CHAPTER - 7

BUCKLING ANALYSIS OF
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7.1. INTRODUCTION

In the previous chapters it was tacitly assumed that the pile-soil systems subjected to generalized load conditions could be analysed through method of superpositioning. That is why the influence of axial load, horizontal thrust, moments and torque was considered independently, respectively in chapters four, five and six. At working loads, it is clear that these basic load conditions could be individually considered as and when required and the corresponding response could be superposed to define the final response against applied load condition.

As the load on the system gets increased or if it be needed for the purpose of designing the system through ultimate load method, it may be necessary to consider coupling between two or all of the above conditions. For example, a system subjected to pure axial load when approaches ultimate load condition, would excite bending of significant order in the system. In fact such a condition defines the problem of buckling or in general a problem of structural stability. In structural analysis it is common practice to analyse such stability problem through formulation of eigen value problem. Therefore it will be of practical design interest to formulate and solve appropriate eigen value problem defining the condition
of stability in the soil-pile system especially by representing the unit behaviour of the soil through subgrade reaction modulii and representing the soil mass through various soil elements discussed in earlier chapters.

Even otherwise it is well known that ultimate merit of a method of structural analysis is judged by its capacity to solve the eigen value problem of the system. Therefore it was considered worthwhile to solve typical set of problems involving buckling of pile-soil system subjected to axial load. In literature some work of this kind was available for purpose of comparison hence these problems were resolved by employing the soil elements.

In this chapter the problem of buckling of soil-pile system subjected to axial load is formulated from the first principles, clearly pointing out various characteristics influencing the complex behaviour of the system and the same is solved.

7.2. FORMULATION OF NON LINEAR GOVERNING EQUATION

The basic feature of a problem having geometric nonlinearity is that equilibrium equations must be written with respect to the deformed geometry which is not known in advance. Strictly, equilibrium equations of the deformed configuration should be used in all problem. Theory on the formulation of such a nonlinear governing equation for a pile-soil system, based on Mallett and Marcal (41), is attempted to develop in this section.
The strain energy stored in the pile can be given by,

\[ \Pi = \int \frac{1}{2} f_i E_{ij} f_j \, dv \quad \ldots (7.1) \]

in which \( f \) denote the functional dependence of strains upon displacement and \( E \) the coefficient matrix in generalized Hooke's law. \( f \) can be given by (ref figure 7.1),

\[ f = \frac{du}{dx} + \frac{1}{2} \left( \frac{dv}{dy} \right)^2 - \mu \frac{d^2 v}{dx^2} \quad \ldots (7.2) \]

\( f \) is the fundamental source of nonlinearity under the assumption that the strain due to midline rotation is not small relative to the direct midline strain.

Substituting equation 7.2 into the strain energy expression and indicating spatial differentiation with subscripts,

\[ \Pi = \int \frac{E}{2} \left( \frac{u_{x \xi}}{} - \mu v_{x \xi \xi} + \frac{1}{2} v_{x \xi}^2 \right)^2 \, dv \quad \ldots (7.3) \]

This strain energy functional for a circular pile section can be represented in matrix notation integrating over pile cross section \( A_p \), as

\[ \Pi = \int \left\{ u_{\xi}, v_{\xi}, v_{\xi \xi} \right\} \left[ \begin{array}{ccc} \frac{(E A_p)}{2} & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_p / A_p \end{array} \right] \left[ \begin{array}{c} \frac{E A_p}{6} \\ \frac{E A_p}{12} \end{array} \right] \left\{ u_{\xi}, v_{\xi}, v_{\xi \xi} \right\} \, d\xi \quad (7.4) \]

which can be written as,
\[
\Pi = \int_{\xi} \left( \{\epsilon\}^T \left[ \frac{E_P A_P}{2} [\hat{K}] + \frac{E_P A_P}{6} [\hat{N}] + \frac{E_P A_P}{12} [\hat{N}^2] \right] \{\epsilon\} \right) \, d\xi \quad (7.5)
\]

Representing axial displacement, \(u\), through linear polynomial and transverse displacement, \(v\), through cubic polynomial, the displacement vector can be expressed through,

\[
\{\mathbf{u}\} = [x] \{\mathbf{\alpha}\} \quad \ldots \quad (7.6)
\]

wherein

\[
[x] = \begin{bmatrix}
1 & \xi & 0 & 0 & 0 \\
0 & 0 & 1 & \xi^2 & \xi^3
\end{bmatrix} \quad \ldots \quad (7.7)
\]

\[
\{\mathbf{u}\} = \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} \quad \ldots \quad (7.8)
\]

and \( \{\mathbf{\alpha}\}^T = \{\alpha_1, \alpha_2, \ldots, \alpha_6\} \quad \ldots \quad (7.8a) \)

With introduction of assumed mode shape and applying proper transformation, displacement vector can be expressed in terms of system gridpoint degrees of freedom \(\{\delta\}\), such that,

\[
\{\mathbf{u}\} = [x] \begin{bmatrix} T_{\mathbf{\alpha}\mathbf{q}}^T \\ T_{\mathbf{q}\delta} \end{bmatrix} \{\delta\} \quad (7.9)
\]

where

\[
\begin{bmatrix} T_{\mathbf{d}\mathbf{q}} \end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-1/L & 0 & 0 & 1/L & 0 & 0 \\
0 & +1 & 0 & 0 & 0 & 0 \\
0 & 0 & +1 & 0 & 0 & 0 \\
0 & -3/L^2 & -2/L & 0 & +3/L^2 & -1/L \\
0 & 2/L^3 & +1/L^2 & 0 & -2/L^3 & +1/L^2
\end{bmatrix} \quad (7.10)
\]
\[
\begin{bmatrix}
  +T_X & +T_Y & 0 \\
  -T_Y & +T_X & 0 \\
  0 & 0 & 1 \\
\end{bmatrix}
\]  \tag{7.11}

and \( T_x = \frac{X_j - X_i}{L} \), \( T_y = \frac{Y_j - Y_i}{L} \)

\( L = \) Length of the element

and \( \{ \xi \}^T = \{ u_i \ v_i \ \theta_i \ u_j \ v_j \ \theta_j \} \)  \tag{7.12}

Vector \( \{ \xi \} \) can be expressed in terms of the assumed functions through a linear transformation of the form

\[ \{ \xi \} = [D]\{ \xi \} \]  \tag{7.13}

Substitution into equation 7.5 yields

\[ \{ \xi \}^T \left[ \frac{1}{2} [\tilde{K}] + \frac{1}{6} [\tilde{N}_1] + \frac{1}{12} [\tilde{N}_2] \right] \{ \xi \} \]  \tag{7.14}

in which

\[
[\tilde{K}] = \int_\xi E_p A_p [D]^T \tilde{[K]} [D] \ d\xi
\]

\[
[\tilde{N}_1] = \int_\xi E_p A_p [D]^T \tilde{[N}_1] [D] \ d\xi
\]

\[
[\tilde{N}_2] = \int_\xi E_p A_p [D]^T \tilde{[N}_2] [D] \ d\xi
\]

\( [\tilde{K}], [\tilde{N}_1], \) and \( [\tilde{N}_2] \) can be defined as,
Transforming equation 7.14 to system grid point degrees of freedom, total potential energy for the system can be given by,

$$\phi_p = \{\delta\}^T \left[ \frac{1}{2} [K] + \frac{1}{6} [N1] + \frac{1}{12} [N2] \right] \{\delta\} - \{\delta\}^T \{P\} \quad (7.17)$$
in which
\[
[K] = [T_{q6}]^T [T_{q6}] + [K][T_{q6}]^T [T_{q6}] + [K_{\text{soil}}] \tag{7.18a}
\]
\[
[N1] = [T_{q6}]^T [N1][T_{q6}]^T [T_{q6}] \tag{7.18b}
\]
\[
[N2] = [T_{q6}]^T [N2][T_{q6}]^T [T_{q6}] \tag{7.18c}
\]

Where \([K]\) is well known linear pile element stiffness coupled with soil element stiffness; \(P\) is the axial load; \([N1]\) and \([N2]\) are first order and second order incremental stiffness matrix respectively. It may be noted that unlike \([K]\) matrix \([N1]\) and \([N2]\) matrices are independent of elastic properties and are instead a function of the element geometry and its internal membrane forces. While \([N1]\) is indirectly a linear function of \(\{\delta\}\), \([N2]\) is indirectly a quadratic function of \(\{\delta\}\).

From principle of virtual work and knowing that \([N1]\) and \([N2]\) matrices are indirectly linear and quadratic functions of \(\{\delta\}\), it can be proved that,
\[
\left[ [K] + \frac{1}{2} [N1] + \frac{1}{3} [N2] \right] \{\delta\} = \{P\} \tag{7.19}
\]

This is the nonlinear governing matrix equation for direct formulation.

Critical load can be found out based on the vanishing of second variation of total potential energy. Direct formulation was based on the first variation and hence second
variation can be obtained applying variation to equation 7.19.

\[
\begin{bmatrix} [K] + [N_1] + [N_2] \end{bmatrix} \begin{bmatrix} \Delta \delta \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \quad \ldots (7.20)
\]

A point of instability can be found out from the existence of a nontrivial solution to equation 7.20. Such a point can be sought computationally by assuming that

\[
\begin{bmatrix} [N_1] + [N_2] \end{bmatrix} \begin{bmatrix} \delta_{cr} \end{bmatrix} = \begin{bmatrix} \frac{P_{cr}}{p} [N_1] + \left( \frac{P_{cr}}{p} \right)^2 [N_2] \end{bmatrix} \begin{bmatrix} \delta \end{bmatrix} \quad (7.21)
\]

where over-symbol (-) indicates equilibrium position.

This is valid when displacement vary linearly with applied load level p and this can be achieved by keeping p sufficiently low. Now from equations 7.20 and 7.21, an approximate critical load can be determined by,

\[
\begin{bmatrix} [K] + \frac{P_{cr}}{p} [N_1] + \left( \frac{P_{cr}}{p} \right)^2 [N_2] \end{bmatrix} \begin{bmatrix} \Delta \delta \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \quad \ldots (7.22)
\]

This is a quadratic eigen value problem, however, for small displacements \([N_2]\) matrix can be neglected and hence in that case equation 7.22 would reduce to linear eigen value problem.

### 7.3. **METHOD OF ANALYSIS**

#### 7.3.1 **Analysis For Critical Load**

As displacement and rotation in case of a pile-soil system will be less due to the support offered by soil, critical load with reasonable accuracy can be achieved through linear eigen value solution given by,
Now representing $\frac{P_{cr}}{P} = \lambda$ equation 7.23 may written as,

$$\begin{bmatrix} K \end{bmatrix} \{ \Delta \delta \} + \lambda \begin{bmatrix} N1 \end{bmatrix} \{ \Delta \delta \} = \{ 0 \} \quad \cdots (7.24)$$

Once more equation 7.24 can be rearranged as,

$$\begin{bmatrix} K \end{bmatrix}^{-1} \begin{bmatrix} N1 \end{bmatrix} \{ \Delta \delta \} = - \frac{1}{\lambda} \{ \Delta \delta \} \quad \cdots (7.25)$$

The negative sign appearing in the right hand side of the equation may be changed to positive if it is understood that compressive membrane stresses are used in forming $[N1]$ matrix. It may be noted that $[K]$ in equation 7.25 indicates total stiffness matrix of the system consisting of linear pile element stiffness matrix and consistent line soil element stiffness, attached to pile elements whereas, $[N1]$ is initial stress stiffness matrix. A matrix iteration scheme was adopted to solve equation 7.25 for $1/\lambda$. The basic computational flow for determining approximate critical load is shown in figure 7.2.

7.3.2 Geometrically Nonlinear Analysis

The direct method was applied to carry out geometrically nonlinear analysis of pile-soil system. This is a direct method of solving non linear problems through the governing equation 7.19. The approach is an iterative nature and the computational flow is shown in figure 7.3.
7.4. RESULTS AND DISCUSSIONS

7.4.1 Critical Load

Critical loads for pile-soil system, using consistent soil line element, were evaluated for five types of boundary conditions, namely, fixed-fixed, fixed-pinned, pinned-pinned, pinned-fixed and free-free (ref. figure 7.4). Three cases of length i.e. 500, 1000 and 2000 cms and three cases of L/D ratios, namely, 10, 25 and 100 were studied. Soil was considered as homogeneous having four types of relative rigidity, viz, \(2 \times 10^2\), \(2 \times 10^3\), \(2 \times 10^4\) whereas three values of \(E_p\) were considered, namely, \(2 \times 10^4\), \(2 \times 10^5\) and \(2 \times 10^6\) kg/cm\(^2\). Thus a total of 540 cases were studied. Poisson's ratio for soil and pile was assumed as 0.45 and 0.15 respectively.

To represent the critical load, few non-dimensional parameters were introduced. They are

\[
I_{cr} = \frac{P_{cr} T^2}{E_p I_p} \quad \ldots (7.26)
\]

Where

\[
T = 4 \sqrt{\frac{E_p I_p}{K_n D}} \quad \ldots (7.27)
\]

and

\[
Y_m = \frac{L}{T}
\]

in which case

- \(P_{cr}\) = Critical load
- \(E_p, I_p\) = Young's modulus and moment of inertia of pile section
- \(K_n\) = Normal subgrade reaction modulus
Knowing critical load, $I_{cr}$ was determined for various cases and is plotted in figure 7.4. Figure 7.4 also indicates the comparison of present analysis with the one available in literature. The comparison reveals an excellent agreement between them. It may be noted that critical load of pile embedded in other type of soil can also be evaluated in the same manner; only employing appropriate subgrade reaction modulii for the respective soil.

7.4.2 Geometrically Nonlinear Analysis

Geometrically nonlinear analysis was carried out for a 40 cm diameter and 1000 cm long pile in homogeneous soil. Young's modulus of pile was assumed as $2 \times 10^5$ kg/cm$^2$ and relative rigidity was assumed as $10^3$. Poisson's ratio for pile and soil was kept constant and was taken as 0.15 and 0.45 respectively. A small lateral force of 50 kg was applied at top to excite in lateral direction. However after few iterations this exiting load can be withdrawn. Top displacements and rotations at various vertical load level are shown in table 7.1. It may be observed that though vertical displacement increases almost linearly with load, horizontal displacement and rotations increases rapidly (and nonlinearly), specially near critical load.
TABLE 7.1

TOP DISPLACEMENTS AND ROTATIONS OF 40CM DIAMETER AND 1000 CM LONG PILE TAKING INTO ACCOUNT GEOMETRIC NONLINEARITY ($E_p = 2 \times 10^5$ kg/cm$^2$, $K = 10^3$).

<table>
<thead>
<tr>
<th>Horizontal load (kg)</th>
<th>Vertical load (kg)</th>
<th>Horizontal Displacement (cm)</th>
<th>Vertical Displacement (cm)</th>
<th>Rotation (RAD)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7 x $10^5$</td>
<td>0.335 x $10^{-2}$</td>
<td>0.679 x $10^1$</td>
<td>0.257 x $10^{-4}$</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>1.2 x $10^6$</td>
<td>0.582 x $10^{-2}$</td>
<td>0.116 x $10^2$</td>
<td>0.5136 x $10^{-4}$</td>
</tr>
<tr>
<td>1.8 x $10^6$</td>
<td>0.178 x $10^{-1}$</td>
<td>0.175 x $10^2$</td>
<td>0.1767 x $10^{-3}$</td>
<td></td>
</tr>
</tbody>
</table>
FIG 7.1 CO-ORDINATE SYSTEM FOR LINE ELEMENT
START

GENERATE/READ ALL DATA, FORM ALL NECESSARY MATRICES

ASSEMBLE [K] = [K] PILE + [K] SOIL

APPLY AN UNIT LOAD AT PILE TOP

SOLVE PILE SOIL-SYSTEM FOR {δj} APPLYING NECESSARY BOUNDARY CONDITIONS

COMPUTE [CN1] MATRIX FOR EACH ELEMENT THROUGH {δj}

ASSEMBLE [CN1] FOR WHOLE SYSTEM

INVERT [KJ] TO DETERMINE [K]ij

ASSUME {δj}/

DIVIDE ALL TERMS OF {δj} BY THE FIRST TERM OF {δj} AND FORM {δj}1

DETERMINE NEW VALUE OF {δj} BY

{δj} = [KJ] [CN1] {δj}1

DIVIDE ALL TERMS OF {δj} BY THE FIRST TERM OF {δj} AND FORM {δj}1

COMPARE ALL TERMS OF NEWLY FORMED {δj}1 WITH PREVIOUS {δj}1 AND CHECK WHETHER ALL OF THEM ARE WITHIN PRESCRIBED LIMIT.

NO

YES

DISPLAY FIRST TERM OF THE CURRENT {δj} AS ALSO {δj}i

STOP

FIGURE 7.2 EVALUATION OF CRITICAL LOAD
START
READ ALL NECESSARY DATA. FORM ALL NECESSARY MATRICES
ASSEMBLE \([K]\) = \([K]\)PILE + 
\([K]\)SOIL
SET P
ASSEMBLE \([N1]\), \([N2]\) THROUGH\(\{\delta\}\)
INTERROGATE \([K1+1/2[N1]+1/2[N2]][\delta]+[P]\)
YES
DISPLAY SOLUTION
REVISE THE NODAL COORDINATES ACCORDING TO DISPLACED POSITION
SET P FOR NEXT STAGE OF LOADING

FIGURE 73 GEOMETRICALLY NON LINEAR ANALYSIS
Fig. 14: Nondimensional representation of critical load of a pile imbedded in homogeneous/two-layered soil.
Fig 7.4 Non-dimensional representation of critical load of a pile embedded in homogenous/two layered soil.