Applications of Fixed Point Theory

5.1 Introduction: Now the study of the topic 'Applications of Fixed Point Theory' we are presented the idea of Fixed point Theory to elucidate the Differential as well as Integral Equations. In excess of the previous some periods, fixed point theory of Lipschitzian mappings had be there established into an significant field of study in both applied as well as pure mathematics. The major thing of this research work is to be presenting some of the basic methods as well as results of this concept.

Fixed point theory is commonly used for simplification, basicity of mathematical settings. In this theory a expression that gives a set $X$ as together for range and domain. A fixed point $f$ which is an point $x$ of $X$ which is for $f(x) = x$. Binary basic theorems regarding to fixed points, those are of Banach as well as Brouwer. In Banach’s theorem, complete metric space $X$ is with metric $d$ also $f:X \rightarrow X$ is mandatory to be a reduction, which is, there maximally occur $L < 1$ such as $d(f(x), f(y)) \leq Ld(x, y)$ foreach and every $x, y \in X$. The assumption is that $f$ is a fixed point. In fact this is precisely one of them.

Brouwer’s theorem needs $X$ for the closing unit sphere in an Euclidean space as well as $f:X \rightarrow X$ is a map, which is a function that is
continuous. Function f is a fixed point is concluding again. But such type of cases a set of fixed point that doesn’t necessity of particular point, infact each and every bounded fulfilled subset of unit sphere which is an fixed point set for certain map. In the Banach theorem, using X as the metric, that is in the crucial hypothesis for the expression which is a contraction.

5.2 Fixed Point Theorems

Fixed point showing interest and takes part in themselves; but those fixed points are also providing a direction towards establishing the reality of solution to an set of functions.

An example, a general equilibrium theory from theoretical economics that denotes the point for an need to understanding whether the solution to that system of expression essentially exists or maximally accurately as per which requirements having for solution essentially exists. As per mathematical investigation of that question typically depends on fixed point theorem.

5.3 Lipschitz Mapping

In the year 1922 Banach has published his fixed point theorem which is also identified as Banach’s Contraction Principle that used for the theory of Lipschitz mapping.

Definition 5.3.1: Here (X, d) is a metric space. The flow \( T: X \rightarrow X \) is supposed as Lipschitzian, now we occur a constant \( \sigma(T) > 0 \) (known as Lipschitz constant) such as

\[
d(T(x), T(y)) \leq \sigma(T)d(x, y) \forall x, y \in X.
\]
**Definition 5.3.2:** An Lipschitzian mapping through Lipschitzian constant $\sigma(T) < 1$ which is known as contraction.

**Theorem 5.3.3:** (Banach Contraction Principle)

Here $(X, d)$ is a whole metric space as well as $T : X \to X$ is a contraction mapping. After that $T$ had unique fixed point $x_0$ as well as for each and every $x \in X$ we get

$$\lim_{n \to \infty} T^n(x) = x_0$$

Furthermore for every $x \in X$, we get

$$d(T^n(x), x_0) \leq \frac{(\sigma(T))^n}{1 - \sigma(T)} d(T(x), x)$$

**5.4 Fixed Points**

**Definition 5.4.1:** Here $(X, d)$ is a metric space as well as $T : M \subset X \to X$ is an map. An explanation of $Tx = x$ is known as fixed point of $T$.

**Example 5.4.2:** Searching an zero of map $F : M \subset X \to X$. Such difficulty can be expressed by various methods as a fixed point problem.

e.g.

i) $T x = x - F(x)$

ii) $T x = x - w F(x), \ w \in R$ (Linear relaxation)

iii) $T x = x - (DF(x))^{-1} F(x)$ (Newton's Method)
5.5 Differential Equations

Now $f(x, y)$ which is a continuous actual valued expression on $[a, b] \times [c, d]$. Here Cauchy IVP (Initial Value Problem) searching a continuous differentiable expression above $[a, b]$ satisfies the diff. eq. $(5.1) \frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$

Here assume that the Banach space $C[a, b]$ of continuous actual valued expression with supremum standard which is defined by $\|y\| = \sup \{|y(x)| : x \in [a, b]\}$. Integrating eq. (5.1), we get an integral eq. (5.2)

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t)) \, dt$$

The eq. (5.1) which is correspondent the problem explaining the integral eq. (5.2).

We express an essential operator $T: C[a, b] \rightarrow C[a, b]$ by

$$Ty(x) = y_0 + \int_{x_0}^{x} f(t, y(t)) \, dt$$

Therefore, an explanation of Cauchy IVP (5.1) that corresponds within a fixed point of $T$. Uniquely easy checks that if $T$ is reduction, after that the eq. (5.1) which is a unique solution.

Here our goal is to execute given situations on $f$ underneath which the integral operator $T$ that gives Lipschitzian through $\sigma(T) < 1$. 
**Theorem 5.5.1:** Here $f(x, y)$ is a continuous expression of $\text{Dom}(f) = [a, b] \times [c, d]$ such as $f$ is Lipschitzian w. r. t. $y$, i. e. there occurs $L > 0$ such as

$$|f(x, u) - f(x, v)| \leq L|u - v| \text{ for all } u, v \in [c, d] \text{ and for } x \in [a, b]$$

Consider that $(x_0, y_0) \in \text{int}(\text{Dom}(f))$. After that for satisfactorily minor $h > 0$, there exists a unique solution of the equation (5.1).

**Proof:** Here $M = \sup \{|f(x, y)| : x, y \in \text{Dom}(f)\}$ as well as $h > 0$ such as $Lh < 1$ also $[x_0 - h, x_0 + h] \subseteq [a, b]$.

Set $C := \{y \in C[x_0 - h, x_0 + h] : |y(x) - y_0| \leq Mh\}$.

After that, $C$ which is a closed subset of the whole metric space $C[x_0 - h, x_0 + h]$ as well as after $C$ is whole. Note $T : C \to C$ that is a contraction mapping.

Certainly, for $x \in [x_0 - h, x_0 + h]$ as well as both continuous expression $y_1, y_2 \in C$, we get

$$\|Ty_1 - Ty_2\| = \left\|\int_{x_0}^{x} f(x, y_1) - f(x, y_2) dt\right\|$$

$$\leq |x - x_0| \sup_{s \in [x_0 - h, x_0 + h]} L|y_1(s) - y_2(s)| \leq Lh\|y_1 - y_2\|.$$ 

Hence, $T$ had an unique fixed point suggesting that the eq. (5.1) has an unique solution.

**5.6 Integral Equations**

Here, we suppose the Fredholm integral eq. for an unidentified equation

$$y : [a, b] \to \mathbb{R}(-\infty < a < b < \infty) \quad \text{(5.3)}$$

$$y(x) = f(x) + \lambda \int_{a}^{b} k(x, t)y(t)dt$$
Where, \( k(x, t) \) is continuous on \([a, b] \times [a, b]\)

As well as \( f(x) \) is continuous on \([a, b]\)

Here we suppose that the Banach space \( X = C[a, b] \) of continuous actual valued expression with supremum condition \( \|y\| = \sup \{|y(x)| : x \in [a, b]\} \) also described an operator \( T: C[a, b] \to C[a, b] \) by (5.4)

\[
Ty(x) = f(x) + \lambda \int_{a}^{b} k(x, t)y(t)\,dt
\]

Therefore, an explanation of Fredholm integral eq. (5.3) that is a fixed point of \( T \).

Now we execute a limit on the real number \( \lambda \) such as \( T \) that turn out to be a contraction.

**Theorem 5.6.1:** Here \( k(x, t) \) is a continuous expression above \([a, b] \times [a, b]\) that with \( M = \sup \{|k(x, t)| : x, t \in [a, b]\} \), \( f \) a contraction expression above \([a, b] \), as well as \( \lambda \) an real number such as \( M(b - a)|\lambda| < 1 \). After that the Fredholm integral eq. (5.3) which is a unique solution.

**Proof:** It is enough to demonstrate that the mapping \( T \) which is defined by eq. (5.4) i.e. a contraction for both continuous expressions \( y_1, y_2 \in C[a, b] \), we get,

\[
\|Ty_1 - Ty_2\| = \sup_{x \in [a, b]} |\lambda| \left| \int_{a}^{b} k(x, t)[y_1(t) - y_2(t)]\,dt \right| \\
\leq |\lambda| \sup_{x \in [a, b]} \int_{a}^{b} |k(x, t)||y_1(t) - y_2(t)|\,dt \\
\leq |\lambda| M \int_{a}^{b} \sup_{t \in [a, b]} |y_1(t) - y_2(t)|\,dt
\]
\[ = |\lambda| M \| y_1 - y_2 \| \int_a^b dt. \]

5.7 Differential Equations with Initial Value

As a non-insignificant example of the contraction mapping theorem, we will solve the IVP (4.1) \( \frac{dy}{dx} = f(t,y), \ y(t_0) = y_0 \)

On behalf of “nice expression \( f \). A proven to eq. (4.1) i.e. an differentiable expression \( y(t) \) which is defined on a neighborhood of \( t_0 \) such as

\[ \frac{dy}{dx} = f(t, y(t)), \ y(t_0) = y_0 \]

As stated previously the uniqueness as well as basic existence of theorem for such type of I. V. P. we appearance at certain cases to increase in the scope of the theorem we determination prove.

**Example 5.7.1:** The I. V. P.

\[ \frac{dy}{dx} = y^2, \ y(0) = 1 \]

Had a single solution which passing through point (0,1) which is found to use separating of integration as well as variables:

\[ y(t) = \frac{1}{1-t} \]
Remind that even though the right side of an diff. eq. makes sense everywhere. The solution rise up in limited time (at \( t = 1 \)). Curve shown as per following figure. The curves as per below given figure are another solutions to \( \frac{dy}{dx} = y^2 \) with \( y(0) > 0 \) (having the form \( y = 1/(c-t) \))

\[ \frac{dy}{dt} = y^2 - t, \ y(0) = 1 \]

Having a proven, but that proved cannot to be stated in terms of elementary purposes or their forms of integrals (theorem of Liouville [18, p.70 ff]). As per previous discussed example, focusing the solution in desired time even though doing sense every-where which is making by the right side. Focusing on the slope of field diagram which is given below to be persuaded visually of the limited blowing up period (time) for solution curve which is passing through point (0,1).

**Example 5.7.3:** The I. V. P. (Initial Value Problem)
\[
\frac{dy}{dt} = y^{2/3}, \quad y(0) = 0
\]

Having two solutions: \( y(t) = 0 \) and \( y(t) = t^3 / 27 \)

Above mentioned both examples shows that for solving eq. (4.1) in any condition of simplicity, we are fully satisfied with that solution which is occurs only locally that is not everywhere that \( f \) makes sense, as well as some constraint are required on \( f \) that having an unique solution nearby \( t_0 \).

We acquire an condition on \( f(t, y) \) for assurance a native unique solution which is Lipschitz condition in the another variable. Now we will consider \( K > 0 \) as a constant such as \( |f(t, y_1) - f(t, y_2)| \leq K|y_1 - y_2| \) for every \( y_1 \) as well as \( y_2 \). The right side cannot includet, therefore that is a Lipschitz condition. Now in another variable this is uniform also in first variable.

Maximum functions aren’t universally Lipschitz. For occurrence the only singlevariable function \( h(x) = x^3 \) does not satisfy

\[
|h(a) - h(b)| \leq K|a - b| \text{ for some } K > 0 \text{ and all } a \text{ and } b.
\]

But in any function that with continuous first derivative that is locally Lipschitz:

**Lemma 5.7.4.** Now \( U \subseteq \mathbb{R}^2 \) is not closed as well as \( f : U \to \mathbb{R} \) being a \( C^1 \)-function.

After that in second variable which is locally Lipschitz also it is uniformly in the first: for each and every \( (t_0, y_0) \in U \), Here \( K > 0 \) is an constant, such as
\[ |f(t, y_1) - f(t, y_2)| \leq K|y_1 - y_2| \]

For every t nearby \( t_0 \) as well as every \( y_1 \) also \( y_2 \) nearby \( y_0 \).

**Proof** :- Here an open set \( U \) containing a rectangle that around \( (t_0, y_0) \), say \( I \times J \). Here \( I \) and \( J \) are closed

Intermission that with \( t_0 \in I, y_0 \in J \), and \( I \times J \subset U \). Put on the mean-value theorem to \( f \) as well as in its another variable, for several \( t \in I \) an \( y_1 \) and \( y_2 \) in \( J \)

\[
f(t, y_1) - f(t, y_2) = \frac{\partial f}{\partial y}(t, y_1)(y_1 - y_2)
\]

Here for several \( y_3 \) in the middle of \( y_1 \) as well as \( y_2 \) which is dependent maybe on \( t \), after that

\[
f(t, y_1) - f(t, y_2) = \left| \frac{\partial f}{\partial y}(t, y_3) \right| y_1 - y_2 \leq \sup_{p \in I \times J} \left| \frac{\partial f}{\partial y}(p) \right| y_1 - y_2
\]

Now \( K = \sup_{p \in I \times J} \left| \frac{\partial f}{\partial y}(p) \right| \), is limited from the time when a continuous function on an compact set which is bounded.

**Example 5.7.5** The expression \( f(t, y) = \sin(ty) \) is \( C^1 \) on all of \( \mathbb{R}^2 \) also is Lipschitz on several perpendicular band \([-R, R] \times \mathbb{R} : if \|p\| \leq R \) then for any \( y_1 \) and \( y_2 \) in \( \mathbb{R} \),

\[
\sin(ty_1) - \sin(ty_2) = t \cos(ty_3)(ty_1 - ty_2)
\]

for some \( y_3 \) between \( y_1 \) and \( y_2 \). Therefore

\[
\sin(ty_1) - \sin(ty_2) = t^2|y_1 - y_2| \leq R^2 (y_1 - y_2).
\]
Example 5.7.6. Now the expression \( f(t,y) = y^{2/3} \) is not \( C^1 \) at \((0,0)\) also in fact that is uneven Lipschitz in several region of \((0,0)\):

If \( |y_1^{2/3} - y_2^{2/3}| \leq K|y_1 - y_2| \) for some \( K > 0 \) and all \( y_1, y_2 \) near 0 then

\( y_2 = 0 \) shows that \( |y^{2/3}| \leq K|y| \) for each and every \( y \) near by 0, but for \( y \neq 0 \) that's the same as \( |y|^{-1/3} \leq K|y|^{-1/3} \)

It is unbounded as \( t \to 0 \)

Controlling the development of hypothetical solution to (4.1) is studied in our next lemma near \( t_0 \).

Lemma 5.7.7 Now \( U \subset \mathbb{R}^2 \) be open as well as \( f: U \to \mathbb{R} \) are continue. For several \((t_0, y_0) \in U\) there are \( r, R > 0 \) such as rectangle \([t_0 - r, t_0 + r] \times [y_0 - R, y_0 + R]\) it is between \( U \) also, for several \( \delta \in (0, r) \), a solution \( y(t) \) to (4.1) which is clear on \([t_0 - \delta, t_0 + \delta]\) must satisfy \( |y(t) - y_0| \leq R \) for \( |t - t_0| \leq \delta \)

Proof :- Consider \( y(t) \) please (4.1) nearby for \( t \) near \( t_0 \), \((t, y(t)) \in U \) since \( U \) is not close as well as \( y(t) \) is continue, so \( f(t, y(t)) \) makes intellect. Here for \( t \) close to
\[ \int_{t_0}^{t} f(s, y(s)) ds \text{ has } t \text{-derivative } f(t, y(t)), \text{ that is also the } t \text{-derivative of} \]

\[ y(t), \text{ so by (4.1) } y(t) \text{ and } \int_{t_0}^{t} f(s, y(s)) ds \text{ vary by a constant close to } t_0. \]
At \( t = t_0 \) the variance is \( y(t_0) - 0 = y_0, \) So

\[ y(t) = y_0 + \int_{t_0}^{t} f(s, y(s)) ds \]

But here \( t \text{ near } t_0. \) take a rectangle

\[ [t_0 - r, t_0 + r] \times [y_0 - R, y_0 + R] \]

In between \( U \) which is focused at \((t_0, y_0)\) also here \( B > 0 \) has been better bound of \(|f|\) which on rectangle. Here \( t \text{ near } t_0, \) (4.2) suggests

\[ (4.3) \quad [y(t) - y_0] \leq B|t - t_0| \]

Consider \( B, r \) \& \( R \) which are resolute completed by \( f \) as well as doesn’t by any solutions to (4.1)

Merger so the expression \( Br \leq R, \) i.e. substitute \( r \) that with \( \min( r, R/B) \). We could instead trying to rise \( R \) to attain this inequality, but also increasing the dimensions of rectangle that would typically alteration \( B \) as well as there is the restraint that rectangle which should be inside of \( U \). Take \( \partial \in (0, r] \) as well as assume there is an solution \( y(t) \) to (4.1) for \( |t - t_0| \leq \partial \). In specific, \( (t, y(t)) \in U \) for \( |t - t_0| \leq \partial \) from (4.3)
Here is the significant being and uniqueness theorem for ODEs.

**Theorem 5.7.8:** Here \( R = [a, b] \times [y_0 - R, y_0 + R] \subseteq \mathbb{R}^2 \) be a rectangle \( f : R \to \mathbb{R} \) which is continuous expression above rectangle that is known as Lipschitz within second variable uniformly that in its first variable. Now \( B > 0 \) which in above on \( |f| \) over that rectangle. That for \( t_0 \in [a, b], \min |t_0 - a|, |t_0 - b| \). The I. V. P. (4.1) having an unique solution by the interval \([t_0 - r_0, t_0 + r_0]\), wherever \( r_0 = \min(r, R / B) \).

In particularly, when \( U \subset \mathbb{R}^2 \) which is not closed as well as \( f : U \to \mathbb{R} \) is \( C^1 \) after that (4.1) having an unique local solution passes throughout several point in its area.

**Proof:** The set

\[ M = \{ h \in C[a, b] : |h(t) - y_0| \leq R \ for \ t \in [a, b] \} \]

The continuous expression
$h(t) = y_0$ which is in $M$, therefore $M \neq \phi$, the plotted graph $\{(t, h(t)) : t \in [a, b]\}$ says that in $R$. That gives the $c[a, b]$ sup norm $\|\cdot\|_\text{sup}, M$ which is a not opened sub space of $c[a, b]$. Hence $c[a, b]$ is completed between the sup norm, $M$ which is completed in the sup norm.

Assuming the condition (4.2), here $F : M \to M$ by

\[(4.4) \quad (Fh)(t) := y_0 + \int_{t_0}^{t} f(s, h(s))ds.\]

Given integration variesability, from $(s, h(s)) \in R$ for all $s \in [a, b]$, also moreover

$$|(Fh)(t) - y_0| \leq B|t - t_0| \leq Br \leq R$$

So the $f(M) \subset M$. Here we are showing the reduction mapping theorem that can useful for finding fixed point of $F$ that in $M$. Here a fixed point of $F$ which is a solution of (4.1) as per the basic theorem for calculus.

The proven of (4.1) as well as (4.2) that are both same. The benefit of (4.2) above (4.1) which is continue expression having continuous anti derivatives but they doesn’t capable for having an derivative integration, it is a “smoother” process on equations than differentiation.

To prove various methods, here we showing, which theorem is used for contraction mapping we are used for finding a fixed point of $F$ by two methods:
(1) Here an repetition of $F^n$ for contraction over $M$ for $n \geq 0$, therefore Theorem 2.2.1 apply now.

(2) Now $F$ is a contraction over $M$ by using a condition replacement on the sup-norm, but that in which $M$ is also still complete.

**Proof for (1):** Now $h_1, h_2 \in M$ and $t \in [a, b]$

\[
(Fh_1)(t) - (Fh_2)(t) = \int_{t_0}^{t} (f(s, h_1(s)) - f(s, h_2(s))) ds,
\]

therefore

\[
(Fh_1)(t) - (Fh_2)(t) \leq \int_{[t_0, t]} [(f(s, h_1(s)) - f(s, h_2(s))] ds,
\]

Here $\int_{[t_0, t]}$ denoted $\int_{[t_0, t]}$ if $t_0 \leq t$ and $\int_{[t, t_0]}$ if $t < t_0$. Now we are considered $f$ is Lipschitz between its another variable uniformly in first, that says

\[
|f(t, y_1) - f(t, y_2)| \leq K|y_1 - y_2| \text{ for } (t, y_1) \text{ and } (t, y_2) \text{ in } R.
\]

Then

\[
|Fh_1 - Fh_2| \leq \int_{[t_0, t]} K|h_1(s) - h_2(s)| ds
\]

\[
\leq K\|h_1 - h_2\|_{sup}(t - t_0)
\]

\[
\leq K\|h_1 - h_2\|_{sup}(b - a)
\]

Therefore $\|Fh_1 - Fh_2\|_{sup} \leq K(b - a)\|h_1 - h_2\|_{sup}$. If $K(b - a) < 1$ then $F$ is a reduction over $M$. This is previously given us proper existence as well as uniqueness of proven by reduction $[a, b]$ slight sufficient all over the $t_0$, therefore $b - a < 1/K$. 

91
But we having for shows there is an solution about our unique \([a, b]\) therefore we continue in another way. To protect the condition \(K(b - a) \geq 1\), we took at how repeat of \(F\) merges distances. Hence

\[
(F^2h_1)(t) - (F^2h_2)(t) = \int_{t_0}^{t} (f(s, (Fh_1)(s)) - f(s, (Fh_2)(s))) \, ds,
\]

we get

\[
\left| (F^2h_1)(t) - (F^2h_2)(t) \right| \leq \int_{[t_0, t]} K \left| h_1(s) - h_2(s) \right| \, ds \quad \text{by the Lipschitz condition}
\]

\[
\leq \int_{[t_0, t]} K \cdot K \left| h_1(s) - h_2(s) \right| \sup \left| s - t_o \right| \, ds \quad \text{by (4.7)}
\]

\[
\leq K^2 \left\| h_1 - h_2 \right\| \sup \int_{[t_0, t]} \left| s - t_o \right| \, ds
\]

Since \(\int_{[t_0, t]} \left| s - t_o \right| \, ds = \frac{(t - t_o)^2}{2}\) (check this separately for \(t \geq t_o\) and \(t < t_o\) we have

\[
\left| (F^2h_1)(t) - (F^2h_2)(t) \right| \leq K^2 \left\| h_1 - h_2 \right\| \sup \frac{(t - t_o)^2}{2} = \frac{(K |t - t_o|)^2}{2} \left\| h_1 - h_2 \right\| \sup
\]

By induction using the formula \(\int_{[t_0, t]} |s - t_o|^n \, ds = |t - t_o|^{n+1} / (n + 1)\) for \(n \geq 0\),

\[
\left| (F^nh_1)(t) - (F^nh_2)(t) \right| \leq K^2 \left\| h_1 - h_2 \right\| \sup \frac{(t - t_o)^2}{2} = \frac{(K |t - t_o|)^2}{2} \left\| h_1 - h_2 \right\| \sup
\]

By induction using the formula \(\int_{[t_0, t]} |s - t_o|^n \, ds = |t - t_o|^{n+1} / (n + 1)\) that for \(n \geq 0\)

\[
\left| (F^nh_1)(t) - (F^nh_2)(t) \right| = \left( \frac{K |t - t_o|}{n!} \right)^n \left\| h_1 - h_2 \right\| \sup .
\]

For each and every \(n \geq 0\) \(t \in [a, b]\). Since \(|t - t_o| \leq b - a\),

\[
\left\| F^n h_1 - F^n h_2 \right\| \sup \leq \frac{(K(b - a))^n}{n!} \left\| h_1 - h_2 \right\| \sup
\]

Whenever \(n \geq 0\), \((K(b - a))^n / n! < 1\), So \(F^n\) is a contraction over \(M\) between the sup - norm. Therefore \(F\) having a unique fixed point in \(M\) by Theorem 3.1.

**Proof for (2) :-** Describe an innovative norm \(\|\|\) on \(C[a,b]\) by

\[
\|h\| := \sup_{t \in [a,b]} e^{-\lambda(t-t_o)} |h(t)|,
\]
Here $K$, it is as Lipschitz constant that for $f$ over $\mathbb{R}$. By checking this, it is an norm over $C[a,b]$ for each and every $t \in [a,b]$ we getting

$$|h_1(t) + h_2(t)| \leq |h_1(t)| + |h_2(t)|$$

so

$$e^{-K|t-t_0|}|h_1(t) + h_2(t)| \leq e^{-K|t-t_0|}|h_1(t)| + e^{-K|t-t_0|}|h_2(t)| \leq \|h_1\| + \|h_2\|$$

Captivating the sup for left side one each and every $t \in [a,b]$ that shows $\|\|$ satisfactory of the triangle inequality. By the another conditions to be an vector spaces norms are simply check. How to doing $\|\|$ comparison to sup-norm on $C[a,b]$? For $t \in [a,b]$

$$e^{-K(b-a)} \leq e^{-K|t-t_0|} \leq 1$$

so for any $h \in [a,b]$

$$e^{-K(b-a)}\|h(t)\| \leq e^{-K|t-t_0|}\|h(t)\| \leq \|h(t)\|$$

Taking the sup over $t \in [a,b]$

(4.8)

$$e^{-K(b-a)}\|h\|_{\text{sup}} \leq \|h\| \leq \|h\|_{\text{sup}}.$$ 

Therefore $\|\|$ also the sup-norm that bounded to each other from both sides of scaling factors, here those both norms having the similar open sets as well as the similar convergent orders in $C[a,b]$. Particularly, $C[a,b]$ & it’s subset $M$ both are fully related to $\|\|$ hence they are fully w. r. t. the sup-norm.

By returning to (4.6) multiplying both sides by $e^{-K|t-t_0|}$:
\[
e^{-K|t-s|}\left|(Fh_1)(t) - (Fh_2)(t)\right| \leq e^{-K|t-s|} \int_{[t_0,t]} K|h_1(s) - h_2(s)| ds
\]

\[
e^{-K|t-s|} \int_{[t_0,t]} Ke^{-K|s-t_o|} e^{-K|s-t_o|} |h_1(s) - h_2(s)| ds
\]

\[
e^{-K|t-s|} \int_{[t_0,t]} Ke^{-K|s-t_o|} |h_1 - h_2| ds
\]

\[
= \|h_1 - h_2\| e^{-K|t-s|} \int_{[t_0,t]} Ke^{-K|s-t_o|} ds
\]

To calculate the integral which is in that final function, we are taking properties which are dependent over whether \( t_0 \leq t \) or \( t_0 > t \).

If \( t_0 \leq t \) then \( |t-t_0| = t-t_0 \) and \( |s-t_0| = s-t_0 \) for \( t_0 \leq s \leq t \), so

\[
\int_{[t_0,t]} Ke^{-K|s-t_0|} ds = \int_{t_0}^{t} Ke^{-K|s-t_0|} ds = e^{K|t-t_0|} - 1
\]

If \( t_0 > t \) then \( |t-t_0| = t_o - t \) and \( |s-t_0| = t_o - s \) for \( t \leq s \leq t_0 \), so

\[
\int_{[t_0,t]} Ke^{-K|s-t_0|} ds = \int_{t_0}^{t} Ke^{K|s-t_0|} ds = e^{K|t-t_0|} - 1
\]

In neither case the value is \( e^{K|t-t_0|} - 1 \), so

\[
e^{-K|t-t_0|} \int_{[t_0,t]} Ke^{-K|s-t_0|} ds = e^{-K|t-t_0|} (e^{K|t-t_0|} - 1) = 1 - e^{-K|t-t_0|}
\]

Therefore

\[
e^{-K|t-t_0|} \left|(Fh_1)(t) - (Fh_2)(t)\right| \leq \|h_1 - h_2\| (e^{-K|t-t_0|} - 1) \leq (1 - e^{-K(b-a)}) \|h_1 - h_2\|
\]

Taking the supremum of the left side over all \( t \in [a,b] \),

\[
\| (Fh_1)(t) - (Fh_2)(t) \| \leq (1 - e^{-K(b-a)}) \|h_1 - h_2\|.
\]

Hence the expression \( K(b-a) > 0, 1 - e^{-K(b-a)} \), so \( F : M \to M \) which is a reduction W. R. T. \( \| \| \). Therefore \( M \) is the completely w. r. to \( \| \| \), it is an unique fixed point of \( F \) that in \( M \) by contracting the map of theorem.
Solution of the Picard’s theorem that says an solution for (4.1) which is close to be founded as edge of order \( \{y_{n+1}(t)\} \) where \( y_0(t) = y_0 \) is antypical function also

\[
y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(s))ds
\]

Theses expressions \( y_n(t) \), clearthat recursively by integration , which are known as Picard iterates. Picard’s solution of theorem 5.7.8 performed in 1890.It is 30 years earlier of Banach specified the general contraction mapping theorem, also the basic hint was already used in an extraordinary example by Liouville in the year 1837.

**Remark** :By the solution of Picard’s theorem, we are introducing an integral function \( F \) that in (4.4) also having two paths of suggesting it has a fixed point. The similar thought can be using to proving by both paths that for each and every continuous expression \( f \in C[a,b] \) the integral equation

\[
y(t) = f(t) + \int_{a}^{t} k(t,s)y(s)ds
\]

Having an unique answer \( f \in C[a,b] \) which is for whichever mentioned continuous \( f \) & \( k \).

**Corollary 5.7.9**:Where \( f : [a,b] \times R \to R \) is Lipschitz is in its another variable consistently in its first variable after that differential equation (4.1) having an unique solution over \([a,b]\) that given several primary condition at point \( t_0 \in [a,b] \).
Proof :- In starting part of proof of Picard’s theorem, the corporate with R that decrease the domain over which is the diff. eq. is deliberate close to that was passed out for putting ourselves in the situation of getting domain wherever f is Lipschitz between its another variable uniformly in first variable also such as we are described M to containing only the continuous expressions whose graph remnants in that domain is for f. Consequently if we are considering from the onset that f have the related Lipschitz property and also its another variable passes over R, now we are dropping the first section of proof of theorem 4.8 by consideration also just taking M = C[a, b]. Define $F : M \to M$ as in (4.4).

Example 5.7.10 The expression $f(t, y) = \sin(ty)$ is Lipschitz here is uniformly in $t$ over several sets of in the form of $[-r, r] \times R$, so diff. eq. that with $y(0) = 1$ has an unique solution over $[-r, r]$ for each and every r. Therefore the proven $y$ is described on each and every of R.

Remark: Here we neglect the known Lipschitz property over $f$ after that we loss of uniqueness of proven to (4.1), as per we given in Example 5.7.3 but also we doesn’t loss of presence. Here we show an theorem of Peano which is as far as $f(t, y)$ it is continuous equation (4.1) willing an solution curve is passes throughout such point $(t_o, y_o)$. As per a method of solution here an proper method is to describe $f$ as constant limit that of polynomial equation $P_n(t, y)$ that uses the theorem of Stone-Weierstrass for Polynomials which are $C^1$, Therefore by the
theorem of Picard’s we fundamentally the reduction mapping theorem which is the I. V. P. \( y'(t) = P_n(t, y), y(t_o) = y_o \) are exclusively locally provable closely to \( t_o \). Signify the provenas \( y_n \). It can be exposedby uses of theorem of compactness in space which is continuous expression that having various subsequence of \( y_n \) that converted in to an solution for original I. V. P. In another proven of theorem of Peano, by using an diff. fixed point theorem.

Theorem of Picard, it is applies for most common first order diff. eq. after that (4.1), such that

\[
(t^2 + y^2 + \left( \frac{dy}{dt} \right)^2) = 1, \ y(0) = \frac{1}{2}
\]  

(4.9)

Situation of \( t = 0 \) yields \( 1/4 + y'(0)^2 = 1 \), \( So \ y'(0) = \pm \sqrt{3}/2 \). We having to made a optimal of \( y'(0) \) beforehand we can pointed out a specific solution, hence the diff. eq. doesn’t govern \( y'(0) = \pm \sqrt{3}/2 \), here we have to prove.

\[
\frac{dy}{dt} = \sqrt{1 - t^2 - y^2}, y(0) = \frac{1}{2},
\]

Although by taking \( y'(0) = -\sqrt{3}/2 \) pointers to the eq.

\[
\frac{dy}{dt} = \sqrt{1 - t^2 - y^2}, y(0) = \frac{1}{2},
\]

Each and every of those having a unique proven close to \( t = 0 \) by using theorem of Picard.
Maximum simply, an primary order ordinary diff. eq. that shows like
\[ g(t, y, y') = 0 \]
that with various primary properties
\[ y(t_0) = y_0 \quad \& \quad y'(t_0) = z_0 (so \ g(t_0, y_0, z_0) = 0) \] . Considering \( g(t, y, z) \) which is a \( C^1 \) expression of 3 variables are as far as \( \left( \frac{\partial g}{\partial z} \right)(t_0, y_0, z_0) \neq 0 \) it is the theorem of implicit function that says us there is not closed set \( U \supset (t_0, y_0) \) in \( R^2 \) in & a \( C^1 \) expression
\[
\begin{align*}
\text{such} \\
z_0 = f(t_0, y_0) \quad \text{and for all } (t, y) \text{ near } (t_0, y_0), \ g(t, y, z) = 0 \quad \text{if and only if } \ z = f(t, y).
\end{align*}
\]
Consequently the diff. eq. \( g(t, y, y') = 0 \) which is locally close to \( (t_0, y_0, z_0) \) looks like \( y' = f(t, y) \) like shows which is precisely the types of diff. eq. we proved locally as well as uniquely by the theorem of Picard. Therefore theorem of Picard that suggests local existence as well as uniqueness of impartially common first-order ODEs. Theorem of Picard simplifies to a first-order ordinary diff. eq. having a vector-valued expression:
\[
\frac{dy}{dt} = f(t, y(t))
\]
Now \( f \) as well as \( y \) are having standards in \( R^n \) also \( f \) is \( C^1 \). Fundamentally the only modification required to spread the proven of theorem of Picard from the one dimensional condition to upper dimensions is using as an integral slightly than differential it from the mean value theorem which shows a \( C^1 \) expression that is generally Lipschitz.
5.8 INEQUALITY DIFFERENTIAL

In the example no. 5.7.2 specified that the I. V. P. \( y'(t) = y(t)^2 - t, \ y(0) = 1 \) it having an proven that setbacks up in limited time. To approximation the expandable time Let \( Y(t) = 1/y(t) \) and displays where \( Y(t) = 0 \). From \( Y'(t) = tY(t)^2 - 1 \) and \( Y(0) = 1 \) a computer algebra package having \( Y(t) = 0 \) at \( t \approx 1.125 \)

**Theorem 5.8.1:** The proven to \( y'(t) = y(t)^2 - t \) satisfying \( y(0) = 1 \) it is indeterminate where beforehand \( t = 1.125 \)

This is not stronger than what numeral’s propose i.e. the expandable time is approximate 1.125 but when solving somewhat louder needs a maximum cautious examine then we desire to improve. An skilled in non-linear ODE’s expressed me that in performs non-entity efforts to proving more loudestimates for blow-up periodsmere being of a blow-up time typically serves).

**Proof :-** Here we know \( y(t) \) is clear for slight \( t > 0 \). Take responsibility \( y(t) \) it is clear for \( 0 \leq t < c \). We are showing for appropriate \( c \) that \( y(t) \geq c/(c - t) \) for \( 0 \leq t < c \), so \( y(t) \to \infty \) as \( t \to c^+ \). Hence \( y(t) \) have to indeterminate for particular \( t \leq c \).

Set \( z(t) = c/(c - t) \), that with \( c \) is quiet to be determined, consequently
\[
\frac{d}{dt}(y - z) = y^2 - t - \frac{dz}{dt}
= y^2 - t - \frac{c}{(c - t)^2}
= y^2 - z^2 + \left(1 - \frac{1}{c}\right)z^2 - t
= (y - z)(y + z) + \frac{(c - 1)c}{(c - t)^2} - t
\]

By calculating \((c - 1)c / (c - t)^2 - t \geq 0 \text{ for } 0 \geq t < c\) as long as \(c - 1 \geq (4/27)c^2\), that which occurs for \(c\) in the middle of the 2 roots of \(x - 1 = (4/27)x^2\), the roots are about 1.2207 as well as 5.5292. Therefore by taking \(c = 1.221\), here we having \((y(t) - z(t))' \geq (y(t) - z(t))(y(t) + z(t))\) for \(0 \leq t < c\) By using this integrating feature, this diff. inequality is the parallel as

\[
\frac{d}{dt} \left\{ -t \int_0^t (y(s) + z(s)) \, ds \right\} (y(t) - z(t)) \geq 0
\]

Hence

\[
- \int_0^t (y(s) + z(s)) \, ds \bigg|_{t=0} = 0, \quad e^{- \int_0^t (y(s) + z(s)) \, ds} (y(t) - z(t)) \geq 0 \text{ for } t \geq 0, \text{ so } y(t) - z(t) \geq 0
\]

Cause of the exponential factor is not negative. Therefore \(y(t) \geq z(t) = c / c(c - t)\) is proved.