Chapter 4

FIXED POINT THEOREMS IN COMPLETE METRIC SPACES

4.1 Introduction: The study of fixed point theory which is commonly established in three orders: simplification of circumstances which confirm presence, as well as if possible, matchlessness, of fixed points; research of the atmosphere of the order of repeats \( \{T^n_x\}_{n=0}^\infty \), wherever \( T: X \to X, X \) a whole metric space, which is plan beneath concern; study of topological things of the established of fixed points, at any time \( T \) takes in excess of single fixed point. Extra exactly we contemplate records \( T:X \to X \), which fulfill circumstances of the form

\[
d(Tx, Ty) \leq \varphi(d(x, y)) + \psi(d(Tx, x)) + \chi(d(Ty, y))
\]

Aimed at every \( x, y \in X \), and for these mappings we verify, improper theories, presence and individuality of fixed points.

During the course of this theme, \( X \) signifies a whole metric space and, \( T:X \to X \), an asymptotically consistent mapping i.e., a meaning sufficient

\[
\lim_{n} d(T^n x, T^{n+1} x) = 0 \quad \text{On behalf of every } x, \in X.
\]

Additionally, we assume that around are present three functions
\( \varphi, \psi, x, \text{from} [0, +\infty] \text{into} [0, +\infty] \),

Which content the expectations:

(I1) \( \varphi(r) < r \text{ if } r > 0 \),

(I2) there subsists \( \lim_{r \to +\infty} \varphi(r) \leq \varphi(\frac{1}{r}) \) \text{ for each } r \in [0, +\infty[ \),

(I3) \( \psi(0) = x(0) = 0 \).

Furthermore, we understand that \( T, \varphi, \psi, \chi \) fulfill the variation

(I4) \( d(Tx, Ty) \leq \varphi(d(x, y)) + \psi(d(Tx, x)) + \chi(Ty, y) \) \text{ for every } x, y \in X.

**Lemma 4.1.1:** Further down the upstairs expectations on \( X \) and \( T \) and if, in adding, \( \psi \) and \( X \) are incessant at \( r = 0 \), then, for all \( x \in X \), there occurs \( z \in X \) such that \( \{T^n x\}_{n=0}^\infty \) congregate to \( z \).

**Proof:** Assume that there occurs \( x \in X \) such as the order of iterates which is not a Cauchy sequence. Which mean that \( \{m(j)\}_j \) \( \{n(j)\}_j \) \( \forall j \in \mathbb{N} \) is fulfill the situations

\[
m(j) > n(j) = \infty \text{ for every } j \in \mathbb{N} \tag{1.1}
\]

\[
\lim_{j \to +\infty} n(j) = +\infty \tag{1.2}
\]

\[
d(T^{m(j)} x, T^{n(j)} x) \geq \varepsilon \tag{1.3}
\]
\[ d(T^{m(j)-1}x, T^{n(j)}x) < \varepsilon \] 

(1.4)

Then, we get

\[ \varepsilon \leq d(T^{m(j)}x, T^{n(j)}x) \leq d(T^{m(j)}x, T^{m(j)-1}x) + d(T^{m(j)-1}x, T^{n(j)}x) < \varepsilon + d(T^{m(j)}x, T^{m(j)-1}x) \]

That suggests

\[ \lim_{j} d(T^{m(j)}x, T^{n(j)}x) = \varepsilon \] 

(1.5)

At the another side

\[ d(T^{m(j)}x T^{n(j)}) \leq d(T^{m(j)}x, T^{m(j)+1}x) + d(T^{m(j)}x, T^{n(j)+1}x) + d(T^{m(j)+1}x, T^{n(j)+1}x) \]

\[ T^{n(j)+1}x \leq d(T^{m(j)}x, T^{m(j)+1}x) + d(T^{m(j)}x, T^{n(j)+1}x) + \phi(d(T^{m(j)}x, T^{n(j)}x)) \]

\[ + \psi(d(T^{m(j)+1}x, T^{n(j)+1}x) + \chi(d(T^{n(j)+1}x, T^{n(j)}x)) \]

Which is

\[ d(T^{m(j)}x, T^{n(j)}x) - \phi(d(T^{m(j)}x, T^{n(j)}x)) \leq d(T^{m(j)}x, T^{m(j)+1}x) + d(T^{n(j)}x, T^{n(j)+1}x) \]

And, allowing \( j \to +\infty \)

\[ \varepsilon - \lim_{j} \phi(d(T^{m(j)}x, T^{n(j)}x)) \leq 0. \]

Hereafter, by (1.3) and (1.5) it tracks that \( \varepsilon = 0 \), a flaw. Meanwhile X is a whole metric space, the compliance of proof.
Here we prove that two fixed point theorems: first is for non-continuous, and another for continuous mappings.

**Theorem 4.1.2:** We have $X, T, \varphi, \psi, \chi$ which is in the Lemma. Additionally, we consider that $x(r) < r$ if $r > 0$. At that point $T$ is an unique fixed point.

**Proof:** Uniqueness is understandable by asset of theories $(I_1), (I_2)$, and $(I_3)$. Therefore we demonstration existence. From the Lemma present as $z \in X$ such as $T^n x \to z, as n \to +\infty$.

For every $x \in X$. Meanwhile

$$d(z, Tz) \leq d(z, T^n x) + d(T^n x, T^{n+1} x) + d(T^{n+1} x, Tz) \leq d(z, T^n x) + d(T^n x, T^{n+1} x)$$

$$+ \varphi(d(T^n x, z)) + \chi(d(Tz, z)) + \psi(d(T^n x, T^{n+1} x))$$

We get

$$d(z, Tx) - \chi(d(z, Tz)) \leq 2d(z, T^n x) + d(T^n x, T^{n+1} x) + \psi(d(T^n x, T^{n+1} x))$$

And, allowing $n \to +\infty$, we have

$$0 \leq d(z, Tz) - \chi(d(z, Tz)) \leq 0;$$

Therefore, $z = Tz$ tracks.

**Remark:** Clearly, the part of the purposes $\psi$ and $x$ can be overturned.

Which for continue function, we proven that the below mentioned result.
**Theorem 4.1.3:** Let \( X, T, \varphi, \psi, \chi \) which is in the Lemma. When, in adding, \( T \) is continuous, then it give a unique fixed point.

**Proof:** Uniqueness gives in Theorem 4.1.2 Here \( z \in X \) which such as \( T^n x \to z, \text{as } n \to +\infty. \)

From continuation of \( T \) then we get \( \lim_{n \to \infty} T^{n+1} x = Tz \)

And , meanwhile \( T \) is asymptotically steady, then we get \( z = Tz. \)

**Remark:** If \( X \) is a shut, non-void, sub section of a Banach space \( B \), Theorem 4.1.3 is getting still true as per the assumption: “\( T \) is a strongly-weakly continuous mapping from \( X \) keen on \( X \)”

**Remark:** As per Lemma, Theorem 4.1.2 as well as Theorem 4.1.3, which follows that repetition of the order \( \{T^n x\}_{n=0}^{\infty} \) meets to the unique fixed point of \( T \), for every \( x \in X. \)

3. In the segment we getting various very well-known results, as corollaries of our earlier theorems.

**COROLARY 4.1.4**[41]: Here \( T \) is a function after \( X \) into \( X \). We supposing that there happens a map \( f, f: [0, +\infty[ \to [0, +\infty[ \) incessant from the right for every \( r \in [0, +\infty[ \), such as

\[
d(Tx, Ty) \leq f(d(x, y)), \text{ for each } x, y \in X
\]
If \( f(r) < r, \) for \( r > 0 \) then the sequence \( \{T^nx\}_{n=0}^\infty \) meets to the unique fixed point of \( T \).

**Proof.** As per Theorem 4.1.3 we get \( \varphi(r) = f(r), \psi(r) = \chi(r) = 0 \) be, for every \( r \in [0, +\infty[. \) Furthermore, if \( v \in N \) occurs such as \( T^{v+1}x = T^v x, \) for each \( x \in X, T \) is a asymptotically continuous flow. On the conflicting, one having for every \( x \in X \) such as that \( v \) doesn’t happen,

\[
d(T^{n+1}x, T^nx) \leq f(d(T^n x, T^{n-1}x)) < d(T^n x, T^{n-1}x) \quad \text{for each } n \in N.
\]

Taking the values \( d = \lim_{n} d(T^{n+1}x, T^nx) \), we have \( d \leq f(d) \leq d \) and so \( d = 0 \).

**COROLARY 4.1.5[36]:** Here \( X, T \) which is in Corollary 4.1.4. Consider that \( f, \) form \( [0, +\infty[ \) it is a increasing equation, continued towards left such as \( d(Tx, Ty) \leq f(d(x,y)), \) for each \( x, y \in X \)

If \( f(r) < r, \) for \( r > 0 \) as well as if \( X \) is bounded, after that \( T \) which is a unique fixed point.

**COROLARY 4.1.6[3]:** Here \( T \) is an asymptotically continue as well as regular expression, such as \( T \) flows \( S \) into \( S \), with \( S \) a non-void not strongly neither open subset of a Hilbert space \( H \). As well as, we consider that \( T \) satisfying

\[
\|Tx - Ty\| \leq \|Tx - x\| + \|Ty - y\|, \quad \text{for each } x, y \in X.
\]

After that, for every \( x \in S \), that routine flow \( \{T^nx\}_{n=0}^\infty \) converting to the unique fixed point of \( T \).
Proof: As per Theorem 4.1.3, we have \( \varphi(r) = 0, \psi(r) = x(r) = r \) for each \( r \geq 0 \).

**COROLARY 4.1.7** [38]: Here \( T : S \rightarrow S \), \( S \) a non-void not strongly neither opened subset of a Hilbert space \( H \) is an asymptotically continue as well as routine stated, such as

\[
\| Tx - Ty \| \leq p \| x - y \| + q (\| Tx - x \| + \| Ty - y \|), \text{ for each } x, y \in X.
\]

Having \( p^2 + q^2 < 1 \). After that, the Corolary 4.1.6 is not false.

Proof: From Theorem 4.1.2, we get \( \varphi(r) = pr, \psi(r) = x(r) = qr \), for each \( r \geq 0 \). We detect that, after using Theorem 4.1.2, we done the distribution with the steadiness of \( T \).

**COROLARY 4.1.8** [43]: Here \( (X, d) \) which is a whole metric space and getting \( a, b, c \) be are positive numbers, with \( a + b + c < 1 \). Also, assume that \( T : X \rightarrow X \) fulfills

\[
d(Tx, Ty) \leq ad(x, y) + bd(Tx, x) + cd(Ty, y), \text{ for each } x, y \in X.
\]

After that, \( T \) which is an unique fixed point.

Proof: - Also we given, Theorem 4.1.2,

\[
\varphi(r) = ar, \psi(r) = br, x(r) = cr, \text{ for each } r \in [0, +\infty[.
\]

Meanwhile, for every \( n \in N \), we having

\[
d(T^{n+1}x, T^nx) \leq ad(T^nx, T^{n-1}x) + bd(T^{n+1}x, T^nX) + cd(T^nx, T^{n-1}x), x \in X,
\]
Which gives that as follows

\[ d(T^{n+1}x, T^nx) \leq \left( \frac{a + c}{1 - b} \right)^n d(Tx, x), x \in X \]

That condition suggests that T which is asymptotically continues, existence \( a + b + c < 1 \).

Remark :- Beneath the expectations of Corollary 4.1.8, we takes, by asset of Remark 3, which is an arrangement as \( \{T^n x\}_{n=0}^\infty \) converted in to an unique fixed point of T.

**COROLARY 4.1.9**[40]: Here \( T, T: X \rightarrow X \), having an order fulfilling the situation \( d(Tx, Ty) \leq (x, y) - \Delta(d(x, y)), \text{ for each } x, y \in X \)

*Where \( \Delta:[0, +\infty[ \rightarrow [0, +\infty[, it is an continuous expression i.e. such as \( \Delta(r) > 0, if r > 0 \). After that, for every \( x \in X \), the order \( \{T^n x\}_{n=0}^\infty \) converts to an unique fixed point of T.*

**Proof :-** Asymptotical orderliness stracks as in Corolary 4.1.4, Here taking, in Theorem 4.1.2, \( \phi(r) = r - \Delta(r), \psi(t) = (r) = 0 \text{ for each } r \in [0, +\infty[ \) once getting the research work. We deserves that can distribute by the help of steadiness of \( \Delta \). In detail it does that \( \Delta \) is a monotonically increasing or accurate incessant function.
At the point of this segment of research work, we assume that the mapping of \( T : X \to X \), is proving each and every assumptions of n.1, with the exclusion of \((I_4)\), which is substituted with

\[
(I_4) \quad d(Tx, Ty) \leq \varphi(d(x, y)) + p(d(x, y)) + q(d(x, y))d(Ty, y) \quad \text{for each } x, y \in X.
\]

Here \( P, P : [0, +\infty) \to [0, +\infty] \), is a function such that

\[
(I_s) \quad \lim_{s \to +\infty} p(s) < +\infty, \quad \text{for each } s \in [0, +\infty] \quad \text{and} \quad q, q : [0, +\infty) \to [0, +\infty], \quad \text{is function such that}
\]

\[
(I_s) \quad \lim_{s \to +\infty} q(s) < 1 \quad \text{and} \quad \lim_{s \to +\infty} p(s) < +\infty, \quad \text{for each } s \in [0, +\infty] .
\]

Beforehand showing the declared fixed point theorems, we note that in such type of cases the lemma having still factual, when we consider that \( \psi, \chi \) are the incessant expression as \( r \to 0^+ \). After that, as per typical opinions, we proves

**THEOREM 4.1.10:** If \( X, T, \varphi, \psi, \chi, p, q \) are satisfies each and every earlier expectations as well as, in adding \( \chi(r) \leq r \) for each \( r \in [0, +\infty] \), after that \( T \) which is a unique fixed point. Furthermore, the arrangement \( \{T^n x\}_{n=0}^\infty \) converting in to the unique fixed point of \( T \).

It is the last theorem 4.1.10oversimplifies the below mentioned result of [45].

“Here \( X, \ d \) is a fulfilled metric space. When \( T,T : X \to X \), Satisfies

\[
d(Tx, Ty) \leq a(d(x, y))d(x, y) + (b(x, y))d(Tx, x) + c(d(x, y))d(Ty, y),
\]

(4.1)
Wherever \( x, y \in X, x \neq y \), as well as \( a, b, c \) which are monotonically non-increasing expressions from \( [0, +\infty] \) in \( [0,1] \) such that \( a(s) + b(s) + c(s) < 1 \), after that T is a unique fixed point”.

At the point of this finalization, we noticed that, first of every, which is T that an regular asymptotically mapping (see [45]).

Furthermore, we also gives

\[
\varphi(r) = \begin{cases} 
0 & r = 0 \\
0 + \rho & r \neq 0
\end{cases} \quad \psi(r) = X(r) = r \quad \text{for each } r \in [0, +\infty]; p(s) = 1
\]

For every \( s \in [0, +\infty] \): since we may assume \( c(s) < \frac{1}{2} \) (see [9]) \( q(s) = \begin{cases} 
1/2 & s = 0 \\
0 & s \neq 0
\end{cases} \)

By the use of Theorem 4.1.10 we can getting that T as unique fixed point in X as well as additionally the sequence that of repeats converting to that unique fixed point.

To conclude, if T which is an satisfies assumptions of continuous mapping as in the Lemma, we proves that proven as Theorem 4.1.3, i.e. without any conditions

\[
\lim_{s \to +\infty} p(s) < +\infty \text{ and } \lim_{s \to +\infty} q(s) < 1.
\]