"The long, slow process, old as the race, through which the frontiers of the known have steadily encroached upon the territory of the unexplored, has been a progressive conquest of new worlds for the imagination."

- John L. Lowes.
CHAPTER II

BENDING OF THIN ELASTIC SHELLS and VIBRATION OF SPHERES OF VARIABLE ELASTIC PROPERTIES

2.1. Introduction

Two problems connected with the theory of bending of thin elastic shells will be considered in the next two sections of this chapter. The first problem is concerned with the bending of the cylindrical shell of anisotropic material while the second one with the shallow spherical shell of isotropic material. In both cases the boundaries are assumed to be simply supported. The bending occurs either under the action of transverse load or by the application of edge couples. Though these problems may appear to be formulated in a conventional manner and their solutions obtained by straightforward methods of Fourier series and separation of variables, the result of the first one has a special technological interest in pointing out the effect of anisotropy, while the procedure in the second problem shows that a simple solution may be obtained on the assumption of shallowness of the thin shell which belong to a otherwise complicated theory. The last section of the chapter deals with the radial vibration of a sphere of the variable modulus of elasticity. Here again, the classical method of perturbation is very conveniently used to find an approximate value of the frequency of vibration. Though attempts have been made to solve similar problems by numerical
methods with the help of high-speed computers and other techniques the simple but analytical method of perturbation as applied here seems to give reasonably good approximate results.

It will not be out of place to mention that the anisotropy and the non-homogeneity in the material are playing gradually more and more important part in practical applications. This fact has been stressed recently in an international symposium [61].

2.2. Thin Cylindrical Shell of Elastically Anisotropic Material

The effect of anisotropy upon the elastic deformation of a thin circular cylindrical shell has been considered when it is bent by transverse loads only. The theory adopted is that given by Love. It is found that for a particular material the magnitude of deflection may differ from that in isotropy by 10 percent or even more depending upon the polar angle. There is a critical angle for which the effect of anisotropy vanishes. For values less than this critical value the effect is negative while for greater values the effect is positive.

Historical Development [1 - 41]

Beginning with the work of Aron (1874), Mathieu (1883), Love (1888) and Basset (1890) as re-examined by Krauss (1929), Trefftz (1935) Odquist (1937) and others, there exists a vast literature about the theory of...
stretching and bending of thin shells of isotropic material.

Though an account of the general theory may be found in books of Love (1926), Flugge (1934), Biesel and Grammel (1939) and Timoshenko (1940), special interest lies in the treatments of Reissner (1912), Meissner (1913), Schwerin (1917), Dischinger (1935), Havers (1935), Jakobsen (1937) as well as in those of Reutter (1942), Byrne (1944), Fadle (1944), Goldenweiser (1945), Rabotnove (1945) and Zerna (1949). The papers by Chien (1944) and Synge and Chien (1941) are remarkable for their complete generality. For fresh derivation and critical reviews, we may refer to papers of Truesdell (1945), Green and Zerna (1949) and Reissner (1941, 1946, 1948, 1950, 1955a, 1955b). Circular cylindrical shells which concern the present section have been dealt with by Teodorescu (1952), Grigolyuk (1958), Hoff and Kempner (1953), Wuest (1954), Ambartsumyan (1954) and Yuan and Ting (1957).

In all cases, however, the material is assumed to be isotropic. In a general survey of orthotropic shells by Hildebrand, Reissner and Thomas (1949), more emphasis has been put on the effect of transverse shear stresses than on the effect of anisotropy. We shall investigate here the modifications to be introduced when anisotropy of the material is taken into account. For the purpose of completeness, the equations of equilibrium, stress-strain relations etc. will be introduced only at the particular stage where required. Then the problem of circular cylinders with transverse loads will be discussed [52].
Equations of Equilibrium

We consider a thin circular cylindrical shell of thickness $2h$, the radius of the middle surface in the unstrained state being $a$. The position of a point on the middle surface may be specified by two coordinates $\xi$ and $\varphi$, where the axis of $\xi$ is taken parallel to the axis of the cylinder and $\varphi$ is the angle between a pair of planes passing through the axis of the cylinder. The origin is taken on the middle surface. We take the axis of $\xi$ along an inward drawn radius. The positive directions of the axes of $\xi$, $\varphi$ and $\zeta$ are so chosen that they form a right-handed system. Thus instead of usual cylindrical coordinates $(\tau, \theta, z)$ we take $(\xi, \varphi, z)$, where $(\tau, \theta, z)$ are replaced by $(a-\xi, \varphi, z)$ respectively. The linear element $ds$ in this system between any two points is given by

$$ds^2 = d\xi^2 + (a-\xi)^2 d\varphi^2 + dz^2. \quad (1)$$

The general equations of equilibrium for the stresses in curvilinear coordinates [19] take the forms,

$$\frac{\partial \tau}{\partial \xi} + \frac{1}{a-\xi} \frac{\partial \tau}{\partial \varphi} + \frac{\partial \gamma}{\partial z} - \frac{1}{a-\xi} \gamma \zeta + F_\xi = 0 \quad (2)$$

$$\frac{\partial \gamma}{\partial \xi} + \frac{1}{a-\xi} \frac{\partial \gamma}{\partial \varphi} + \frac{\partial \tau}{\partial z} - \frac{2}{a-\xi} \tau \varphi + F_\varphi = 0 \quad (2)$$

$$\frac{\partial \gamma}{\partial \zeta} + \frac{1}{a-\xi} \frac{\partial \gamma}{\partial \varphi} + \frac{\partial \tau}{\partial z} - \frac{1}{a-\xi} (\tau \zeta - \gamma \varphi) + F_\zeta = 0.$$
We multiply these equations by \((a-z)\) and integrate with respect to \(z\) for the limits \((-h, h)\). We also multiply the first two equations by \((a-z)\) and integrate for the same limits. The second operation is not performed on the third equation, for that will introduce quantities not required in the present theory. It is obtained that

\[ \frac{\partial N_x}{\partial x} + \frac{1}{a} \frac{\partial N_{y}^x}{\partial \phi} + b_z = 0 \]

\[ \frac{\partial N_{y}^x}{\partial x} + \frac{1}{a} \frac{\partial N_{y}^x}{\partial \phi} - \frac{1}{a} \frac{\partial \phi}{\partial y} + b_y = 0 \]  \hspace{1cm} (3)

\[ \frac{\partial M_x}{\partial x} + \frac{1}{a} \frac{\partial M_{y}^x}{\partial \phi} - \phi_x + M_x = 0 \]

\[ \frac{\partial M_{y}^x}{\partial x} - \frac{1}{a} \frac{\partial N_{y}^x}{\partial \phi} + \phi_y - m_y = 0 \]  \hspace{1cm} (4)

where the stress-resultants \(N_x, N_y, \phi_x\) etc. and the stress couples \(M_x, M_y\) etc. are defined as follows.

\[ N_x = \int_{-h}^{h} \left(1 - \frac{z}{a} \right) \phi_x \, dz \]

\[ N_y = \int_{-h}^{h} \phi_y \, dz \]

\[ N_{y}^x = \int_{-h}^{h} \left(1 - \frac{z}{a} \right) \phi_y \, dz \]

\[ N_{x}^y = \int_{-h}^{h} \phi_{x} \, dz \]
\[ q_x = \int_{-h}^{h} \left(1 - \frac{x}{a}\right) \tau_{xz} \, dz, \quad \phi = \int_{-h}^{h} \tau_{\phi z} \, dz \]

\[ M_x = \int_{-h}^{h} x \left(1 - \frac{x}{a}\right) \tau_{xz} \, dz, \quad M_\phi = \int_{-h}^{h} z \tau_{\phi z} \, dz \]

\[ M_{x\phi} = -\int_{-h}^{h} x \left(1 - \frac{x}{a}\right) \tau_{x\phi} \, dz, \quad M_{\phi x} = \int_{-h}^{h} x \tau_{\phi x} \, dz \]  \hspace{1cm} (5)

\[ f_n = \int_{-h}^{h} \left(1 - \frac{x}{a}\right) \tau_{xz} \, dz + \int_{-h}^{h} \left(1 - \frac{x}{a}\right) F_x \, dz \]

\[ b_\phi = \int_{-h}^{h} \left(1 - \frac{x}{a}\right) \tau_{\phi z} \, dz + \int_{-h}^{h} \left(1 - \frac{x}{a}\right) F_\phi \, dz \]

\[ k_x = \int_{-h}^{h} \left(1 - \frac{x}{a}\right) \tau_{xz} \, dz + \int_{-h}^{h} \left(1 - \frac{x}{a}\right) F_x \, dz \]

\[ m_x = \int_{-h}^{h} x \left(1 - \frac{x}{a}\right) \tau_{xz} \, dz + \int_{-h}^{h} x \left(1 - \frac{x}{a}\right) F_x \, dz \]

\[ m_\phi = \int_{-h}^{h} x \left(1 - \frac{x}{a}\right) \tau_{\phi z} \, dz + \int_{-h}^{h} x \left(1 - \frac{x}{a}\right) F_\phi \, dz \]  \hspace{1cm} (6)
Elimination of $\alpha_\varphi$ and $\alpha_x$ between (3) and (4) gives

$$\begin{align*}
\frac{\partial N_x}{\partial x} + \frac{1}{2} \frac{\partial N_{\varphi}}{\partial \varphi} + \frac{1}{x} = 0 \\
\frac{\partial N_{\varphi}}{\partial x} + \frac{1}{2} \frac{\partial M_{\varphi}}{\partial \varphi} - \frac{1}{2} \frac{\partial N}{\partial x} + \frac{1}{2} \varphi + \frac{1}{2} \frac{M_{\varphi}}{\partial \varphi} - \frac{1}{x} \frac{M_{\varphi}}{\partial \varphi} + \frac{1}{2} \frac{M}{\partial \varphi} = 0 \\
\frac{\partial M_x}{\partial x} + \frac{1}{2} \frac{\partial M_{\varphi}}{\partial \varphi} + \frac{1}{2} \frac{\partial M}{\partial x} + \frac{1}{2} \frac{M_{\varphi}}{\partial \varphi} + \frac{1}{2} \frac{M}{\partial \varphi} = 0.
\end{align*}$$

These equations may be deduced from the general formulae given by Reissner [25] and Hildebrand, Reissner and Thomas [14]. These have been deduced in a different manner by Timoshenko [35].

**Strain-displacement Relations**

Using the general formulae in curvilinear coordinates [19], the strain components are expressed in terms of displacement components as

$$\begin{align*}
\varepsilon_x &= \frac{\partial u_x}{\partial x} + \frac{1}{2} \frac{\partial u_{\varphi}}{\partial \varphi} - u_z, \\
\varepsilon_\varphi &= \frac{1}{2} \left( \frac{\partial u_x}{\partial \varphi} + \frac{\partial u_{\varphi}}{\partial x} \right), \\
\varepsilon_z &= \frac{\partial u_z}{\partial z} \\
\gamma_{\varphi z} &= \frac{1}{2} \left( \frac{\partial u_x}{\partial \varphi} + \frac{\partial u_{\varphi}}{\partial x} \right), \\
\gamma_{zz} &= \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}, \\
\gamma_{x \varphi} &= \frac{\partial u_{\varphi}}{\partial x} + \frac{1}{2} \frac{\partial u_z}{\partial \varphi}.
\end{align*}$$

(8)
Stress-strain Relations

For cylindrically anisotropic material the strain energy function $W$ is given by

$$2W = c_{44} \varepsilon_x^2 + c_{42} \varepsilon_x \varepsilon_y + c_{55} \varepsilon_y^2 + 2c_{3i} \varepsilon_x \varepsilon_z + 2c_{6i} \varepsilon_z \varepsilon_y$$

$$+ 2c_{12} \varepsilon_x \varepsilon_y + c_{66} \varepsilon_y^2 + c_{55} \varepsilon_z^2 + c_{66} \varepsilon_y^2 + c_{56} \gamma_{2z}^2$$

$$c_{ij} = c_{ji}.$$

The corresponding stress-strain relations are

$$\sigma_x = c_{11} \varepsilon_x + c_{12} \varepsilon_y + c_{13} \varepsilon_z$$

$$\sigma_y = c_{21} \varepsilon_x + c_{22} \varepsilon_y + c_{23} \varepsilon_z$$

$$\sigma_z = c_{31} \varepsilon_x + c_{32} \varepsilon_y + c_{33} \varepsilon_z$$

$$\gamma_{2z} = c_{44} \gamma_{2z}, \quad \gamma_{3z} = c_{55} \gamma_{3z}, \quad \gamma_{xx} = c_{66} \gamma_{xx}.$$

In a thin shell the normal stress $\sigma_z$ is of smaller order of magnitude in comparison with the normal stresses $\sigma_x$ and $\sigma_y$ \([11]\). Hence it is customary to take, in the stress-strain relations,

$$\sigma_z = 0. \quad (10)$$

When the assumption (10) is used, the elimination of $\varepsilon_z$ reduces the first three relations of (9) to

$$\sigma_x = \lambda_{11} \varepsilon_x + \lambda_{12} \varepsilon_y, \quad \sigma_y = \lambda_{22} \varepsilon_y$$

where

$$\lambda_{11} = c_{11} - \frac{c_{13}^2}{c_{55}}, \quad \lambda_{22} = c_{22} - \frac{c_{23}^2}{c_{55}}$$

$$\lambda_{12} = c_{12} - \frac{c_{13} c_{23}}{c_{55}}. \quad (11)$$
Usual Assumptions

Let it be assumed that

\[
\begin{align*}
\dot{u}_x(x, \varphi, z) &= u(x, \varphi) + z \dot{u}_1(x, \varphi) \\
\dot{u}_\varphi(x, \varphi, z) &= u(x, \varphi) + z \dot{v}_1(x, \varphi) \\
\dot{u}_z(x, \varphi, z) &= \omega(x, \varphi)
\end{align*}
\]  

(12)

where \((u, v, \omega)\) are the displacement components of the middle surface.

We also assume that the transverse shear stresses \(\gamma_{xz}\) and \(\gamma_{\varphi z}\) are zero and hence

\[
\gamma_{xz} = \gamma_{\varphi z} = 0.
\]  

(13)

The assumptions (13) imply that straight lines initially normal to the unstrained middle surface remain straight after deformation and become normal to the strained middle surface (Bernoulli’s law) \([38]\) while the third assumption of (12) puts a severe restriction on the strain components \(\varepsilon_x\) and \(\varepsilon_\varphi\) viz, \(\varepsilon_{13} \varepsilon_x + \varepsilon_{32} \varepsilon_\varphi = 0\) as is evident from (8) and (9). This restriction is not retained in the classical theory of thin shell, thus introducing one of its drawbacks. Substituting (12) in the fourth and fifth relations of (8) and then using (13) we get

\[
\dot{u}_1 = -\frac{\partial \omega}{\partial x} \\
\dot{v}_1 = -\frac{1}{\alpha} \left( \frac{\partial \omega}{\partial \varphi} + \varphi \right).
\]  

(14)

The remaining relations of (3) with the help of (12) and (14) give
\[e_x = e_1 - x \xi_x, \quad e_\varphi = (e_2 - 2 x \xi_\varphi)/(1 - \frac{x}{a}) \]

\[\gamma_{x\varphi} = [\omega - (2 - \frac{x}{a}) x \xi_{x\varphi}]/(1 - \frac{x}{a}) \tag{15} \]

where

\[\varepsilon_1 = \frac{3u}{\phi_x}, \quad \varepsilon_2 = \frac{1}{a} \left( \frac{3u}{\phi_\varphi} - \omega \right), \quad \omega = \frac{3u}{\phi_x} + \frac{1}{a} \frac{3u}{\phi_\varphi} \]

\[\xi_x = \frac{3u}{\phi_x}, \quad \xi_\varphi = \frac{1}{a} \frac{3u}{\phi_\varphi} \left( \frac{3u}{\phi_x} + \omega \right) \]

\[\tau_{x\varphi} = \frac{1}{a} \frac{3u}{\phi_x} \left( \frac{3u}{\phi_\varphi} + \nu \right) \tag{16} \]

The formulae (15) may be deduced from the general expressions given by Love [19, p. 527].

Love's First Approximation

If the thickness of the shell is small in comparison with the radius, we may put in (5) and (15)

\[1 - \frac{x}{a} \approx 1 \tag{17} \]

Consequently from (15)

\[\varepsilon_x = \varepsilon_1 - x \xi_x, \quad \varepsilon_\varphi = \varepsilon_2 - x \xi_\varphi, \quad \gamma_{x\varphi} = \omega - 2 \varepsilon \xi_{x\varphi} \tag{18} \]

It is seen from (5), (9) and (13) that

\[\varepsilon_x = \varepsilon_\varphi = 0 \tag{19} \]
Even when (19) is not satisfied the relations (19) are assumed to be approximately correct. In that case $\theta_x$ and $\theta_\varphi$ are determined from (4). The other stress-resultants and stress-couples are determined from (5), (11), (18) and the last relation of (9) as

\begin{align*}
N_x &= 2k \left( \sigma_{11} \epsilon_1 + \sigma_{12} \epsilon_2 \right), \quad N_\varphi = 2k \left( \sigma_{12} \epsilon_1 + \sigma_{22} \epsilon_2 \right) \\
N_x \theta_\varphi &= N_\varphi \theta_x = 2k c_{66} \varphi \\
M_x &= -\frac{1}{3} k^3 \left( \sigma_{11} \chi_x + \sigma_{12} \chi_\varphi \right), \\
M_\varphi &= -\frac{2}{3} k^3 \left( \sigma_{12} \chi_x + \sigma_{22} \chi_\varphi \right), \\
M_x \varphi &= -M_\varphi x = \frac{4}{3} k^3 c_{66} \chi_x \varphi.
\end{align*}

The effective transverse shear forces $V_x$ and $V_\varphi$ and the effective tangential shears forces $V_{x\varphi}$ and $V_{\varphi x}$, defined as [19, p. 537]

\begin{align*}
V_x &= \theta_x - \frac{1}{a} \frac{\partial N_x}{\partial \varphi}, \\
V_\varphi &= \theta_\varphi + \frac{\partial M_{\varphi x}}{\partial x}, \\
V_{x\varphi} &= N_{x\varphi} + \frac{1}{a} M_{x \varphi}, \\
V_{\varphi x} &= -N_{\varphi x} + \frac{1}{a} M_{\varphi x},
\end{align*}

are obtained by utilising (4) and (20). These are

\begin{align*}
V_x &= -\frac{2}{3} k^3 \left[ \sigma_{11} \frac{\partial \chi_x}{\partial a} + \frac{\partial \chi_\varphi}{\partial a} + k \left( c_{66} / a \right) \frac{\partial \chi_\varphi}{\partial \varphi} \right] + m_x \\
V_\varphi &= -\frac{2}{3} k^3 \left[ \left( \sigma_{12} / a \right) \frac{\partial \chi_x}{\partial \varphi} + \left( \sigma_{22} / a \right) \frac{\partial \chi_\varphi}{\partial \varphi} + 4 c_{66} \frac{\partial \chi_\varphi}{\partial a} \right] + m_\varphi \\
V_{x\varphi} &= -V_{\varphi x} = 2k c_{66} \left[ \varphi + \frac{1}{3} \left( \sigma^2 / a \right) \chi_\varphi \chi_* \right].
\end{align*}
Substituting (16) in (20) and then (20) in (7) we get three differential equations satisfying the displacement components \((u, v, w)\) of the middle surface as follows.

\[
\begin{align*}
\alpha_1 \frac{\partial^4 u}{\partial x^4} + (\alpha_1 c_{12} + \alpha_2 c_{66}) \frac{\partial^2 u}{\partial x^2 \partial \phi} + \frac{1}{\alpha} (\alpha_2 c_{12} + \alpha_6 c_{66}) \frac{\partial^2 u}{\partial \phi^2} - \frac{\partial^2 u}{\partial \phi^2} + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial \phi} &= 0 \\
\frac{1}{\alpha} (\alpha_1 + \alpha_2) \frac{\partial^2 u}{\partial x^2 \partial \phi} + c_{66} \frac{\partial^4 u}{\partial \phi^4} + \frac{\alpha_2}{\alpha} \frac{\partial^2 u}{\partial \phi^2} + \frac{\partial u}{\partial \phi} \left(\frac{\alpha_1 + \alpha_2}{2\alpha} \frac{\partial u}{\partial \phi} \right) \frac{\partial^2 u}{\partial \phi^2} &= 0 \\
\frac{1}{\alpha} (\alpha_2 + \alpha_6) \frac{\partial^2 u}{\partial x^2 \partial \phi} + \frac{\partial u}{\partial \phi} \left(\frac{\alpha_1 + \alpha_6}{\alpha} \frac{\partial u}{\partial \phi} \right) &= 0.
\end{align*}
\]

In writing the second equation of (22) the assumption (17) has been used. The results for isotropic materials will be obtained by setting \(c_{11} = c_{22} = c_{33} = \lambda + 2\mu\), \(c_{44} = c_{55} = c_{66} = \mu\), \(c_{23} = c_{31} = c_{12} = \lambda\) where \((\lambda, \mu)\) are Lamé constants.

**Simply Supported Shell with Sinusoidal Loads**

We consider the shell of which the edges are given by \(x = 0\), \(x = l\), \(\phi = 0\) and \(\phi = \alpha\). Let it be bent...
by a load only so that \( \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0 \) and:

\[
\sigma_{zz} = \sigma_0 \sin \frac{\pi y}{a} \sin \frac{\pi z}{a}.
\]  

(24)

If the shell be simply supported the boundary conditions are [19, p.537]

\[
\begin{align*}
\omega &= M_x = N_x = 0 \quad \text{on } x = 0, \; x = l, \\
\omega &= N_y = N_y = 0 \quad \text{on } y = 0, \; y = a, \\
\nu_x \phi &= 0 \quad \text{on } x = 0, \; x = l, \\
\nu_y \phi &= 0 \quad \text{on } y = 0, \; y = a.
\end{align*}
\]  

(23)

(25)  

(26)  

(27)

The displacement components

\[
\begin{align*}
u &= A \sin \frac{\pi x}{a} \cos \frac{\pi z}{l}, \\
v &= B \cos \frac{\pi x}{a} \sin \frac{\pi z}{l}, \\
\omega &= C \sin \frac{\pi x}{a} \sin \frac{\pi z}{l}.
\end{align*}
\]  

(29)

where \( A, B, \) and \( C \) are constants may be easily verified to satisfy the conditions (25) and (26). Substitution of (29) in (23) yields
Equations in (30) determine the constants $A$, $B$, and $C$. It may be pointed out that the state determined by (29) and (30) does not satisfy the boundary conditions (27) and (28) exactly. However, the average values of $V_x \phi$ and $V_{\phi x}$ on the corresponding edges as determined by the integrals 

$$\int V_x \phi \, d\alpha \quad \text{and} \quad \int V_{\phi x} \, d\alpha$$

vanish. This shows that each of these tangential shear forces forms a self-equilibrating system on the separate edges of the shell. Hence, by Saint Venant's principle we may assume that the boundary conditions (27) and (28) are satisfied so long as the edge-effects are not considered. Using (29) and (16) in (20) and the first two equations of (22) it is obtained that

$$N_x = \frac{2}{3} \pi x_0^3 \left[ \frac{\partial_2}{\pi \alpha} - B + \left( \frac{\partial_2}{\alpha^2} + \frac{\partial_1}{\alpha^2} \right) \right] \sin \frac{\pi \Omega}{\alpha} \sin \frac{\pi \Omega}{\ell}$$

$$M_\phi = \frac{2}{3} \pi x_0^3 \left[ \frac{\partial_2}{\pi \alpha} - B + \left( \frac{\partial_2}{\alpha^2} + \frac{\partial_1}{\alpha^2} \right) \right] \sin \frac{\pi \Omega}{\alpha} \sin \frac{\pi \Omega}{\ell}$$

$$N_{x \phi} = -N_{\phi x} = \frac{2}{3} \pi x_0^3 \left[ \frac{\partial_1}{\pi \alpha} B + C \right] \cos \frac{\pi \Omega}{\alpha} \cos \frac{\pi \Omega}{\ell}$$
In the particular case when \( \alpha = \frac{L}{a} \) the equations in (30) simplify to

\[
\begin{align*}
(1 + \lambda_{12}) A + (1 + \lambda_{12}) B + \frac{4}{\pi a} \cdot \lambda_{12} C &= 0 \\
(1 + \lambda_{12}) A + (1 + \lambda_{22}) B + \frac{1}{3} \cdot \frac{4}{a^2} (2 + \lambda_{12} + \lambda_{22}) \frac{\pi a}{L} C &= 0 \\
\lambda_{12} A + \left[ \lambda_{22} + \frac{4 a^2}{3 \pi^2} (4 + \lambda_{12} + \lambda_{22}) \right] B + \frac{1}{\pi a} \cdot \frac{\lambda_{22}}{L} + \frac{\pi^2 a L^2}{3 \pi^2} \\
(4 + \lambda_{12} + 2 \lambda_{12} + \lambda_{22}) C - \frac{a L b_0}{2 \pi a c_{66}} &= 0
\end{align*}
\]

where

\[
\lambda_n = \frac{\lambda_n}{c_{66}} \quad \lambda_{22} = \frac{\lambda_{22}}{c_{66}} \quad \lambda_{12} = \frac{\lambda_{12}}{c_{66}}
\]

In the isotropic material

\[
\begin{align*}
\lambda_{11} &= \lambda_{22} = 4 \left( c_{12} + c_{66} \right) / \left( c_{12} + 2 c_{66} \right) \\
\lambda_{12} &= 2 c_{12} / \left( c_{12} + 2 c_{66} \right)
\end{align*}
\]
Numerical Results

For barytes we have

\[ c_{11} = 907, \quad c_{22} = 860, \quad c_{33} = 1074 \]
\[ c_{23} = 273, \quad c_{31} = 275, \quad c_{12} = 468 \]
\[ c_{44} = 122, \quad c_{55} = 293, \quad c_{66} = 283 \]  

(35)

where the constants are expressed in terms of an unit stress of grammes' weight per square centimetre. These give

\[ \lambda_1 = 2.9561, \quad \lambda_{22} = 2.5816, \quad \lambda_{12} = 1.4067. \]  

(36)

In the isotropic case, calculating \( \lambda_{11} \) and \( \lambda_{12} \) from (34) and (35) we get

\[ \lambda_{11} = \lambda_{22} = 2.9052, \quad \lambda_{12} = 0.9052. \]  

(36a)

Since it is generally conceded that the shell is no longer thin when \( \sqrt{a_1 a} > 1/3 \), we take an extreme value, viz. \( \sqrt{a_1 a} = 0.03 \). The equations (32) now give \( A_1, B_1 \) and \( C \) as

\[ (A_1, B_1, C_1) = (\bar{\omega}_1, \bar{\omega}_1, \overline{\omega}_1) \frac{a^{2 f_0}}{2.5 c_{11}} \]  

(37)

\[ (A_i, B_i, C_i) = (\bar{\omega}_i, \bar{\omega}_i, \overline{\omega}_i) \frac{a^{2 f_0}}{2.5 c_{11}} \]  

(37a)

where \( A_i \) and \( A_{0a} \) denote the values of \( A \) for the isotropic and anisotropic cases respectively with similar notations for other quantities, and \( \bar{\omega}_1, \overline{\omega}_1 \ldots \) are the
amplitude constants depending upon $\alpha$. For transverse loads, as in the present illustration, the normal deflection $\omega$ is of primary consideration and hence the anisotropic effect on the amplitude of $\omega$ is shown in the following Table 2.1

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>$\bar{\omega}_i$</th>
<th>$\bar{\omega}_o$</th>
<th>Percent variation in $\bar{\omega}_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.001784</td>
<td>0.001681</td>
<td>- 5.773</td>
</tr>
<tr>
<td>0.10</td>
<td>0.02644</td>
<td>0.02516</td>
<td>- 4.841</td>
</tr>
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<td>0.20</td>
<td>0.1919</td>
<td>0.1968</td>
<td>+ 2.554</td>
</tr>
<tr>
<td>0.50</td>
<td>0.3214</td>
<td>0.3526</td>
<td>+ 9.709</td>
</tr>
<tr>
<td>0.70</td>
<td>0.3256</td>
<td>0.3530</td>
<td>+ 9.949</td>
</tr>
</tbody>
</table>

It is seen from the table that the anisotropic effect vanishes at a polar angle in between $\alpha = 0.1 \pi$ and $\alpha = 0.2 \pi$. For values of $\alpha$ greater than this critical value the amplitude increases while for values less than it the amplitude decreases. At a polar angle $\alpha = 0.70 \pi$, the increase is as far as 10 percent approximately.

2.3. Bending of Thin Shallow Spherical Shells

The transverse displacement of a thin shallow spherical shell bent by couples applied at its edge and by uniform pressures on its faces, has been obtained when the...
boundary is simply supported. The solution is expressed in the form of a Fourier series whose coefficients depend on Kelvin functions.

Preliminaries

The general theory of thin elastic shell even with the assumption of small strain (an example of which has been given in section 2.2) is too involved to solve special problems. Some simplification is obtained in the theory of the so-called shallow shell in which the ratio of the maximum rise of the shell to its maximum span is much less than unity. General treatment of such a shallow shell was given by Marguerre [42] while Federhofer [43], Reissner [44-47], Johnson and Reissner [48,49] and others considered the bending, transverse vibration and deformation of shallow shells of spherical form. The present section concerns with the bending of a shallow spherical shell by edge couples as well as by uniform pressures on its face. The boundary of the shell is assumed to be simply supported. The solution is obtained in the Fourier series with its coefficients in terms of Kelvin functions.

Fundamental Equations and Boundary Conditions

In the bending of a shallow spherical shell by transverse load the differential equations to be solved have been put by Reissner [45,48] in the form

$$\nabla^4 f - \frac{Eh}{R} \nabla^4 w = 0$$

(1)
\[ \varphi^2 \psi + \frac{1}{R} \varphi^2 \psi = \psi (r, \theta) \]

(2)

where \( \varphi \) = Airy's stress function.

\( E \) = Young's modulus

\( t \) = wall thickness of the shell

\( R \) = radius of the middle surface of the shell

\( \psi \) = transverse (axial) displacement of the middle surface

\( D = \frac{E t^3}{12 (1 - \nu^2)} \), \( \nu \) = Poisson ratio

\( \psi \) = transverse load

\[ \nabla^2 \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \]

Here \( (r, \theta) \) are the polar co-ordinates. The stress resultants and stress couples required in the present discussion are given by

\[ N_r = \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \]

(3)

\[ N_{r\theta} = -\frac{\partial}{\partial \theta} \left( \frac{1}{r} \frac{\partial \psi}{\partial r} \right) \]

(4)

\[ M_r = -D \left( \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \right) \]

(5)

Let \( a \) be the radius of the base of the spherical shell.

If the boundary is simply supported, the conditions are
There is the further condition that all functions are finite at \( r = 0 \).

The simultaneous differential equations (1) and (2) can be written as two separate equations, each satisfying a function, in the form

\[
\nabla^2 \left( \nabla^4 + \lambda^4 \right) \omega = \frac{1}{\beta} \nabla^2 \phi \tag{10}
\]

\[
\nabla^2 \left( \nabla^4 + \lambda^4 \right) F = \lambda^4 R \phi \tag{11}
\]

where

\[
\lambda^4 = \frac{Eh}{R^3} \tag{12}
\]

Bending by Edge Couples

We assume that

\[
\phi = 0, \quad \omega = \sum_{n=0,1,2,\ldots}^{\infty} \omega_n(\theta) \cos n \theta, \quad F = \sum_{n=0,1,2,\ldots}^{\infty} F_n(\theta) \cos n \theta \tag{13}
\]

These values satisfy the differential equations (2), (10).
and (11) provided,
\[ \nabla^4 \omega + \frac{1}{R^2} \nabla^2 \omega = 0 \]  
(14)
\[ \nabla^2 (\nabla^4 + \lambda^4) \omega = 0 \]  
(15)
\[ \nabla^2 (\nabla^4 + \lambda^4) f = 0 \]  
(16)
where
\[ \nabla^2 = \frac{1}{r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{2}{r^2} \]  
(17)

The solution of (15) is obtained by the linear combination of the solutions of the equations
\[ \nabla^2 \omega = 0 \]
\[ (\nabla^2 + i \lambda^2) \omega = 0 \]
\[ (\nabla^2 - i \lambda^2) \omega = 0 \]

If the regularity condition at \( \nu = 0 \) is used we have the solution
\[ \omega = C_\nu \gamma_n + \Phi_n (\lambda \nu, \nu^2) + K_n (\lambda \nu, \nu^2) \]  
(18)
where \( J_n \) and \( I_n \) denote the Bessel function and the modified Bessel function of order \( n \) respectively. Introducing the Kelvin functions \([49,50]\) defined as
\[ J_n(x i \nu^2) = \text{ber}_n x + i \text{bei}_n x \]
\[ I_n(x i \nu^2) = e^{-\frac{1}{2} \pi i \nu} (\text{ber}_n x - i \text{bei}_n x) \]  
(19)
and using the results

\[ J_n (ix) = i^n J_n (x) \quad J_n (x) = J_n (-x) \]

\[ I_n (x) = I_n (-x) \]

we obtain

\[
\begin{align*}
J_n (\lambda r \sqrt{2}) &= \lambda e_i \lambda r - i \lambda e_i \lambda r \\
I_n (\lambda r \sqrt{2}) &= i^{-n} (\lambda e_i \lambda r + i \lambda e_i \lambda r).
\end{align*}
\]

(20)

Consequently (18) reduces to

\[
\omega_n = \xi_n r^n + A_n \lambda e_i \lambda r + B_n \lambda e_i \lambda r
\]

(21)

where \( A_n = \bar{A}_n + i^{-n} \bar{B}_n \), \( B_n = (-\bar{A}_n + i^{-n} \bar{B}_n)i \).

Similarly the solution of (16) is

\[
F_n = \gamma_n r^n + \alpha_n \lambda e_i \lambda r + \beta_n \lambda e_i \lambda r.
\]

(22)

As a consequence of the result

\[
\nabla^2 (A_n \lambda e_i \lambda r + B_n \lambda e_i \lambda r) = \lambda^2 (B_n \lambda e_i \lambda r - A_n \lambda e_i \lambda r)
\]

(23)

Substitution from (21) and (22) in (14) gives

\[
\nabla^2 \left[ \lambda^2 \left( B_n \lambda e_i \lambda r - A_n \lambda e_i \lambda r \right) + \frac{i}{R_D} (\alpha_n \lambda e_i \lambda r + \beta_n \lambda e_i \lambda r) \right] = 0.
\]
This being true for all values of $\nu$, the coefficients of $\text{Re}_n \lambda^p$ and $\text{Im}_n \lambda^p$ must separately vanish.

Hence

$$\frac{\alpha_n}{R^D} = -\lambda^2 B_n, \quad \frac{\beta_n}{R^D} = \lambda^2 A_n.$$  \hspace{1cm} (24)

so that (22) reduces to

$$f_n = Y_n \nu^n - \lambda^2 R^D \left( B_n \text{Re}_n \lambda^p - A_n \text{Im}_n \lambda^p \right).$$  \hspace{1cm} (25)

The boundary conditions (6), (7) and (9) are all satisfied if

$$\frac{\partial f_n}{\partial n} = \frac{\partial f_n}{\partial \nu} = 0 , \quad \frac{\partial f_n}{\partial \nu} - f_n = 0 , \quad \omega_n = 0 , \quad a^+ - \nu = a.$$  \hspace{1cm} (26)

These give with the help of (21) and (25) the following equations

$$n Y_n a^n + a \lambda^3 R^D \left( A_n \text{Re}_n \lambda a - B_n \text{Re}_n \lambda a \right) = 0.$$  \hspace{1cm} (27)

$$Y_n a^n + \lambda^2 R^D \left( A_n \text{Re}_n \lambda a - B_n \text{Re}_n \lambda a \right) = 0.$$  \hspace{1cm} (28)
Let it be assumed that
\[ M_\nu(a,\sigma) = \sum_{n=0,1,2,\ldots} M_n a^n \sigma^n \quad (29) \]

Substituting the value of \( c^* \) from the second equation of (13) in equation (5), putting \( \sigma = a \) and then comparing it with (29) we get
\[ \left[ \frac{3}{2 \sigma} + \nu \left( \frac{3}{2 \sigma} - \frac{\sigma}{\nu} \right) \right] \omega_n = -\frac{M_n}{D} \quad \sigma = a. \quad (30) \]

With the help of (21) and (23) the above condition gives
\[ \eta(1-\eta) c_n a^n + A_n (\lambda a \cos \nu a - \eta^2 \cos \nu a + \frac{\lambda^2 a^2}{1-\nu} \sin \nu a) \\
+ B_n (\lambda a \sin \nu a - \frac{\lambda^2 a^2}{1-\nu} \cos \nu a - \eta^2 \sin \nu a) \\
= \frac{M_n a^2}{D(1-\nu)} \quad (31) \]

Equations (26), (27), (28) and (31) determine the coefficients as
\[ A_n = -\frac{M_n a^2 X_n}{D(1-\nu) A_n} \quad B_n = -\frac{M_n a^2 \chi_n}{D(1-\nu) A_n} \]
\[ C_n = \frac{M_n}{D(1-\nu) a_n^{-2} A_n} (X_n \cos \nu a + \chi_n \sin \nu a) \quad (32) \]
\[ \gamma_n = -\frac{\lambda^2 R M_n Z_n}{(1-\nu) a_n^{-3} A_n} \]
where
\[ \chi_n = n \Lambda_n \lambda - \lambda \Lambda_n \lambda + \lambda \Lambda_n \lambda \]
\[ \psi_n = n \Lambda_n \lambda - \lambda \Lambda_n \lambda + \lambda \Lambda_n \lambda \]
\[ \zeta_n = \Lambda_n \lambda - \Lambda_n \lambda - \Lambda_n \lambda - \Lambda_n \lambda \]
\[ \Delta_n = \chi_n^2 + \psi_n^2. \]

As an example, if the shell be bent by edge couple \( M_n(\alpha, \beta) \) given by
\[ M_n(\alpha, \beta) = k \cdot (a \text{ constant}), \quad 0 < \beta \leq 2\pi - \beta \]
then from (29) we have
\[ M_n = \frac{2k \sin n \beta}{n}. \] (34)

The displacement and stresses can now be easily calculated with the help of (21) and (25).

**Bending by Transverse Load**

If the shell is bent by uniform pressure \( p = p_0 \) (a constant) distributed on the face, all functions will be independent of \( \beta \) and the equations (1) and (2) reduce to
\[ V_0^4 F = - \frac{F_k}{R} V_0^2 \omega = 0, \quad D V_0^4 \omega + \frac{1}{R} V_0^2 F = 0. \] (35)

where \( V_0^2 = \frac{3}{2} \frac{\partial^2}{\partial \tau^2} + \frac{1}{2} \frac{\partial^2}{\partial \tau^2} \), in conformity with (17), if we note that in this case \( n = 0 \). A particular solution
The complementary functions corresponding to (35) are the solutions of (15) and (16) and are of the forms (21) and (25) with $n=0$. Thus writing $\overline{\omega}$ for $\omega$ (when $n=0$), we have

$$\overline{\omega} = s_1 + s_2 \text{ber} \; \lambda \varphi + s_3 \text{bei} \; \lambda \varphi$$

$$F = \frac{1}{4} R \phi_0 r^2 + A_4 - \frac{1}{2} R^2 \left( 2 \text{ber} \; \lambda \varphi - n_2 \text{bei} \; \lambda \varphi \right)$$

(36)

if the Kelvin functions of order zero are written in Thomson's notations

$$\text{ber}_0 \; x = \text{ber} \; x, \quad \text{bei}_0 \; x = \text{bei} \; x.$$

Since only derivatives of the stress function $F$ are required we may take $A_4$ to be identically zero. The condition (7) is automatically satisfied while conditions (6) and (9) give

$$s_3 \text{ber}' \lambda a - s_2 \text{bei}' \lambda a = \frac{1}{2} \frac{A\phi_0}{\lambda \beta \rho} \quad (37)$$

$$s_1 + s_2 \text{ber} \lambda a + s_3 \text{bei} \lambda a = 0. \quad (38)$$

With the help of (5) and the identities

$$\text{ber}'' x = -\text{bei} x - \frac{x}{2} \text{ber}' x, \quad \text{bei}'' x = \text{ber} x - \frac{x}{2} \text{bei}' x$$

the condition $M_{p\rho} = 0$ on $r = a$ gives
Equations (37 - 39) determine the coefficients as

\[ s_1 = \frac{(\alpha \phi_0 s_1)}{(2 \lambda^3 \Delta)} \]

(40)

\[ s_2 = -\frac{(\alpha \phi_0 s_2)}{(2 \lambda^3 \Delta)} \]

(41)

\[ s_3 = \frac{(\alpha \phi_0 s_3)}{(2 \lambda^3 \Delta)} \]

(42)

where

\[ \Delta = (1-v) \left\{ \left( \text{er}^2 \lambda a \right)^2 + \left( \text{er}^3 \lambda a \right)^2 \right\} + \lambda a \left( \text{er} \lambda a \cdot \text{er}^2 \lambda a \right. \]

\[ - \left. \text{er}^2 \lambda a \cdot \text{er}^3 \lambda a \right\} \]

\[ s_1 = (1-v) \left( \text{er}^2 \lambda a \cdot \text{er}^3 \lambda a - \text{er} \lambda a \cdot \text{er}^2 \lambda a \right) \]

\[ - \lambda a \left\{ \left( \text{er}^2 \lambda a \right)^2 + \left( \text{er}^3 \lambda a \right)^2 \right\}, \]

\[ s_2 = (1-v) \cdot \text{er}^2 \lambda a - \lambda a \cdot \text{er} \lambda a, \]

\[ s_3 = (1-v) \cdot \text{er}^3 \lambda a + \lambda a \cdot \text{er} \lambda a. \]

Thus the displacement function and the stress function being known the stresses and the moments can be easily calculated.

2.4: Radial Vibration in a Sphere of Variable Modulus of Elasticity

The radial vibration of a sphere whose modulus of elasticity varies linearly as the distance from the centre has been investigated by the method of perturbation. It is

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seen that corresponding to a variation of about ten per cent in the modulus of elasticity, the variation in the angular frequency is approximately two per cent.

Introductory Remark

In the classical theory of elasticity it is generally assumed that the elastic property of the medium is the same at all points throughout the body. Such an assumption not only simplifies the mathematical complexity to a great extent but also gives a quite satisfactory results in practical applications. In recent years however much attention is being paid to the non-homogeneity of the material [51]. Statical deformations in a medium whose elastic properties are functions of the position have been considered by a number of authors [53,55-60,66,67]. There exists an extensive literature, both experimental and theoretical, on the propagation of wave motion in elastically heterogeneous bodies, a comprehensive account of which may be found in a recent monograph [54] and elsewhere [62]. In fact, investigations into seismic waves on the assumption that the modulus of elasticity of the earth varies with depth have resulted in the explanation of many geophysical phenomena. This again has stimulated the work on the vibration of spherical bodies whose elastic properties depend upon the radial distance and it is hoped that the period of oscillation of the earth may be determined by such a model [53-65].
In the present section we shall consider the free radial vibration of a sphere of which the Young's modulus is a linear function of the radial distance, as is generally assumed in the earth model. The Poisson ratio is however treated as a constant and this assumption is consistent with many experimental facts [54]. The problem is then reduced to the solution of a differential equation whose coefficients are functions of the independent variables. It is a common experience to encounter with differential equations having variable coefficients in problems of elastically heterogeneous bodies. These may be solved numerically with the help of high-speed computers or by other techniques. But in the present case we have tried to find an approximate solution analytically by applying the method of perturbation when the vibration parameter is small compared to $\omega$. The results are thus obtained in a simple and straightforward way. It is found that the angular frequency varies by two per cent corresponding to a variation of ten per cent for the modulus of elasticity [69].

Fundamental Equations

We use the spherical coordinates $(r, \theta, \phi)$. For the radial vibration of a sphere all functions will depend only on the radius $r$ and time $t$. The displacement components in this case are $u_r = u(r, t)$, $u_\theta = u_\phi = 0$ and consequently the non-vanishing strain components are $\varepsilon_{rr} = \frac{2u}{r^2}$ and $\varepsilon_{\theta\theta} = \varepsilon_{\phi\phi} = \frac{u}{r^2}$. The stress-strain relations by Hooke's law reduce to
where $\nu$ is the constant Poisson ratio. We assume that
\[
e = E_o \left[ 1 + \epsilon \left( \nu / \nu_o \right) \right]
\]

where $\nu_o$ is the radius of the sphere, $\epsilon$ is the variation parameter and $E_o$ is the value of $E$ at $\nu = \nu_o$. The above equations give
\[
\begin{align*}
\sigma_{rr} &= \frac{E}{(1+\nu)(1-2\nu)} \left\{ (1-\nu) \epsilon_{rr} + 2\nu \epsilon_{\theta \theta} \right\}, \\
\sigma_{\theta \theta} &= \frac{E}{(1+\nu)(1-2\nu)} (\nu \epsilon_{rr} + \epsilon_{\theta \theta}).
\end{align*}
\]

(1)

In the absence of body forces the equation of motion is
\[
\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} (r \sigma_{rr} - \sigma_{\theta \theta}) = \hat{\rho} \frac{\partial u}{\partial t}
\]

(2)

where the density $\hat{\rho}$ is assumed as a constant. For a periodic motion, $u = u(r) e^{i\omega t}$ and the relations (1) and (2) give
\[
\frac{\partial}{\partial r} \left[ \frac{E}{(1-2\nu)} \left( \frac{du}{dr} + \frac{2u}{r} \right) \right] + \nu \left[ \frac{f(1+\nu)}{r^2} - \frac{2\epsilon'}{\nu} \right] u = 0
\]

(3)

where $\nu = (1-2\nu)/(1+\nu)$. A prime on a function denotes its differentiation with respect to $r$. If the function $\nu$ is defined as
\[
\left[ \frac{f(1+\nu)}{r^2} - \frac{2\epsilon'}{\nu} \right] u = \frac{d\nu}{dr}
\]

(4)

the relation (3) is satisfied provided
Elimination of $\psi$ between (4) and (5) determines the differential equation for $\psi$.

\[ \left\{ f(1+\psi) b^2 - \frac{2E'}{E} \int \frac{d^2 \psi}{dr^2} + \frac{\nu}{E} \left( \frac{b}{r} \right)^2 \frac{1}{\frac{E'}{E}} \int \frac{d^2 \psi}{dr^2} \right\} = 0. \]

For a complete sphere all physical quantities must be finite at $r = 0$. The condition for the surface being free from stresses is that on $r = r_0$ we have $\sigma_r = 0$. The latter condition gives

\[ \left\{ f(1+\psi) b^2 - \frac{2E'}{E} \int \frac{d^2 \psi}{dr^2} + \frac{\nu}{E} \left( \frac{b}{r} \right)^2 \frac{1}{\frac{E'}{E}} \int \frac{d^2 \psi}{dr^2} \right\} = 0. \]

Perturbation Solution

When the variation parameter is small we put

$\psi = \psi_0 + \varepsilon \psi_1 + \varepsilon^2 \psi_2 + \cdots$ and $\rho = \rho_0 + \varepsilon \rho_1 + \varepsilon^2 \rho_2 + \cdots$.

Confining our attention to terms up to the first power of $\varepsilon$, it is found from (6) that $\psi_0$ and $\psi_1$ satisfy the equations

\[ \frac{\partial^2}{\partial r^2} (r \psi_0) + \lambda_0 (r \psi_0) = 0, \quad \lambda_0^2 = \frac{f(1+\psi_0) b^2}{\varepsilon_0}. \]

(8)
In deducing (9) the second derivative $\psi_0''$ has been eliminated with the help of (8). Again, if the coefficients of different powers of $\epsilon$ are equated to zero the boundary condition (7) reduces to

$$2\nu_1' \psi_0' + 4\nu_1 \psi_0 = 0$$

on $r = r_0$. \hspace{1cm} (10)

$$2\nu_1 (r \psi_1)' + (4\nu \epsilon^2 - 2\nu_1) \psi_1 - \left(\frac{4\nu \epsilon^2}{\nu_0} - \frac{2\nu \epsilon^2}{\nu_0} + \frac{2\nu_1}{\nu_0}\right) (r \psi_0) = 0$$

on $r = r_0$. \hspace{1cm} (11)

The second derivatives $\psi_0''$ and $(r \psi_1)''$ have again been eliminated from (10) and (11) with the help of (8) and (9). The solution of (8) for a complete sphere may be taken as

$$r \psi_0 = A \sin \left(\frac{\nu_0}{r_0} r\right)$$

where $A$ is a constant. The condition (10) is satisfied if

$$\tan\left(\frac{\nu_0}{r_0} r_0\right) = \frac{(2\nu_1 \nu_0 \psi_0)'}{(2\nu_1 - 4\nu \epsilon^2)}$$

which is the classical period equation for $\nu_0$. \hspace{1cm} (12)

The complementary function for the equation (9) is taken as $r \psi_1 = B \cos \left(\frac{\nu_0}{r_0} r\right)$, where $B$ is a constant. The other solution $r \psi_1 = -\sin \left(\frac{\nu_0}{r_0} r\right)$ is omitted, because a similar term has already been taken in $\psi_0$. The particular integral is written in the operator form as
\[ r(y) = \frac{1}{y^2 + k^2} \left[ \frac{2y^2 \Psi_0'}{k} + \left( \frac{2k^2}{r_0} - \frac{2k^2}{r_0} + \frac{2y}{r_0} \right) \Phi_0 \right] \]

\[ D = \frac{d}{dr} \]  

(13)

In order to evaluate the right side of (13) we substitute the value of \( \Psi_0 \) and use the following formulae and results:

\[ \int \frac{\sin y}{y} \, dy = -\frac{\sin y}{y} + Ci y, \]
\[ \int \frac{\cos y}{y} \, dy = -\frac{\cos y}{y} - Si y, \]
\[ \int \frac{\sin y}{y^2} \, dy = Si y, \quad \int \frac{\cos y}{y^2} \, dy = Ci y \]

\[ \frac{1}{y^2 + k^2} \left( \frac{\Psi_0'}{y} \right) = -\frac{1}{2} \frac{\sin(2\pi y)}{y}. \]

Here \( Si y \) and \( Ci y \) denote sine and cosine integrals [Ref. 51, pp.10, 115, 136]. The complete solution for \( y_1 \) is then obtained by adding the particular integral with the complementary function. Thus

\[ r(y_1) = B \cos(\alpha y_1) + A \int - \frac{\sin\alpha y_1}{\alpha y_1} - \frac{\sin\alpha y_1}{\alpha y_1} \left( \frac{\sin\alpha y_1}{\alpha y_1} - \frac{\sin\alpha y_1}{\alpha y_1} \right) \]

\[ + \frac{\cos\alpha y_1}{\alpha y_1} \right] + \frac{\psi_1}{\psi_0} \int \cos\alpha y_1 \left\{ Ci(\alpha y_1) - \ln y_1 \right\} \]

\[ + \sin\alpha y_1 \cdot Si(\alpha y_1) \]  

(14)

We are to satisfy the condition that \( y_1 \) must be finite as \( y \to 0 \). Remembering again that for small values of there hold the results \( \frac{\sin\alpha y_1}{\alpha y_1} \approx \frac{\alpha y_1}{y_0} \) and
where \( y = 1.78 \) is the Euler constant \([61]\) we obtain from (14)

\[
\left( \frac{\delta}{\eta} \right) = \frac{\nu}{\eta \eta_0} \left[ 1 - \ln (2\eta_0y) \right].
\]

The condition (11) with the help of (14) and (15) gives the period equation for \( \phi_1 \) as

\[
(\psi_1/\phi_0) = \left[ \frac{\gamma \cos \delta + 2(1-\nu) \sin \delta}{1 - \ln (2\eta_0y)} \right] + \frac{\nu \gamma}{1 - \ln (2\eta_0y)} \left[ \cos (\gamma \cos \delta - \gamma \sin \delta) \right] + \frac{1}{4} \gamma^2 \left[ \cos (\gamma \cos \delta + \gamma \sin \delta) \right].
\]

where \( \gamma = \frac{\psi_1}{\phi_0} \).

In the particular example if \( \nu = \frac{1}{4} \) i.e. \( \nu_1 = \frac{5}{3} \) it is known that \([19, p.285]\) \( \gamma = 0.8160 \pi \) for the lowest mode. Substituting these values of \( \nu_1 \) and \( \gamma \) in (16) it is obtained that \( (\psi_1/\phi_0) = 0.2098 \). If \( \epsilon = 0.1 \) the angular frequency \( \phi \) has the value \( \phi = \phi_0 (1 + 0.0210) \).

Thus for a variation of 10 per cent in the modulus of elasticity the variation in the angular frequency is approximately 2 per cent.
27. __________, (1948), ............... 27, 240.
35. Timoshenko, S., (1940), Theory of plates and shells.


