Chapter VIII

"The human mind is not exhausted; it searches and continues to find so that it may know that it can find indefinitely."

— Bossuet.
CHAPTER VIII

ROTATORY FLOW OF VISCOPLASTIC MATERIAL OF 
BINGHAM TYPE

Rotatory flow of viscoplastic material of Bingham type between two coaxial circular cylinders has been investigated for the constitutive equations developed by Oldroyd. For steady flow both 'free' and 'plug' motions have been discussed, while in unsteady flow only free motion has been considered. Particular emphasis has been put on determining the critical value of the Bingham number above or below which the flow pattern is a possible one.

8.1. Introduction

Since Bingham and Green[2] discovered in 1919 that oil paint (before drying) is a plastic solid and not a viscous liquid, attempts have been made to explain the actual mechanical behaviour of similar material by combining linearly the laws governing the Newtonian viscous liquid and the perfectly plastic Mises solid. Following Bingham [1] and Houwink [6], a material which can support a finite stress elastically without flow and which flows with constant mobility (or plastic fluidity) when the stresses are sufficiently great, is called a viscoplastic solid of Bingham type.
The mathematical formulation of the equations of flow for this solid has been given by Oldroyd [10]; an account may also be found in works of Reiner and Prager [18,5]. As illustrations of the general theory, special problems solved so far, either exactly or approximately, include the steady rectilinear flow (i) through a circular pipe by Buckingham and Reiner, (ii) between two parallel planes by Prager and (iii) flow between co-axial rotating circular cylinders by Reiner and Rivlin [3,17,5]. In a series of papers Oldroyd [11, 12,14,15] has considered steady rectilinear flow between co-axial cylinders, eccentric circular cylinders, confocal elliptic cylinders and cylinders of arbitrary cross-section as well as unsteady rectilinear flow between parallel planes and through a circular pipe. He [13] has also introduced the concept of plastic boundary layer and Bingham number in analogy with Prandtl viscous boundary layer and Reynolds number respectively. The only uniqueness theorem concerning stress and velocity fields is due to Prager [16]. Ilyushin and Ishlinky [7,8] have discussed several problems regarding the stability of viscoplastic flow, an account of which may be found in the work of Nadai [9]. Recently Sliber and Paslay [20] has solved a problem of flow between coaxial cylinders. In the present section the rotatory flow both steady and unsteady, between coaxial circular cylinders has been investigated [21]. The inner cylinder is supposed to be at rest while the flow is caused by the rotation of the outer one. Though the characteristic feature for the steady motion is known already, its fresh derivation and discussion with reference to Bingham number clarifies ideas that follow.
The steady flow occurs throughout the whole region if the Bingham number is less than a critical value, while for its greater values, flow region and elastic region (plug motion) occur side by side. In the unsteady flow we have considered the case when the whole region is in motion. It is found that for a prescribed Bingham number, such a motion is possible only for a particular interval of time.

3.2. Fundamental Equations

The rheological equations of state are [10, 18]

\[ p_{ik} = 2 \kappa \lambda (\text{in elastic regions only, } \frac{1}{2} \delta_{ik} \pi_{ik} \leq \nu^2) \]  
\[ p_{ik} = 2 \mu \epsilon_{ik} (\text{in elastic regions only, } \frac{1}{2} \delta_{ik} \pi_{ik} \leq \nu^2) \]  
\[ p_{ik} = 2 \eta \epsilon'_ik (\text{in flow regions only, } \frac{1}{2} \delta_{ik} \pi_{ik} > \nu^2) \]

with

\[ \eta = \eta_1 \nu (2 \epsilon_{ik} \epsilon_{ik})^{-\frac{1}{2}} \]  

In the above equations, \( \eta_1 \) is the (constant) reciprocal mobility, \( \eta \) is the (constant) yield value, \( \mu \) is the (constant) rigidity modulus in the elastic region and \( \kappa \) (not necessarily constant) is the bulk modulus, whereas \( \epsilon_{ij} \) is the strain tensor, \( \epsilon'_{ik} \) is the rate of strain tensor, \( \epsilon_{ii} \) is the dilatation, \( \pi_{ij} \) is the stress tensor, and primes denote deviatoric components of tensors. For example
\[ p_{\kappa} = p_{\kappa} + p \delta_{\kappa} \quad p = -\frac{1}{2} \varepsilon_{\kappa} \]  

(5)

\( \delta_{\kappa} \) being the substitution tensor. On the yield surface, which is defined as the surface of separation between elastic regions and plastic flow regions, the transition conditions are that \( \varepsilon_{\kappa} \) must vanish identically and the velocity must be continuous, the elastic region being treated as rigid. The equations of motion and continuity are:

\[ \rho \frac{du}{dt} = \frac{\partial p}{\partial \kappa} - \frac{\partial p}{\partial \kappa} + \rho \kappa, \]  

(6)

\[ \frac{\partial p}{\partial t} + \rho \kappa \kappa = \rho \varepsilon, \]  

(7)

where \( u \) is the velocity vector, \( \kappa \) is the external force vector, \( \rho \) is the density and \( \frac{\partial}{\partial t} \) denotes differentiation with respect to time following the material particle. We also have

\[ \varepsilon_{i,j} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \]  

(8)

The above equations with appropriate boundary conditions on stresses, velocity and pressure determine the velocity field as well as the yield surface as a part of the solution.
8.3. Steady Flow

We assume the velocity components in the cylindrical coordinates \((r, \theta, z)\) to be

\[
\begin{align*}
    u_1 &= u_r = 0, \\
    u_2 &= u_\theta = 0, \\
    u_3 &= u_z = u(r),
\end{align*}
\]

where \(u\) is a function of \(r\) only. From (3), (4) and (8) the nonvanishing components of strain-rate and stress tensors are

\[
\begin{align*}
    \varepsilon_{\theta\theta} &= \varepsilon_{\theta\theta} = \frac{1}{2} \left( \frac{du_r}{dr} - \frac{u_\theta}{r} \right), \\
    \sigma_{\theta\theta} &= \sigma_{\theta\theta} = 2\eta \varepsilon_{\theta\theta} + \frac{\varepsilon_{\theta\theta}}{|\varepsilon_{\theta\theta}|}.
\end{align*}
\]

If \(\varepsilon_{\theta\theta}\) is assumed to be positive everywhere in the flow region, the latter equation can be written as

\[
\sigma_{\theta\theta} = \sigma_{\theta\theta} = 2\eta \varepsilon_{\theta\theta} + u_r.
\]

The equation of continuity (7) is satisfied if \(\rho\) is taken as a constant, while the equations of motion (6) in the absence of body forces reduce to

\[
\begin{align*}
    \frac{d}{dt} \left( \rho \frac{u_r}{r} \right) - \frac{\partial}{\partial r} \left( \rho \frac{u_r}{r} \right) + \frac{\partial}{\partial \theta} \left( \rho \frac{u_\theta}{r} \right) - \frac{1}{r} \frac{\partial}{\partial \theta} \left( \rho \frac{u_\theta}{r} \right) - \frac{1}{r^2} \frac{\partial}{\partial r} \left( \rho \frac{u_r}{r^2} \right) = 0, \\
    \frac{\partial}{\partial r} \left( \rho \frac{u_\theta}{r} \right) - \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( \rho \frac{u_\theta}{r^2} \right) = 0.
\end{align*}
\]

which show that \(\rho\) can be taken as a function of \(r\) only. Substituting (10) in the second relation of (11) and...
Integrating we get
\[ 2 \eta_1 e_{10} + \varphi = \frac{2c}{r^2} \]  
\( (12) \)

where \( c \) is a constant. Again, substitution of (9) in (12) and then integration give

\[ \eta_1 u = D r - \frac{c}{r} - a r \ln r, \]
\( (13) \)

where \( D \) is another constant. If the flow occurs throughout the whole region \( r_1 < r < r_e \), the boundary conditions \( u = 0 \) on \( r = r_1 \) and \( u = U \) on \( r = r_e \) determine \( \eta_1 \) and \( D \) as

\[ C = \frac{\eta_1 U r_e}{1 - a^2} \left( a - B \log a \right), \]
\( (14) \)

\[ D = \frac{\eta_1 U}{(1 - a^2) r_e} \left[ 1 - B a \log a + \frac{B}{a} (1 - a^2) \log r_e \right], \]

where \( B = \frac{2\varphi r_e}{\eta_1 U} \) is the dimensionless Bingham number, and \( a = r_1/r_e \). Relations (13) and (14) imply

\[ \frac{U}{U} = \frac{1}{1 - a^2} \frac{r_e}{r} \left( 1 - B a \log a \right) - \frac{a}{1 - a^2} \frac{a}{r_e} \left( a - B \log a \right) + \frac{B}{a} \frac{r_e}{r} \left( \frac{B}{2} \right), \]
\( (15) \)

The relation (12) now reduces to

\[ e_{10} = \frac{a U r e}{1 - a^2} \left( a - B a \log a \right) \frac{1}{r^2} - \frac{B U}{2 a r e} \]
\( (16) \)

It is to be noted that \( e_{10} \) decreases as \( \gamma \) increases.
In order to satisfy the condition of flow viz \( \eta > 0 \) in the whole region \( \eta_i < \eta < \eta_e \), we replace \( \eta \) by \( \eta_e \) in (16) and thus get a critical value \( B = B_1 \) such that

\[
B < B_1 = \frac{2a^3}{1 - a^2 + 2a^2 \log a} \quad \text{(12)}
\]

If \( B > B_1 \), the flow is confined in a certain region \( \eta_i < \eta < R \) \( (R < \eta_e) \) and the region \( R < \eta < \eta_e \) moves as a rigid body. In this case the boundary conditions are

\[
u = 0 \text{ on } \eta = \eta_i \quad \text{and} \quad \frac{\partial \eta}{\partial \eta} = \frac{vR}{\eta_e} \text{ on } \eta = R,
\]

while relations (14), (15) and (16) are modified as

\[
c = \frac{\alpha_1 \eta e}{1 - a^2} \left[ \frac{a - B \log a}{\eta_e^2} \right],
\]

\[
d = \frac{\eta e}{(1 - a^2) \eta_e} \left[ 1 + \frac{B}{a} \left( \log R - a^2 \log \eta_i \right) \right],
\]

\[
u = \frac{1}{1 - a^2} \left( \frac{\eta}{\eta_e} \left( 1 - \frac{B a^2}{a} \log a + \frac{B}{a} \log \frac{a}{a_i} \right) \right)
\]

\[
- \frac{\eta}{1 - a^2} \left( \frac{\eta}{\eta_e} \left( a - B \log a \right) + \frac{B a}{\eta_e} \log \left( \frac{\eta}{\eta_e} \right) \right),
\]

\[
\frac{\partial \eta}{\partial \eta} = \frac{\alpha_1 \eta e}{1 - a^2} \left( a - B \log a \right) \frac{1}{\eta^2} - \frac{B v}{2a \eta_e},
\]

where \( a_1 = \frac{\eta_i}{R} \).

To satisfy the condition \( \eta e > 0 \) for \( \eta_i < \eta < R \), we replace \( \eta \) by \( R \) in (16a) and get a critical value \( B = B_2 \) such that.
\[ B < B_2 = \frac{2a \alpha_1}{1 - \alpha_1^2 + 2\alpha_1^2 \ln \alpha_1} \] 

(17a)

It is to be noticed that \( \alpha_1 \) lies between the least value \( \alpha \) and the greatest value unity (since \( \tau_1 < \alpha < \tau_2 \)), and \( B = (2a \alpha_1)/(1 - \alpha_1^2 + 2\alpha_1^2 \ln \alpha_1) \) is an increasing function of \( \alpha_1 \). The value of \( B \) increases from the least value \( B \) (corresponding to \( \alpha_1 = \alpha \)) to infinity (corresponding to \( \alpha_1 = 1 \)). Conversely, \( \alpha_1 \) may be defined as the increasing function of \( B \), and \( \alpha_1 \) tends to zero as \( B \) tends to zero while the line \( \alpha_1 = 1 \) serves as an asymptote for large values of \( B \). A Newtonian liquid is a limiting case of a Bingham solid with \( \beta = \psi = \phi = \sigma = \infty \) and consequently \( \alpha_1 = 0 \) i.e. \( R \) infinite. As \( B \) increases from zero the possible region of flow contracts gradually, and when a critical value \( B_1 \) is passed, flow in the whole region between the cylinders ceases to be possible. A plug of elastic solid then forms at the outer boundary, the region of flow contracts as \( B \) increases further and is confined to a thin sheath surrounding the inner cylinder for very large values of \( B \). However, for finite values of \( B \) (greater than \( B_1 \)), \( R \) is finite i.e. there is a marked yield surface. Table 8.1 gives values of \( B \) for different values of \( \alpha_1 \) when \( \alpha_1 \) is prescribed. Since \( \alpha_1 \) is equal to, \( B \) for flow throughout the whole region \( \tau_1 < \tau < \tau_2 \), the numbers written diagonally in Table 8.1 correspond to the critical value \( B = B_1 \), and these can also be verified from the result (17).
Table 8.1

Values of \( B \) as a function of: \( \alpha \) and \( \alpha_i \):

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>0.2</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
</tr>
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<td>0.2</td>
<td>0.0192</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>0.4</td>
<td>0.1171</td>
<td>0.2342</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>0.5</td>
<td>0.2479</td>
<td>0.4958</td>
<td>0.6197</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>0.6</td>
<td>0.5290</td>
<td>1.0580</td>
<td>1.3221</td>
<td>1.5870</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>0.8</td>
<td>3.4453</td>
<td>6.8904</td>
<td>8.6133</td>
<td>10.3359</td>
<td>12.7812</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>0.9</td>
<td>15.7480</td>
<td>31.4960</td>
<td>39.3701</td>
<td>47.2441</td>
<td>62.9921</td>
<td>70.8662</td>
<td>-</td>
</tr>
</tbody>
</table>

8.4. Unsteady Flow

The velocity components are taken to be \( u_1 = u_2 = 0 \) and \( u_3 = u_0 = u(\tau, t) \), where \( u_0 \) is now a function of \( \tau \) and \( t \). The non-vanishing components of rate of strain and stress are still given by (9) and (10). The equation of continuity is satisfied by taking \( \rho \) as a constant. If \( p(\tau, t) \) is assumed to be a function of \( \tau \) and \( t \) only, the third equation of motion is identically satisfied, while the first two reduce to

\[
\frac{\partial u}{\partial x} = \frac{\partial p}{\partial x}, \quad \frac{\partial \rho}{\partial x} = \frac{\partial p}{\partial x} + \frac{\partial \rho}{\partial x}.
\]

When expressed in terms of velocity component with the help of (9) and (10) the second of the above equations implies

\[
\rho \frac{\partial u}{\partial t} = k \eta_1 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} \right) + \frac{\partial \rho}{\partial x}. \tag{18}
\]
We consider the simple case in which the initial state is one of rest and the motion is set up by a uniform acceleration $\gamma$ of the outer boundary. We also suppose that motion takes place in the whole of the region $r < r_e$. The boundary conditions are then

$$u = 0, \quad t = 0, \quad r < r_e,$$
$$u = \gamma t, \quad t > 0, \quad r = r_e,$$
$$u = 0, \quad t > 0, \quad r = r_0.$$  \hspace{1cm} (19)

The Laplace transform $\tilde{u}(r, \beta) = \int_0^\infty u(r, t) e^{-\beta t} \, dt$ of (18) and (19) reduces the problem to the solution of the differential equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - (\lambda^2 + \frac{1}{r^2}) u = - \frac{2 \gamma}{\eta \beta} r,$$
$$\lambda = \frac{r_0}{\eta},$$

with the boundary conditions $\tilde{u} = 0$ on $r = r_0$ and

$$\tilde{u} = g |r|^2$$ on $r = r_e$.

When this is solved we have $[19]$

$$\tilde{u} = C_1 I_1(\lambda r) + C_2 K_1(\lambda r) + \frac{2 \gamma}{\eta \beta^2} r,$$  \hspace{1cm} (21)

where $I_1$ and $K_1$ are modified Bessel functions of order one of the first and the second kinds respectively and constants $C_1$ and $C_2$ are given by

$$C_1 = \left( \frac{\gamma}{r_0} - \frac{2 \gamma^2}{\eta \beta^2 r_e^2} \right) K_1(\lambda r_0) + \frac{2 \gamma}{\eta \beta^2} K_1(\lambda r_e),$$  \hspace{1cm} (22)
\[ E_0^2 = - \left[ \left( \frac{1}{2} \right) - \frac{32}{15} \right] I_1(\lambda \tau_t) + \frac{23}{15} \right] I_1(\lambda \tau_t) \]  

(23)

with

\[ E = I_1(\lambda \tau) K_1(\lambda \tau) - I_1(\lambda \tau_t) K_1(\lambda \tau_t). \]  

(24)

The inverse Laplace transform of (21) gives the velocity distribution \( u \). Since motion for only small values of time is of interest we expand various expressions for large values of \( \xi \), and then take the Inversion.

If we use the asymptotic expansions [19]

\[ I_1(x) = \frac{e^x}{\sqrt{\pi x}} \left( 1 - \frac{3}{8x} + \cdots \right), \]

\[ K_1(x) = \sqrt{\frac{\pi}{2x}} e^x \left( 1 + \frac{3}{8x} + \cdots \right), \]

and the formulae \( \sinh x = \cosh x = \frac{1}{2} e^x \) for large \( x \), we get

\[ K_1(\lambda \tau_t) I_1(\lambda t) - I_1(\lambda \tau_t) K_1(\lambda t) \]

\[ = \frac{1}{2 \lambda \sqrt{\tau_t \tau}} \left\{ 1 + \frac{3}{8 \lambda} \left( \frac{1}{\tau_t} - \frac{1}{\tau} \right) \right\} e^{\lambda(\tau - \tau_t)}, \]  

(25)

\[ \frac{1}{E} = 2 \lambda \sqrt{\tau_t \tau} \left\{ 1 - \frac{3}{8 \lambda} \left( \frac{1}{\tau_t} - \frac{1}{\tau} \right) \right\} e^{\lambda(\tau - \tau_t)}, \]  

(26)

and similar results, neglecting terms of \( \lambda^{-2} \) and higher orders. With the help of results from (22) to (26), the relation (21) simplifies to
\[ \bar{u} = \sqrt{\frac{\nu}{\pi}} \cdot \frac{1}{3} \left( 3 - \frac{32}{\pi^2} \right) \left\{ 1 - \frac{2}{3} \sqrt{\frac{\pi}{6}} \cdot \frac{1}{16} \left( \frac{1}{\nu} - \frac{1}{\nu_1} \right) \right\} e^{-\frac{\nu_1^2}{\nu}} (x - y) \]
\[ + \frac{22}{3^2} \cdot \frac{1}{16 \gamma} \left\{ 1 - \frac{2}{3} \sqrt{\frac{\pi}{6}} \cdot \frac{1}{16} \left( \frac{1}{\nu_1} + \frac{1}{\nu_2} \right) \right\} e^{-\frac{\nu_1^2}{\nu}} (2x - y - y_0) \]
\[ + \frac{22}{3^2} \cdot \frac{1}{16} \]  

(27)

The inversion of the above relation gives

\[ \frac{u}{V} = \frac{1}{\sqrt{\alpha V_1}} \left( (\alpha_1 - 2\alpha_0) \frac{4\pi^2}{\alpha V_1} \text{erfc} \frac{\alpha - \nu_1}{2\sqrt{\alpha}} + 3 \left( \frac{\alpha - \nu_1}{2\sqrt{\alpha}} \right)^2 \text{erfc} \frac{\alpha - \nu_1}{2\sqrt{\alpha}} \right) \]
\[ + \frac{2\pi}{\alpha V_1} \left[ 4\pi \text{erfc} \frac{\alpha - \nu_1}{2\sqrt{\alpha}} - 3 \left( 1 + \frac{\alpha - \nu_1}{2\sqrt{\alpha}} \right)^2 \text{erfc} \frac{\alpha - \nu_1}{2\sqrt{\alpha}} \right] \]
\[ + \frac{2\pi}{\alpha V_1} \]

(23)

where \( \alpha_1, \gamma \) and \( \gamma_0 \) are dimensionless quantities defined by

\[ \gamma = \gamma \alpha_1, \quad \gamma_1 = \gamma \alpha_2, \quad \gamma_0 \gamma_1 u = \frac{\nu_1}{\alpha} \]

while \( \alpha \) and \( \beta \) are defined as before, and \( V \) in this problem is be treated as a reference velocity used to define \( B \). In order to satisfy the condition of flow, viz. \( \nu \gamma_0 > 0 \) in the whole region \( \gamma_1 < \gamma < \gamma_0 \), we write

\[ \frac{u}{V} = \frac{4\pi}{\sqrt{\alpha V_1}} (\alpha_1 - 2\alpha_0) \text{erfc} \frac{\alpha - \nu_1}{2\sqrt{\alpha}} + \frac{8\pi}{\alpha V_1} \text{erfc} \frac{\alpha - \nu_1}{2\sqrt{\alpha}} \]
\[ + \frac{2\pi}{\alpha V_1} \]

(28a)

neglecting \( \text{erfc} \), which means the same as retaining terms up to \( \frac{1}{5^2} \) in (27). Nothing the results.
Table 8.2 gives the values of \( 10^2 \frac{F(t)}{\sqrt{2}} \) for different values of \( \sqrt{2} \). While Fig. 8.1 gives the graphical representation. The negative values corresponding to \( \sqrt{2} > 1 \) are inadmissible, \( \frac{\partial}{\partial t} \) being essentially positive. It is seen from the figure that as \( \tau \) increases, \( \frac{F(t)}{\sqrt{2}} \) gradually increases, takes a maximum value at a point in between \( \sqrt{2} = 0.56 \) and \( \sqrt{2} = 0.62 \) and then gradually decreases. The implies that for prescribed value of \( \frac{\partial}{\partial t} \), there are two values of \( \tau \) between which \( \frac{F(t)}{\sqrt{2}} \) holds, the inequality, \( \{F(t)/\sqrt{2}\} \geq \{\partial/\partial t\} \), and hence the flow pattern is a possible one within this period only. The flow does not exist if \( \frac{\partial}{\partial t} \) is greater than 4.24 approximately.

<table>
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<th>( 10^2 \frac{F(t)}{\sqrt{2}} )</th>
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<td>-16.60</td>
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References

18. Reiner, M., Ten lectures on Theoretical Rheology (1943) 43 and 133.