
\[ u_z = \frac{1}{\xi^2 (p_0 + 4)} \left[ \kappa \int_0^\infty \frac{q_1}{\xi} \eta \cdot J_0 (\eta) \cdot e^{-\xi Z} \, d\eta \right. \\
+ \kappa \int_0^\infty \frac{q_2}{\xi} \eta \cdot J_0 (\eta) \cdot e^{-\xi Z} \, d\eta \right. \\
- N \left( \int_0^\infty \frac{q_0}{\xi} \eta \cdot J_0 (\eta) \cdot e^{-\xi Z} \, d\eta \right] , \]

where

\[ \kappa = \frac{\alpha \xi}{\xi} - \alpha u \xi - c \]

\[ \kappa = \frac{\alpha \xi}{\xi} - \alpha u \xi - c . \]  

The recurrence formulae

\[ J_0 (\eta) + J_1 (\eta) = 0 , \quad \eta \cdot J_1 (\eta) + J_1 (\eta) = \eta \cdot J_0 (\eta) , \]

have been used above.

From (4) and (5),

\[ \sigma_2 = 2N \left( \frac{\partial u}{\partial Z} + \eta \right) + \eta \cdot \right. \]

\[ \sigma_{\gamma Z} = N \left( \frac{\partial u_y}{\partial Z} + \eta \frac{\partial u_z}{\partial Z} \right) , \]

\[ \sigma_1 = \eta \cdot \right. \]

With the help of (4), the first of these can also be written as

\[ \sigma_2 = 2N \left( \frac{\partial u_z}{\partial Z} + \left( \frac{\alpha}{\eta} \right) \right) + \frac{\alpha}{\eta} \sigma_1 . \]  

The boundary conditions (18) - (20) transform to
if it is remembered that the Laplace transform of Dirac delta function is unity. We write the condition (46) in the Sine integral form as [29]

\[
\sigma_z = -\frac{1}{\alpha^4} \int_0^\infty J_1(\alpha \eta) J_0(\eta \eta) \, d\eta, \quad z = 0.
\]

**Determination of Coefficients**

When the values of \( e', u', y' \) are substituted from (37), (42) and (43) in the last two relations of (45) as well as in (45a), and the conditions (46a), (47) and (48) are used, we obtain the following three equations for the determination of the coefficients \( e_1, e_2, \) and \( e_3 \):

\[
\begin{align*}
\left( \chi_5 - 2NR \chi_5 \eta^2 \right) e_1 + \left( \chi_6 - 2NR \chi_4 \eta^2 \right) e_2 & = -\frac{poR}{\eta} \eta^2 (\eta \eta + 6 \lambda^2) \\
2\eta (\chi_5 s_1, e_1 + \chi_4 s_2 e_2) & = N (\eta \chi^2 + 6) (\lambda^2 + \eta^2) e_3,
\end{align*}
\]

(49)
where
\[ J_5(t) = (AR - q^2) \delta^2 (p_3 t + 4t) - 2NR_3, \]
\[ J_4(t) = (AR - q^2) \delta^2 (p_3 t + 4t) - 2NR_3, \]
\[ J_3(t) = \frac{AR_x J_3 + 4 \delta_0 \delta^2 - c_1 \delta}{AR_x J_3 + 4 \delta_0 \delta^2 - c_1 \delta}. \]

\[ \delta_0 = Q_\alpha_3 - R_\alpha_2. \]

From (49) - (51):

\[ G_1 = -\mu G_2, \]
\[ G_2 = -\frac{4AR\delta^2}{\eta} (p_3 t + 4t) (\lambda^2 + \eta^2) J_1(\lambda\eta), \]
\[ G_3 = -\frac{2\lambda_0 AR \delta^2}{N(p_3 t + 4t)\Delta} (p_3 t + 4t) (N_{32} - \mu N_{33}) J_1(\lambda\eta), \]

where
\[ \Delta = (\lambda^2 + \eta^2) \left[ \lambda_6 - \mu \lambda_5 - 2NR_3 \eta^2 (N_4 - \mu N_3)^2 \right] + 4NR \lambda \eta^2 \left( N_{32} - \mu N_{33} \right). \]

The expressions for stresses and displacements can now be written down. From the first relation of (45), the transformed normal stress in the axial direction is obtained, after a little lengthy calculation, as,
\[ \sigma_2' = -2P_0aRN \int_0^\infty \left( \mu \chi_3 \xi^2 e^{-\xi^2} - \chi_4 \xi^2 e^{-\xi^2} \right) \left( \lambda^2 + \eta^2 \right) \]
\[ + 2 \left( \chi_4 \xi^2 - \mu \chi_3 \xi^1 \right) \lambda \eta e^{-\xi^2} J_0(\eta \xi) J_1(\eta \xi) \frac{d\eta}{\eta} \]
\[ - \frac{\rho_0aRA}{\xi^2} \left( \xi_1 + \lambda \xi \right) \int_0^\infty \left( \chi_1 \xi^2 - \alpha_2 \xi^2 - C_1 \right) e^{-\xi^2} \]
\[ \times \left( \lambda^2 + \eta^2 \right) J_0(\eta \xi) J_1(\eta \xi) \frac{d\eta}{\eta} \]  

The inverse Laplace transform of (55) can be written as

\[ \sigma_2 = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \omega e^{\xi t} dt \]  

**Deduction for Non-Dissipative System**

The above results are simplified, while dealing with a medium where there is no dissipation. In this case, since \( \lambda = 0 \) one gets from (28), (31), (44) and (52):

\[ C_1 = C_2 = 0, \]
\[ 2 \alpha_1 \xi^1 = \beta_1 \xi^2, \quad 2 \alpha_1 \xi^2 = \beta_2 \xi^2, \]
\[ (\beta_1, \beta_2) = \alpha_4 \pm (\alpha_4^2 - 4 \alpha_1 \rho_0)^{1/2}, \]
\[ X_3 = \frac{1}{2} (\beta_1 - 2\alpha_0)^x, \quad X_0 = \frac{1}{2} (\beta_2 - 2\alpha_0)^x, \]

\[ 2\alpha_0 X_5 = \beta_3 x^3, \quad 2\alpha_0 X_6 = \beta_4 x^3, \]

\[ \beta_3 = 2\alpha_1 (\alpha R - q^2) t_0 - NR\psi_1 (\beta_1 - 2\alpha_0), \]

\[ \beta_4 = 2\alpha_1 (\alpha R - q^2) t_0 - NR\psi_2 (\beta_2 - 2\alpha_0), \]

\[ C_1 = (\eta^2 + \frac{\beta_1}{2\alpha_0} x^2)^{\frac{1}{2}}, \quad C_2 = (\eta^2 + \frac{\beta_2}{2\alpha_0} x^2)^{\frac{1}{2}}, \]

\[ \lambda = (\eta^2 + \frac{\beta_1}{2\alpha_0} x^2)^{\frac{1}{2}}, \quad \eta^2 = \frac{\psi_0}{N\pi^2}, \]

\[ \mu z = \frac{R\psi_2 + 2\beta_0}{R\psi_1 + 2\beta_0} = \text{constant}. \]

The relation (55) reduces to

\[ \frac{\partial^2 \psi}{\partial z^2} = -b_0 a R N \frac{\varepsilon}{2} \int_0^\infty \left( 2\eta^2 + \frac{\beta_1}{2\alpha_0} x^2 \right) \left( \frac{\beta_1}{2\alpha_0} x^2 \right) e^{-\psi} d\psi \]

\[ - (\beta_2 - 2\alpha_0) \left( \eta^2 + \frac{\beta_1}{2\alpha_0} x^2 \right) e^{-\psi} d\psi \]

\[ + 2\lambda \eta \left[ (\beta_2 - 2\alpha_0) \xi_2 - \mu (\beta_1 - 2\alpha_0) \xi_1 \right] e^{-\lambda \eta} \]

\[ \int_0^\infty \frac{\partial}{\partial \eta} \left( \frac{\partial}{\partial \psi} \right) e^{-\psi} \eta \left( \frac{\partial}{\partial \eta} \right) d\psi \]

\[ (58) \]

where \( \Delta \) is now given by
The exact evaluation of the Inverse Laplace transform of (55) or (58) is out of the question. However we are interested in small values of time and in that case it is always possible to evaluate the Inversion integral up to any degree of accuracy by applying the same method as was discussed in chapter III.
References

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