CHAPTER - 2

REVIEW OF LITERATURE
2.1. INTRODUCTION

The study, how the load is transmitted to the soil surrounding an pile or pile group, is an important aspect of foundation engineering. As such a large bulk of literature comprising the details of field and laboratory tests on piles as also some literature on analysis of structural response of pile-soil system is available. The literature concerning analytical approach to the problem of pile design is of great importance and hence some important aspects of the works on analysis of pile-soil system have been presented in this chapter.

The analytical developments in the design of pile-soil systems have been through numerical methods, employing high speed digital computers. In that either the concept of normal sub-grade reaction modulus were employed whereby the piles are assumed to be supported at discrete points through isolated springs or by establishing displacement compatibility at the interface of pile and soil where displacement of pile could be determined through standard structural procedure and soil through Mindlin's equation (44). Of late, two dimensional and three dimensional finite element analysis has been developed for analysing the pile-soil system.

The analysis of pile-soil system would primarily consist of the following -
A review of literature has been done in detail on the above mentioned topics and some selected literature as also various methods of analysis of pile-soil system are presented, in brief, in subsequent sections.

2.2. AXIALLY LOADED SINGLE PILE

Many investigators outlined approximate method based on elastic theory for analysing different aspects of the load displacement behaviour of single axially loaded piles and piers. D'Appolonia and Romualdi (21) determined load transfer in 14 BP 89 and 14 BP 117 end bearing steel H - piles of 40 ft length. The piles were driven through sand and gravel and were fixed with strain gauges located along the length of the piles. A theoretical load transfer between a point-bearing steel pile and an elastic medium was also calculated from fundamental compatibility concepts in the theory of elasticity. In his analysis following three simplifying assumptions were made.

(a) The soil trapped between the flanges of the pile was assumed to act integrally with the pile.
(b) The tip of pile was assumed not to move.
(c) The surrounding soil was assumed to be a semi-infinite elastic solid.

In his analysis the pile was divided into \( n \) equal segments of length \( h \). The interaction shear stress between the pile and soil was assumed constant over the length \( h \) and the resultant force, \( F \), was assumed to act at the midpoint of the interval.

The pile was assumed to be free to move within the soil and the vertical displacement at any interval mid point \( i \) was assigned as \( \Delta_i \). This is the displacement of the pile relative to the soil. The criterion from which the interaction forces were calculated is that there be no relative motion between the pile and soil except in those cases in which the interaction stress exceeds the shear strength between pile and soil, in which case the force is the product of the maximum shear strength and the pile surface area over the interval \( h \). The condition that the interaction forces be of such magnitude that there be no relative displacement between the pile and soil at any position \( i \) is then

\[
\sum_{j=1}^{n} d_{ij} F_j + \sum_{j=1}^{n} d'_{ij} F_j = \Delta_i \quad \ldots (2.1a)
\]

or

\[
\sum_{j=1}^{n} (d_{ij} + d'_{ij}) F_j = \Delta_i \quad \ldots (2.1b)
\]
An equation of this form could then be written for every point \( i \), leading to a system of \( n \) simultaneous equations for the interaction forces \( F_1, F_2, \ldots, F_n \). The interaction forces were assumed to be the force of the soil on the pile (negative upward) or its equal and opposite reaction, the force of the pile on the soil (positive downward). \( d_{ij} \) is the vertical displacement of the soil at any location \( i \), due to unit force at location \( j \). This value should actually be along the pile-soil interface but, because the pile was narrow compared with its length, all positions were referred to the pile centerline. In a similar manner, \( d'_{ij} \) could be defined as the deflection of the pile at any point \( i \), due to a unit load on the pile at joint \( j \).

Expressions for the coefficient \( d_{ij} \) was obtained from a solution for displacements due to a force in a semi-infinite elastic solid given by Mindlin (44, see also appendix B). The coefficients \( d_{ij} \) were calculated in a straightforward manner. The deflection at the point of a unit load on the pile was given by,

\[
d'_{jj} = W = \frac{L'}{A_p E_p} \quad \ldots \quad (2.1c)
\]

in which \( L' \) was the distance from the unit load point to the base of the pile, \( A_p \), the area of the steel and \( E_p \), the modulus of elasticity of pile. Points below the load point displace an amount proportional to the length of pile below the point in question and points above the load point
displace an amount equal to the displacement of the load point. The computed transferred load, using a maximum shear strength limitation for soil, was in good agreement with test results, where the maximum load that could be transferred from the pile to the surrounding soil over an interval $h_i$ is

$$F_i = \gamma \alpha (4B) \int_{Z_o}^{Z_1} Z \, dz$$

in which $Z_o$ and $Z_1$ are the depths to the top and bottom of the interval, respectively, $4B$ is the surface area of the pile per unit length of pile, $\alpha$ is the coefficient of friction and was assumed as 0.6 and $\gamma$ is the unit weight of soil. The analysis suggests that pile capacity can be predicted by rational analysis and a reasonable knowledge of pertinent soil properties. It also suggests that ultimate total load on a pile can be determined by adding the point load to the computed transferred load as computed from fundamental concepts of bearing capacity for foundations.

Thurman and D'Appolonia (80) analyzed, based on the same elastic theory, the behaviour of single compressible floating and end bearing piles. However the authors, mentioned so far, considered only the specific cases, and general characteristics of the behaviour of a single compressible pile have not been considered.

Poulos and Davis (55) and Mattes and Poulos (42) employed linear elastic theory to analyze the behaviour of a
incompressible and compressible floating pile of circular cross section in an ideal elastic soil mass respectively and proposed exhaustive design charts as functions of various non-dimensional parameters. Their method of analysis had following features:

In the analysis, they considered a circular pile of length $L$ and outer diameter $d$ with area of pile section equal to $A_p$. The pile was divided into $n$ equal cylindrical elements, with any element $j$ being acted upon by a uniformly distributed vertical shear stress $p_j$ around the periphery. The base was considered to be a rigid circular disc of diameter $d_p$ acted upon by a uniform vertical stress $p_b$. The surrounding soil layer was assumed to have constant elastic parameters $E_s$ and $\mu_s$ which remained unaltered by the presence of the pile.

The analysis was based on the compatibility of vertical displacement of pile and adjacent soil where, these displacements were calculated at the mid point of the outer space of each element. Referring to figure 2.1(a) and assuming vertical displacements as positive, the displacement at $i$ due to the shear stress $p_j$ on element $j$ was expressed as

$$\delta_{ij} = \frac{d}{E_s} I_{ij} p_j \quad \ldots \quad (2.2a)$$

in which $I_{ij} = \text{the influence factor for the displacement at } i \text{ due to shear stress } p_j \text{ on element } j$. 
The displacement at \( i \) due to all \( n \) elements of the pile, plus the base, is therefore

\[
\rho_i = \frac{d}{E_s} \left( \sum_{j=1}^{n} I_{ij} P_j + I_{ib} P_b \right) \quad \ldots (2.2b)
\]

in which \( I_{ib} \) = the influence factor for displacement at \( i \) due to a vertical stress on the rigid base of the pile and was obtained by double integration over a circular area of the Mindlin equation (44) for vertical displacement.

The displacement of soil under the pile base may be given by,

\[
\rho_b = \frac{d}{E_s} \left( \sum_{j=1}^{n} I_{bj} P_j + I_{bb} P_b \right) \quad \ldots (2.3)
\]

The displacement of all elements of pile could be written as:

\[
\{ \rho \} = \frac{d}{E_s} \left[ sI \right] \{ p \} \quad \ldots (2.4)
\]

in which

\[
\left[ sI \right] = \begin{bmatrix}
I_{11} & I_{12} & \cdots & I_{1n} & I_{1b} \\
I_{21} & I_{22} & \cdots & I_{2n} & I_{2b} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
I_{n1} & I_{n2} & \cdots & I_{nn} & I_{nb} \\
I_{b1} & I_{b2} & \cdots & I_{bn} & I_{bb}
\end{bmatrix}_{(n+1)x(n+1)}
\]

where \( I_{ij}, I_{ib}, I_{bj}, \) and \( I_{bb} \) are soil displacement influence factors and were obtained by double integration of the Mindlin equation and
Now considering the vertical equilibrium of a small cylindrical element (figure 2.1b) of the pile, following relationship could be established:

\[
\begin{align*}
\frac{\partial \sigma}{\partial z} &= -\frac{p A_d}{A_p} \\
&= -\frac{4p}{R_A d} \quad \ldots \quad (2.7)
\end{align*}
\]

in which \(\sigma\) = normal stress in the pile; \(p\) = vertical shear stress on pile periphery; and \(R_A = A_p/(\pi d^2/4)\).

Considering the axial strain of the element

\[
\frac{\partial P_i}{\partial z} = -\frac{\sigma}{E_p} \quad \ldots \quad (2.8)
\]

in which \(P_i\) = the displacement of the pile and \(E_p\) = Young's modulus of the pile material.

Differentiating equation 2.8 with respect to \(z\) and substituting in equation 2.7 yields

\[
\frac{\partial^2 P_i}{\partial z^2} = \frac{4p}{d} \frac{1}{E_p R_A} \quad \ldots \quad (2.9)
\]

For an element \(i\), where \(n-1 \gg i \gg 2\), equation 2.9 may be written in finite difference form, so that shear stress, \(P_i\), can be expressed as

\[
P_i = \frac{d}{4} E_p R_A \left( P_{i-1}^{i-2} + P_{i}^{i+1} \right) \quad .. \quad (2.10)
\]
in which $\delta$ = length of an element = $L/n$

Similarly, considering an imaginary soil element above first real soil element, shear stress $p_1$ on element 1 may be expressed as

$$p_1 = \frac{d}{4} \frac{E_p R_A}{\delta^2} \left( -p_1 + p_2 \right) + \frac{P}{\pi d^2} \frac{n}{L/d} \ldots (2.11)$$

For the bottom cylindrical element $n$ of the pile, shear stress $p_n$ could be related to the displacements of the midpoint of elements $n-2$, $n-1$, $n$ and the base, using equation 2.9 and a finite difference formula for points with non-uniform spacing. Using a formula with an error of order $\delta^2$ to be consistent with the finite difference approximation in equation 2.10, equation 2.9 becomes,

$$p_n = \frac{d}{4} \frac{E_p R_A}{\delta^2} \left( -0.2 p_{n-2} + 2 p_{n-1} - 5 p_n + 3.2 p_b \right) \ldots (2.12)$$

Normal stresses at base may be given by,

$$p_b = \frac{d}{4} \frac{E_p R_A}{\delta^2} \frac{L}{n R_A} \left( -1.33 p_{n-1} + 12 p_n - 10.67 p_b \right) \ldots (2.13)$$

Now shear stresses and base pressure could be written in matrix form as,

$$\mathbf{p} = \frac{d}{4} \frac{E_p R_A}{\delta^2} \left[ \mathbf{p} \mathbf{I} \right] \left\{ \mathbf{p} \mathbf{C} \right\} + \left\{ \mathbf{Y} \right\} \ldots (2.14a)$$

in which

$$\left[ \mathbf{p} \mathbf{I} \right] = \begin{bmatrix} -1 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & -0.2 & 2 & -5 & 3.2 \\ 0 & 0 & 0 & 0 & \ldots & 0 & -1.33f & 12f & -10.67f \end{bmatrix} \ldots (2.14b)$$
\[ f = \frac{L/d}{n R_A} \]

\[
\begin{pmatrix}
\frac{E}{\lambda d^2} & \frac{n}{L/d} \\
\end{pmatrix}
\begin{pmatrix}
\vdots \\
\vdots \\
\end{pmatrix}
\]

\[ \{y\} = \begin{pmatrix}
\vdots \\
\vdots \\
\end{pmatrix} \quad \text{(n+1, 1)} \]

(2.14c)

As the vertical displacement of pile and soil are compatible i.e. \( \{s \} = \{p \} \), equation 2.4 and 2.14 yields

\[
\{p\} = \left( \left[ I \right] - \frac{n^2}{4(L/d)^2} \cdot K \cdot \left[ p \right] \left[ s \right] \right)^{-1} \{y\} \quad \text{(2.15)}
\]

in which \( \left[ I \right] \) = the unit matrix of order \((n+1) \times (n+1)\)
and \( K = \frac{E_p}{E_s} \cdot R_A \quad \text{(2.16)} \)

Equation 2.15 may be solved to obtain unknown stresses acting on the pile. The displacement distribution may then be calculated from equation 2.4.

The above mentioned method was extended by Poulos (59) to analyze the behaviour of axially loaded pile in Gibson soil where, the modulus of elasticity of soil varies linearly with depth. The non-homogeneity of the soil mass was accommodated by modifying the value of \( E_s \), in equations 2.4 and 2.15 as,

\[
E_s \text{ (modified)} = 0.5 \left( E_{si} + E_{sj} \right) \quad \text{(2.16a)}
\]

in which \( E_{si} \) and \( E_{sj} \) are the values of the soil modulus at the influenced element \( i \) and the influencing element \( j \), respectively. However, the soil below the pile tip may also
be non-homogeneous, and in that case the displacement influence factors were modified conveniently by using an extension of the steinbrenner approximation (77). A series of parametric solutions were presented and analysis was also found to be applicable to piles in a layered soil, except in the case where the pile is founded in soil, that is more compressible than the overlaying strata.

Poulos and Mattes (54) studied the behaviour of axially loaded end bearing piles employing once more the same method of analysis. To allow for the reduction in the soil displacement due to the presence of the bearing stratum, a mirror image element $j'$ of an element $j$ was introduced (figure 2.2). Element $j'$, in that manner was acted upon by a shear stress $Kp_j$ acting in opposite direction to $p_j$ where $0 \leq K \leq 1$. For a floating pile where the bearing stratum is identical with the soil layer, $K$ was set equal to zero, while for the other extreme case of a rigid bearing stratum, $K$ was assumed to be 1.

Booker and Poulos (10) studied the problem of the settlement of a vertically loaded pile in a soil exhibiting creep properties by converting viscoelastic problem to an equivalent elastic problem by means of Laplace transform. The solution was then found in Laplace transform space in the form of eigen value expansion and transformed back to obtain a numerical solution to the original problem. Some typical solutions were presented for the settlement and base load transfer in a pile for two types of creep response.
All these analyses of Poulos were based on the following assumptions, in addition to those of ideal elasticity:
(a) The pile shaft load was replaced by uniform vertical shear stress on the surface of each of a suitable number of small cylindrical pile elements.
(b) The base was assumed to be a smooth disc, not necessarily of the same diameter as the shaft, across which the base load was uniformly distributed.
(c) The disturbance of the continuity of the elastic half space due to the presence of the piles was ignored.

Butterfield and Banerjee (15) studied the response of rigid and compressible single as also group of piles, embedded in a homogeneous isotropic elastic medium, by applying a rigorous analysis based on Mindlin's solution for a point load in the interior of an ideal elastic medium. The analysis was capable of eliminating assumptions (b) and (c) mentioned above and had shown that the assumption (b) has a significant effect on the response of underreamed piles and short plain piles and that appreciable error occur in the calculated values of radial stress components in the immediate vicinity of loaded pile due to assumption (c).

Banerjee and Davies (7) employed, within a boundary element algorithm, the integration of an appropriate elementary point force solution for the soil medium (Mindlin's solution for homogeneous soil) over the discretized surface elements of the pile-soil interface and then coupled the equations relating displacements and surface tractions for the soil
domain with compressibility and flexibility equations of the pile to generate the final systems of equations for single pile in homogeneous soil.

\[ \mathbf{P}^0 \mathbf{p}^0 = \mathbf{U}^0 \quad \ldots \quad (2.17) \]

where \( \mathbf{P}^0 \) is a \((2a+b)\) square matrix of coefficients

\( \mathbf{p}^0 \) is the vector of surface tractions at the pile-soil interface.

\( \mathbf{U}^0 \) is the vector of displacement boundary condition

\( a \) is the number of shaft segments

\( b \) is the number of base elements.

The basic feature of the boundary element method is presented herein.

The Boundary Element Method (3) utilizes the discretisation of the surface enveloping each homogeneous zone of the body. If a region \( \mathcal{D} \) is considered to be bounded by a surface \( S \), the displacements \( u_i \) and tractions \( p_i \) at a point \( A \) on \( S \) due to traction vectors \( \mathbf{\varphi}_j(B) \) at \( B \) over \( S \) can be obtained from:

\[ u_i(A) = \int_S K_{ij}(A,B) \varphi_j(B) \, ds \quad \ldots \quad (2.18) \]

\[ p_i(A) = \int_S \Gamma_{ij}(A,B) \varphi_j(B) \, ds + \int_S \mathbf{\varphi}_j(A) \varphi_j(B) \, ds \quad \ldots \quad (2.19) \]

where

\[ K_{ij}(A,B) = \frac{A_1}{r} \left( \delta_{ij} (3-4\mu) + \frac{\mathbf{\xi}_i \cdot \mathbf{\xi}_j}{r^2} \right) \]

\[ \Gamma_{ij}(A,B) = \frac{A_2}{r^2} \left( \frac{1-2\mu}{r} \left( n_j \mathbf{\xi}_i - n_i \mathbf{\xi}_j \right) \right) + \left( (1-2\mu) \delta_{ij} + \frac{3}{r^2} \mathbf{\xi}_i \cdot \mathbf{\xi}_j \right) \frac{\mathbf{\xi}_k \cdot \mathbf{n}_k}{r} \]

\[ A_1 = \frac{1}{16 \pi G (1 - \mu)} \quad , \quad A_2 = \frac{1}{8 \pi (1 - \mu)} \]
\[ \xi_i = y_i - x_i, \quad \xi_j = y_j - x_j, \quad \xi_k = y_k - x_k \]
\[ r^2 = \xi_i \xi_i, \quad p_i = \sigma_{ij} n_j \]

\( n \) = number of layers in soil along pile length.

\[ \sigma_{ij} = 1, \text{ for } i = j \]
\[ = 0, \quad i \neq j \]

\( \sigma_{ij} \) = stresses at \( A \)

\( G \) and \( \mu \) are the shear modulus and Poisson's ratio for the region \( D \).

The surface \( S \) can then be represented by a series of triangular or rectangular flat elements (ref. figure 2.3) and equations 2.18 and 2.19 for the centroids of each of the elements of the surface in matrix notation can be written as

\[ u = K \varphi \quad \ldots \quad (2.20) \]
\[ p = \Gamma \varphi \quad \ldots \quad (2.21) \]

By eliminating \( \varphi \) between these sets, one can obtain

\[ p = \Gamma K^{-1} u = F u \quad (2.22) \]

where the matrix \( F \) can be regarded as, in the terminology of the finite element analysis, the stiffness matrix for the region \( D \). Instead of the conventional relationship between the nodal forces and nodal displacements, a relationship between the surface tractions and surface displacements are obtained. Moreover because the governing differential equations are satisfied exactly by equations 2.18 and 2.19 for points within \( D \), the region may be made as large as required.

For any layer \( i \) equation 2.22 can be written for the pile domain 1 and the soil domain 2 as
\[ p_1^i = F_1^i u_1^i \quad \ldots \quad (2.23) \]

and
\[ p_2^i = F_2^i u_2^i \quad \ldots \quad (2.24) \]

By utilizing the equilibrium and compatibility at the pile-soil interface elements, these equations can be assembled to form a stiffness matrix for the pile-soil system for \( i \)th layer.

\[ p_i^i = L_i^i u_i^i \quad \ldots \quad (2.25) \]

It is interesting to note at this stage that because the thickness of each layer was assumed to be equal, the matrix \( F_1^k \) for the \( k \)th layer is obtainable from that of the \( j \)th layer from

\[ F_1^k = \Lambda F_1^j \quad \ldots \quad (2.26) \]

where \( \Lambda = E_i^k / E_i^j \), the ratio of the Young's moduli. From equation 2.25 for each of the layers in succession and assembling the system of equations 2.25 for each layer of the problem by satisfying the equilibrium and compatibilities at the layer interfaces as before; the final system can be written, as mentioned before,

\[ \mathbf{P}^o = L^o \mathbf{u}^o \quad \ldots \quad (2.17a) \]

where the matrix \( L^o \) is a block banded matrix. The system of equations then can be solved for surface displacements.

This method of analysis was then modified to solve for piles in nonhomogeneous soil. In that case, for each position of the load point, the soil medium was idealized by a two layer medium (figure 2.4) and the magnitudes of the layer modulii \( E_{s1} \) and \( E_{s2} \) were so adjusted that the displacements, parti-
cularly in the neighbourhood of the point load, in both systems were equal. From a series of trial computations, they specified the value of $E_s_1$ and $E_s_2$ in the following manner:

$$E_{s1} = E_s(0) + 0.5 mc$$

$$E_{s2} = E_s(0) + 3.0 mc \quad \text{for} \quad C < 0.4L \quad \text{(2.27)}$$

$$E_{s2} = E_s(0) + m (L + 0.5c) \quad \text{for} \quad C > 0.4L$$

where $L$ is the embedded length of the pile. Numerically integrated solutions, accurate to within three significant figures in the limiting homogeneous case, to the two-layer problem (figure 2.4) have been given recently by Davies and Banerjee (20). These solutions, combined with equation 2.27, were incorporated into the algorithm in the manner described above.

The work of Poulos, Banerjee, Butterfield, et al. has a limitation that unless the soil stratum is isotropic, homogeneous, elastic, direct Mindlin's theory has no validity. Consequently, their method has to be modified (as could be noted in case of Gibson soil) in all cases other than soil being isotropic, homogeneous and elastic. For the field conditions with the soil mass being heterogeneous, the method developed by Coyle, Reese and their associates (18) is versatile. Their method depends on the data from tests on instrumented piles. It had following features:

Their analysis aims at developing the load settlement curve and evaluating the load shared by the pile and soil at various
depth Z in figure 2.5. For analysis, the pile was divided into number of segments such as shown in figure 2.6. The analysis begins with the bottom most segment by assigning a small tip movement at the base of the segment. With the assumed tip movement and the forces acting on the system such as skin friction, bearing pressure etc, the movement at the top of the segment as also the load of the top of the segment were estimated by employing the standard structural procedure. The movement of the top of the base segment is equal to the movement of the tip of the segment above, so as to satisfy the structural compatibility between the segments. This gives the tip settlement for the section above. Once more the procedure adopted in base segment was employed for the next segment. In this manner the pile was worked up and the load shared at various depth as well as load at the pile top, corresponding to the assumed tip movement at the base of the pile, was estimated. For various assumed tip movements, different values of the load at the top of pile \( Q_0 \) and displacement of the pile top \( S \) were evaluated. This ultimately established the required load-settlement curve and the load coming up on each segment helps in establishing the load transfer mechanism. A typical such load transfer data is shown in figure 2.7.

In performing above kind of analysis, over and above the data obtained from the tests on soil samples, the relationship between the load transferred to the soil, shear strength of the soil and the pile movement is also needed. Such data
can only be obtained through the tests on instrumental piles. This retards the practical utility of the method of analysis. Further, the method does not account for phenomenon of local limiting equilibrium, which may involve the relative displacements between pile and the soil at various points over the interface.

Ellison et al (26) adopted axi-symmetric finite element analysis for piles in clay. This finite element method of analysis removed all the limitations inherent in the method proposed by Reese and Coyle. The analytical results were verified by carrying out field test on bored piles in "London clay". The authors used triangular and quadrilateral elements along with spring elements which could act as rigid, elastic or free connection between adjacent elements, depending on the assumed radial and vertical stiffness coefficients, $K_r$ and $K_z$, respectively. The method could be used to predict accurately the load capacity and load-deformation behaviour of a single pile. A trilinear approximation of the stress-strain curve obtained from undrained triaxial test was used. The formation of a tension crack which developed at the tip of the pile was shown, invalidating the classical plasticity solutions for tip capacity. The usual assumption of adhesion failure occurring at the pile-soil interface was supported. Randolph and Wroth (66) obtained an approximate closed form solution to a problem of a vertically loaded pile in a linear elastic soil. This solution was derived by uncoupling the load transfer for the pile into separate shaft
and base components. The final form of the solution gives the load settlement ratio of the pile in terms of the pile geometry and stiffness and soil stiffness. The analysis was also used to back analyze pile tests, enabling immediate estimates of the soil stiffness profile to be made from the measured load-settlement curve of the pile.

Pal and Parikh (49,50) performed axi-symmetric finite element analysis of axially loaded pile embedded in nonhomogeneous cohesive and stress dependent cohesionless soils respectively. A complete parametric investigation, covering a wide range of practical importance, was carried out and load-settlement characteristics were presented in terms of various non-dimensional parameters. However a further refinement of analysis, incorporating infinite element, was carried out by the author and have been presented in respective chapter.

2.3. AXIALLY LOADED PILE GROUPS

The majority of the theoretical solutions for prediction of load-settlement behaviour of pile groups were based on Mindlin's elastic half space concept. Pichumani and D'Appolonia (53) have presented solutions for evaluating the distribution of load and the displacements in the case of square pile groups both floating and end bearing, embedded in soil which exhibits perfect elastic-plastic behaviour.

Analysis of the settlement of axially loaded incompressible pile groups was carried out by Poulos (57). His method of analysis had following features:
Analysis was made of the settlement interaction between two identical piles in an elastic mass. This was carried out by taking influence of adjacent pile on the displacement $\rho_i$ of the soil adjacent to the centre of the periphery of an element i on the influenced pile and could be given by,

$$\rho_i = \sum_{j=1}^{n} p_j (1_{ij} + 2_{ij}) + p_b (1_{ib} + 2_{ib}) \quad \cdots (2.28)$$

where $1_{ij}$ is the displacement influence factor at element i due to a uniform ring load on element j on pile 1.

$2_{ij}$ is the displacement influence factor at element i due to a uniform ring load on element j on pile 2.

$1_{ib}$ is the displacement influence factor at element i due to a uniform load on the base of pile 1, and similarly for $2_{ib}$. It may be noted that pile 1 and pile 2 are influenced and influencing pile respectively.

A similar expression may be obtained for the displacement $\rho_b$ of soil directly beneath the base of the pile

$$\rho_b = \sum_{j=1}^{n} p_j (1_{bj} + 2_{bj}) + p_b (1_{bb} + 2_{bb}) \quad \cdots (2.29)$$

where $1_{bj}$ is the displacement factor for the pile base due to a uniform ring load on element j on pile 1, and similarly for $2_{bj}$.

$1_{bb}$ is the displacement factor for the pile base due to a uniform load on the base of pile 1, and similarly for $2_{bb}$. 
For the case of a semi-infinite mass, the displacement factors $I_{ij}$ and $I_{ij}^2$ may be obtained by integration of the Mindlin equation for vertical displacement, over the cylindrical ring elements $j$ on pile 1 and pile 2 respectively, for the appropriate points on pile 1. The factors $I_{ib}$ and $I_{ib}^2$ may similarly be obtained by integration of the Mindlin equation over the circular base of pile 1 and pile 2.

For all the elements on pile 1 the vertical displacements of the soil adjacent to the pile may be expressed in matrix form as

$$\{ \theta \} = \begin{bmatrix} [I_1] + [I_2] \end{bmatrix} \{ p \} + \begin{bmatrix} [I_{ib}] + [I_{ib}^2] \end{bmatrix} \{ p_b \}$$  \hspace{1cm} (2.30)

The soil displacement at each element may now be equated to the displacement of each element of the pile. If this displacement is assumed to be unity, then for the pile elements

$$\{ \theta \} = \{ 1 \} \hspace{1cm} \text{\ldots} \hspace{1cm} (2.31)$$

From equations 2.30 and 2.31

$$\begin{bmatrix} [I_1] + [I_2] \end{bmatrix} \{ p \} + \begin{bmatrix} [I_{ib}] + [I_{ib}^2] \end{bmatrix} \{ p_b \} = \{ 1 \}$$  \hspace{1cm} (2.32)

Equation 2.32 was solved to obtain the distribution of shear stress along each pile and the stress on the base, for unit displacement of the pile, whence the displacement of each pile for a unit load was calculated.

The increase in settlement due to adjacent equally loaded identical pile was expressed in terms of an interaction factor $\lambda$. Employing these interaction factors and assuming
that superposition holds good, pile groups in general were analysed for the case of a rigid pile cap and flexible pile cap.

Poulos and Mattes (56,63,58) have extended their analysis for the case of compressible pile also. They developed design charts for evaluating load-displacement behaviour of pile group-s. Solutions are provided for symmetrically placed two-pile groups, assuming the piles to be identical in behaviour. Two pile group analysis was extended to four and bigger pile groups. For estimating settlements of pile groups, certain group reduction factors were recommended. Values of these reduction factors have been provided for floating and point bearing pile groups for different widths of pile groups and number of piles in a group.

A rigorous elastic analysis of bonded compressible plain and underreamed piles and compressible piles in general groups under a rigid floating cap has been presented by Butterfield and Banerjee (15), in which the truly rigid pile base and radial deformation compatibility conditions could be included. They have extended their analysis (16) to consider pile group-pile cap interaction effects and provided design curves (figure 2.8) for estimating pile group settlement for different spacing to diameter and length to diameter ratios of piles. These curves are for pile groups having different numbers of piles.
Banerjee and Davies (6) studied load displacement and load distribution behaviour of axially loaded symmetrical pile groups embedded in Gibson soil by employing the concept of the interaction factors. Boundary element method, on isolated single piles and pile groups embedded in a non-homogeneous three-dimensional solid whose modulus of elasticity increases linearly with depth (Gibson soil), was applied to derive these factors. Results for settlement and load distribution characteristics have been presented in non-dimensional forms and were compared with experimental results. This revealed that boundary element method of analysis gives more realistic predictions of pile group behaviour than those described by Poulos (63) and Butterfield and Banerjee (15).

Ottaviani (48) has analysed the problem of settlement of pile groups by employing finite element method of analysis and has presented some interesting results for distribution of stress around a group. Such results are useful in clarifying the mechanism of load transfer from the piles to the surrounding soil and from that point of view, are very worthwhile. However the considerable effort of data preparation and the high cost, especially when non-linear soil behaviour is to be considered, may render such finite element analysis uneconomical for obtaining predictions of pile group settlement and load distribution for most practical cases.

Sowers et al (74) have determined the bearing capacity of friction pile groups in homogeneous clay from model studies. Saffery and Tate (70) have done model test on pile group in
a clay soil with particular reference to the behaviour of
the group when it is loaded eccentrically. The paper had
considered the results of loading tests on 3x3 groups of \( \frac{1}{4} \)
inch diameter model piles driven into a remoulded clay. The
trends of the values of load efficiency and settlement
characteristics for the various pile group conditions and
degrees of applied load eccentricity were discussed with
reference to the test results.

2.4. SUBGRADE REACTION MODULII

The concept of normal subgrade reaction modulus was introd-
ced into structural mechanics for the first time by Winckler
(85) and subsequently used by Zimmermann (89) for the purpose
of computing the stresses in railroad ties which rest on
ballast over their full length. During the following decades
the theory was expanded to include the computation of the
stresses in flexible foundations, such as continuous footing
or rafts and in concrete pavements acted upon by wheel loads.
Concept of normal subgrade reaction has also been used by
many investigators for computing the stresses in the piles and
sheet piles subjected to horizontal load. Hayashi (33), in
his comprehensive treatise on the subject, mentioned that
the normal subgrade reaction should be determined by loading
test. Hetenyi (32), the acknowledged pioneer on analysis on
beams on elastic foundation, while advocating the use of
normal subgrade reaction in such an analysis does not mention
anything regarding the factors which determine the numerical
value of the coefficients of subgrade reaction. Terzaghi (79)
published the factors which determine the value of coefficient
of normal subgrade reaction modulii in vertical and horizontal
directions for cohesionless sand and stiff clay, and numerical values were proposed for the constants which appeared in the equations defining these coefficients. He suggested an effective spring stiffness constant with depth for cohesive soils and linearly increasing with depth from zero for granular soils. Furthermore, he proposed how these spring stiffness may be determined from a plate bearing test on a 1 ft square horizontal plate, and suitably modified for width, length or depth effects. Biot (9) found that, taking for the coefficient of normal subgrade reaction \( K \) the value

\[
k (\text{lbs/inch}^3) = \frac{1.23}{B} \left[ \frac{E_S b^4}{C(1-\mu_s^2)E_b I} \right]^{0.11} \frac{E_S}{1-\mu_s^2}
\]

(2.33)

the maximum bending moment of a flexible beam of infinite length on a semi-infinite, homogeneous elastic, isotropic solid under a concentrated load, as found by conventional analysis will be the same as obtained by the more rigorous theory, where \( B \) is the width of the beam in inches, \( b \) is the half width of beam in inches, \( E_S \) and \( E_b \) are Young's modulus for subgrade and beam in p.s.i. respectively, \( I \) is the moment of inertia of the beam, \( \mu_s \) is the Poisson's ratio of subgrade and \( C = 1.00 \) if the distribution of pressure across the beam is uniform; \( 1.00 \leq C \leq 1.13 \) if the deflection across the beam is uniform.

However, bending moments other than the maximum one, as well as other influences in the beam (deflections, shearing forces, contact pressures) were not evaluated by Biot.
Vesic (84) analysed all these parameters and found that if coefficient of normal subgrade reaction,

\[ k_\infty = \frac{k_2}{B} = \frac{0.90}{C.B} \left( \frac{E_b}{E_d} \frac{b^4}{I} \right)^{0.083} \frac{E_s}{1-\mu_s^2} \]  

all the curves representing these parameters, obtained by conventional analysis would become congruent to the corresponding curves obtained by the rigorous solution of the same problem. This expression differs somewhat from that found by Biot by simply equating maximum bending moments. Introducing B instead of 2b and taking C = 1.10 as a sufficiently correct value for any practical purposes, the following expression for \( K_\infty \) was obtained

\[ k_\infty B = K_\infty = 0.65 \frac{12}{E_b} \frac{b^4}{I} \frac{E_s}{1-\mu_s^2} \]  

Infinity in the suffix indicates infinity of beam.

In some instances the value of the coefficient \( k \) for a square plate having width B is known. Denoting \( K B = \overline{K} \)

\[ K_\infty = 0.52 \frac{12}{E_b} \frac{b^4}{I} \frac{E_s}{\overline{K}} \]  

For possible range of values

\[ K_\infty = 0.20 - 0.65 \overline{K} \]  

where the lower limit is for very rigid beams and upper limit is for very flexible beams.

Vesic has recommended various methods of analysis depending upon the length characteristics, \( \Lambda L \), and has shown errors
involved in them. His results indicated that provided the length of the beam exceeds 
\[ \frac{4\sqrt{\frac{\lambda^2 E_b I}{K_0}}} \]
there is very little error in assuming that the foundation behaves as a spring medium for any type of loading. It may be noted though these normal subgrade reaction modulus was used for analysis of beams on the surface of an elastic isotropic solid, yet the method itself lends to the analysis of laterally loaded piles.

Broms (12) had considered the methods for the evaluation of the coefficient of normal subgrade reaction for long as also short piles. Indicating \( \lambda = 0.52 \frac{12\sqrt{E_b I}}{E_b I} \) in equation 2.36 (\( E_b \) and \( I \) now indicates Young's modulus and moment of inertia of pile respectively), \( K \) for long piles can be given by \( K = \lambda K \) ... (2.38)

Numerical calculations by Brom have indicated that the coefficient \( \lambda \) can only vary between narrow limits for steel, concrete or timber piles. It was given approximately by the expression \( \lambda = n_1 \frac{E_b I}{\rho} \) ... (2.39) in which \( n_1 \) and \( n_2 \) are functions of the unconfined compressive strength of the supporting soil and the pile material respectively and was indicated in various tables.

For short piles (\( \beta L < 2.25 \) where \( \beta = 4 \sqrt{\frac{K}{4 E_b I}} \)) a lateral load \( P \) at the mid height was applied and if \( y_p \) be the pure translation of the pile, the coefficient \( K_p \)
for lateral load $P$ could be given by,

$$K_p = \frac{P}{L y_p} \quad \ldots \quad (2.40)$$

However for low load levels when the deflection is proportional to the applied loads, $K_p$ can be given by,

$$K_p = \frac{E_s \sqrt{5}}{m (1 - \mu_s^2) \sqrt{L}} \quad \ldots \quad (2.41)$$

where $m$, a numerical factor, depends on $L/D$ ratio and its value was indicated in tabular form. He also suggested that the coefficient $K_m$ for the lateral deflection caused by moment could be evaluated from equations 2.40 or 2.41, assuming equivalent length $L'$ equal to $1/10$th the total length of the pile.

Brom has also indicated how coefficient of normal subgrade reaction could be evaluated approximately by plate load tests with the aid of theory of elasticity.

Francis (28) suggested that in the pile problem the medium extends on both sides of the pile, and not on one side only, as in the problem considered by Vesic, and if his expression is doubled to allow for this it becomes

$$K_0 = 1.30 \frac{12}{12} \frac{E_s B^4}{E_p I_p} \frac{E_s}{1-\mu_s^2} \quad \ldots \quad (2.42)$$

The term $\frac{12}{12} \frac{B^4}{I_p}$ does not vary greatly for different sections. If it is assumed as 1.24
\[ K_\infty = \frac{1.60 \left( E_s \right)^{13/12}}{\left( \frac{E_p}{E_s} \right)^{1/12} \left( 1 - \mu_s^2 \right)} \] (2.43)

in which \( \left( \frac{E_p}{E_s} \right)^{1/12} \) varies between about 3.25 for timber and 4.20 for steel in pounds per square inch values.

Another approach to the problem of estimating \( K \) was made by Granholm (31) who used Boussinesq's expression for the displacement of a point in an infinite elastic medium due to a concentrated load applied at another point, and assumed that the lateral pressure along the length of the pile under consideration was uniform and that the deformation of the pile outside this length had no effect on displacement within the length. This analysis was related to the problem of pile buckling under axial loads. More recently Glick (30) and Gibson have carried out more refined analyses, making use of Mindlin's expression for the displacement of a point in the interior of a semi-infinite elastic medium, and for the case of a pile bending in a series of sinusoidal half-waves (the buckling problem) Gibson's work leads to the following equation for \( K \):

\[ K = \frac{22.25 E_s \left( 1 - \mu_s \right)}{(1 + \mu_s) \left( 3 - 4 \mu_s \right) \left( 4.6 \log_{10} \frac{2L}{B} - 0.44 \right)} \] (2.44)

in which \( L \) is the half wave-length of the pile in its buckled form.

A pile whose length is more than \( 2 \times \frac{4 \sqrt{\frac{\pi^4 E_p I_p}{K}}}{K} \) will
buckle in half-waves of length approximately \( 4 \sqrt{\frac{k^4 E_p I_p}{\mu}} \).

Noting that \( 4 \sqrt{\frac{k^4 E_p I_p}{\mu}} \) does not vary greatly for different sections, it can be shown that a reasonable approximation to equation 2.44 is as follows:

\[
K (7.8 - \log_{10} K) = 19.5 \frac{E_s}{(1 + \mu_s)(3 - 4 \mu_s)} \quad \text{(for concrete piles)}
\]

\[
K (9.2 - \log_{10} K) = 19.5 \frac{E_s}{(1 + \mu_s)(3 - 4 \mu_s)} \quad \text{(for steel piles)}
\]

Although equation 2.45 apply to the problem of pile buckling, it gives values of \( K \) of the same order as does equation 2.43.

Jampel (35) gives a theoretical formula for the effect of interference of another pile on \( K \), which suggests that \( K \) may be reduced by a factor of 3 or more for very close pile spacing. Miyahara and Ergatoudis (45) obtained some values of these reaction modulii experimentally.

Parikh and Pal (51) carried out plane strain finite element analysis for circular sections and established various subgrade reaction modulii (shear, normal and torsion) for semi-infinite homogeneous and two layered soil system. They also investigated interference effect on these subgrade reaction modulii, in case of group of piles. Various mathematical expressions were also established to define these subgrade reaction modulii. The subgrade reaction modulii, derived in this manner, were employed to investigate soil-pile system. Comparison of results with available literature revealed an excellent agreement.
2.5. **LATERALLY LOADED SINGLE PILE**

Many investigators (68, 43, 22, 12, 13) employed the theory of subgrade reaction, through normal subgrade reaction coefficients, to determine the deflection and moment in case of laterally loaded pile. Non-dimensional solutions for laterally loaded pile have been presented by Reese (68) in which the soil modulus increases in simple proportion to depth. While this simple form appeared to be applicable to most laterally loaded pile problems, some cases had been encountered where some other type soil modulus function found suitable. Matlock and Reese (43) presented general solutions for laterally loaded piles which are supported by an elastic medium. Their method of analysis had following features:

A group of non-dimensional parameters were defined which will have the same numerical value for any pair of structurally similar cases or for any model and its prototype (also refer figure 2.9). They were

- **Depth coefficient**, \[ Z = \frac{x}{T} \] ...
  \[ (2.46) \]

- **Maximum depth coefficient** \[ Z_{\text{max}} = \frac{L}{T} \] ...
  \[ (2.47) \]

- **Soil modulus function** \[ \varphi (Z) = \frac{E_s T^4}{E_p I_p} \] ...
  \[ (2.48) \]

- **Deflection coefficient for horizontal load** \[ P_t \] only \[ A_y = \frac{Y_A E_s I_p}{P_t T^3} \] ...
  \[ (2.49) \]

- **Deflection coefficient for moment load** \[ M_t \] only \[ B_y = \frac{Y_B E_s I_p}{M_t T^2} \] ...
  \[ (2.50) \]
Where $Y_A$ and $Y_B$ are deflections due to horizontal force $P_t$ and moment $M_t$ respectively.

Thus from equations 2.46 to 2.50, for (1) similar pile-soil stiffness systems, (2) similar positions along the piles and (3) similar pile lengths, the solution of the problem could be expressed as,

$$y = \frac{P_t T^3}{E_p I_p} A_y + \frac{M_t T^2}{E_p I_p} B_y \ldots \ (2.51)$$

Substituting the definitions of non-dimensional parameters contained in equations 2.46 to 2.50, a non-dimensional equation was written as,

$$\frac{d^4 A_y}{d Z^4} + \phi (Z) A_y = 0$$

and

$$\frac{d^4 B_y}{d Z^4} + \phi (Z) B_y = 0 \ldots \ (2.52)$$

Now specifying $\phi (Z)$ and defining relative stiffness factor, $T$, the differential equations could be solved for $A_y$ and $B_y$. The resulting $A_y$ and $B_y$ were then be used to compute deflections, from where, slopes, moments etc were determined.

He extended his analysis for rigid pile theory and in both cases of elastic-pile theory and rigid pile theory methods of computation were given, by which non-dimensional solutions may be computed for any desired form of variation (according to various power laws) of soil modulus with respect to depth. Typical solutions were presented and recommendations given for their use in design problems.
Davisson and Gill (22) investigated analytically the effect of a two layer soil system on the engineering behaviour of a laterally loaded pile in terms of non-dimensional terms. A modulus of subgrade reaction was used to define soil stiffness; the stiffness of the surface layer was defined in terms of that of underlying layer. The complete range of relative stiffness and relative thickness of the two layers was investigated. The analysis which was carried out in non-dimensional terms, allows general qualitative conclusions to be drawn from results. Brome (12,13) presented methods for calculations of lateral deflections at working load and the ultimate lateral resistance of laterally loaded piles driven into cohesive saturated as well as cohesionless soils. The lateral deflections at working loads were calculated utilizing the concept of subgrade reaction theory, whereas, ultimate lateral load of laterally loaded piles was calculated assuming that the piles were transformed into a mechanism through the formation of plastic hinges.

Davisson and Robinson (23) developed an approximate procedure for treating the problem of bending and buckling of partially embedded piles. The procedure was developed with use of theoretically correct solutions applicable to a partially embedded pile subjected to moment, shear and axial loads, when acting separately. It was shown that a partially embedded pile may be represented as a free-standing pile with a fixed base at a depth below the ground surface equal to 1.4R or
1.8T, where \( R = \frac{4EIP}{K} \) (sub grade modulus constant with depth) and \( T = \frac{5EIP}{n_h} \) (sub grade modulus increasing linearly with depth and \( n_h \) is the proportion of linearity).

The above mentioned investigators employed subgrade reaction theory, which though simple, assumes soil as Winckler or isolated spring medium. This assumption, in many cases, leads to unsatisfactory results as the continuity of the soil mass is not taken into account and for the same reason the results cannot be related directly to any of the material properties of the soil. Thus it becomes difficult to use the observed response for a given set of soil and pile characteristics to predict the response for a new set of conditions. More satisfactory analysis in which the soil was assumed to be an elastic continuum had been presented by Douglas and Davis (25) and Spillers and Stoll (75).

Douglas and Davis found the displacements and rotation of a thin rigid vertical plate buried in an elastic medium when acted upon by a moment and horizontal load applied to its upper edge. The first stage in the analysis was to integrate the equations of Mindlin for the displacements due to horizontal load acting at a point below the surface of a semi-infinite elastic medium (ref. Appendix B), in order to give the horizontal displacement of a vertical plane due to uniform normal stress over a finite area of the plane.
The second stage was to employ a numerical method to calculate the distribution of normal stress required to produce planar rotation about some horizontal axis of that portion of the plane representing the buried plate or footing. Integration of these normal stresses provided the external horizontal load and moment required to produce the assumed rotation. Model tests using gelatine and paraffin wax as embedding materials were performed and shown to provide satisfactory confirmation with the theory. In the second part of the paper, the displacements in a vertical plane within an elastic medium due to uniform loading over a portion of the plane were evaluated. These theoretical results were then used to give the mean horizontal movement of a vertical rigid plate buried beneath the surface when acted upon by a horizontal load applied at the centre.

Spillers and stoll (75) have analysed the lateral response of piles by using Mindlin equation for displacements at any point in the soil. Using beam theory the displacement of the pile at any point may be written and equating the displacement of soil and pile as also considering the equilibrium of pile, a set of equation was generated. Solution of these equations determined the loads at various depth and permitted direct calculation of the displacements, shears and moments for the pile. The authors had extended their analysis for local yielding of soil also and developed a plot of surface load-deflection relationship. The nonlinear form of this curve is quite similar to the observed results for lateral load tests performed on scale piles driven in the field.
However no systematic study appears to have been made of the characteristics of the behaviour of a single laterally loaded pile till Poulos (60) made an extensive study and presented various load displacement characteristics of a vertical pile subjected to lateral loading and moment, through various influence factors covering a wide range of pile flexibilities and length to diameter ratios, for both free-head and fixed-head piles. In the analysis soil and pile displacements were evaluated and equated at the element centers except for the two extreme elements, for which displacement were calculated at the top and tip of the pile respectively. Once more soil displacement have been evaluated from Mindlin equation (ref. Appendix B) for horizontal displacement due to a horizontal load within a semi-infinite mass. The pile displacements have been obtained from the equation of a flexure of a thin strip, expressed in finite difference form. By equating soil and pile displacements at each of the points mentioned above, and using the appropriate equilibrium conditions, sufficient equations were obtained to solve for the unknown horizontal displacements at each element. The elastic analysis was extended to include the effect of local yield between the soil and pile as also to study the effect of socketing of pile tip (62). In the analysis the stress state at the interface of the pile and elastic media was assumed to consist of only normal stresses and the true pile-elastic media is only approximately modeled. Poulos solution are not rigorously valid for layered media, although he had approximated the influence of a hard sublayer.
Banerjee and Davies (7) made an approximate elastic analysis of the working load responses of single piles embedded in a soil of linearly increasing modulus with depth. The formulation employed, within a boundary element algorithm, the fundamental solution for point loads acting at the interface of a two layer elastic half-space. The analysis is of comparable generality and computational efficiency to earlier elastic solutions of the corresponding homogeneous problem and has distinct advantages in practical applications over the simpler homogeneous soil model, namely, improved predictions of the response due to lateral loads and the rationalization of the selection of soil modulii.

To solve difficult boundary condition mechanics problems, finite element method of analysis has extensively been used and an efficient finite element formulation applicable to non axi-symmetrically loaded axisymmetric elastic bodies is available (83,87). Kuhlemeyer (37) had modified this method to use the same for accurate analysis of the transversely loaded beam problem in which the beam bending aspect is accurately modeled; the modification involved the use of the reduced integration technique (90). Utilizing the formulation for a good approximation to a bending finite element, he obtained an efficient finite element solution to the three-dimensional problem of static and dynamic transversely loaded piles. The solutions were presented in a form such that one relatively simple closed form expression could be used to represent both the static and steady state dynamic cases. Static loading results were presented in the form of flexibility coefficients for a range of two layer soil system.
2.6. **LATERALLY LOADED PILE GROUPS**

Several analyses, employing subgrade reaction theory, have been presented in the literature for displacement and load distribution within pile group, subjected to lateral loads and moments. A recent series of contributions from about 1945 has attempted to update the earlier works by modernizing the mathematical formulation, including soil-pile interaction, and by imposing less restriction on the problem properties.

Among the latest, Hrennikoff (34) published a planar analysis which included hinged or fixed connections at the cap, use of batter piles, and lateral resistance by the soil to pile movement. An approximation method useful in design were also described. Aschenbrenner (1) presented three dimensional analysis taking into account both lateral and longitudinal resistance of soil, but is limited to hinged connections. While above mentioned authors presented derived formulas, Rising, Roth and Anderson (69) included matrix formulation as well, although their problem is severely limited to hinged connections and no lateral soil resistance. Twizunski (82) considered matrix methods but is more restrictive than Rising et al. in that, in addition to their limitations, only planar loading was considered. Juhl (36) used matrix methods but, again, was more restrictive than Rising, et al. in that, in addition to the limitations already existing, only vertical piles were considered. Asplund (2) used matrix methods in a fairly general formulation but sacrifices usefulness, introducing complicating fictitious devices for operating at the
inflection points of the piles.

Francis (28) presented a procedure for the analysis of groups of piles subjected to loading in one plane. The piles were assumed to be either pinned at both head and tip or fixed at cap and fully or partially embedded in soil. The analysis procedure had following features:

**Piles pinned at both ends:** A group of piles, forming a foundation under a bridge, pier or building, was considered as shown in figure 2.10. The piles may be of various lengths and cross-sections and have different batters, but all were assumed to lie in one vertical plane. The following assumptions were made by him,

1. Each pile is pinned at top and bottom and behaves elastically. The relation between the load \( P \) which it carries and its overall changes in length \( \Delta \) is
   \[
   \Delta = f \cdot P \quad \ldots \quad (2.53)
   \]
   in which \( f \) is the change in length of the pile per unit axial load (\( f \) having a particular value for each pile) and \( P \) is the axial load of the pile.

2. The pile cap acts as a rigid body.

Refering to figure 2.10 and applying standard structural exercise, the force in the pile was given by,
\[
P = \frac{1}{f} \left( u \sin \theta + (v + \lambda x) \cos \theta \right) \quad \ldots \quad (2.54)
\]
and following three equations of equilibrium, expressed in terms of displacements as unknowns, were obtained,
\[
H - \sum \frac{1}{f} \left( u \sin^2 \theta + (v + \lambda x) \sin \theta \cos \theta \right) = 0 \quad (2.55)
\]
\[ V - \sum \frac{1}{f} (u \sin \theta \cos \theta + (v + \delta x) \cos^2 \theta) = 0 \quad (2.56) \]

and

\[ M - \sum \frac{1}{f} (ux \cos \theta \sin \theta + (Vx + \delta x^2) \cos^2 \theta) = 0 \quad (2.57) \]

In each equation the summation covers all the piles of the group. The quantities, \( u, v \) and \( \delta \) can be calculated from equations 2.55 to 2.57 and force \( P \) in any pile is then given by equation 2.54.

**Pile fixed at cap and fully or partially embedded in soil:**

It was assumed that each pile was fixed at the head and also effectively fixed at some depth \( l_e \) down the pile from its head. The distance \( l_e \) will vary with the type of soil, and means of calculation of \( l_e \) was also described. A distinction was made between the length \( l_{e1} \) for the case of rotation of the head without translation and the length \( l_{e2} \) for the case of translation of the head without rotation.

The restraint against free rotation of the pile at its ends, one causes forces additional to the axial forces to be set up in the pile. These additional forces are illustrated in figure 2.11 for the three components of deformation \( u, v \) and \( \delta \) of the pile cap. The forces exerted by the pile on the cap due to displacement \( u \) are,

\[
\begin{align*}
X_h &= - \frac{12 E_p I_p}{L_{e2}} \frac{u \tau \cos^2 \theta}{3} \\
Y_h &= + \frac{12 E_p I_p}{L_{e2}} \frac{u \tau \sin \theta \cos \theta}{3} \quad (2.58)
\end{align*}
\]
Similarly the forces induced by v and \( \chi \) are,

\[
X_v = + 12 E_i \frac{v L}{p_2} \cos \theta \sin \theta \cos \theta
\]

\[
Y_v = - 12 E_i \frac{v L}{p_2} \sin \theta \cos \theta
\]

\[
M_v = - 6 E_i \frac{v L}{p_2} \sin \theta
\]

\[
X\chi = 12 E_i \frac{\chi L}{p_2} \sin \theta \cos \theta + 6 E_i \frac{\chi L}{p_2} \cos \theta
\]

\[
Y\chi = - 12 E_i \frac{\chi L}{p_2} \sin \theta \cos \theta - 6 E_i \frac{\chi L}{p_2} \cos \theta
\]

\[
M\chi = - 6 E_i \frac{\chi L}{p_2} \sin \theta - 4 E_i \frac{\chi L}{p_2}
\]

The equations of equilibrium, i.e., equations 2.55 to 2.57, were modified and was written as,

\[
H = \sum \frac{1}{f} \left( u \sin^2 \theta + (v + \chi x) \sin \theta \cos \theta \right)
\]

\[
- \sum \frac{12 E_i L}{p_2} \left( u \frac{3}{L} \cos^2 \theta - (v + \chi x) \frac{3}{L} \sin \theta \cos \theta - \frac{\chi L}{p_2} \frac{3}{2 L} \right) = 0
\]

\[
v = \sum \frac{1}{f} \left( u \sin \theta \cos \theta + (v + \chi x) \cos^2 \theta \right)
\]

\[
- \sum \frac{12 E_i L}{p_2} \left( -u \sin \theta \cos \theta + (v + \chi x) \frac{3}{L} \frac{3}{2 L} \sin^2 \theta + \frac{\chi L}{p_2} \frac{3}{2 L} \sin \theta \right) = 0
\]
Values of $\ell, T, L_1, L_2$ etc were defined, through various tables and expressions, for both uniform soil as well as soil with stiffness proportional to depth. Information was also given on methods of estimating the lateral resistance of given soils and on the stability of partially embedded piles.

Saul (72) have presented a general method of analysis by direct stiffness of three-dimensional pile foundations for static loading or dynamic response. The method presented by him assumes a rigid piling cap and elastic behaviour of the system. The analysis can also take into account fixed or hinged connections at the cap; piles with different bending stiffness about their principal axes; any degree of linear torsional or axial stiffness, or lateral soil resistance, including zero; any position and batter of piles; or piles of different sizes, materials and end conditions in the same foundation. In the analysis the generalized forces, $\{F\}_i$, and generalized displacement, $\{X\}_i$, were related by

$$\{F\}_i = [b]_i \{X\}_i \ldots \text{(2.62)}$$
in which $i$ identifies a particular pile; and the elements of the elastic pile constants matrix $[b]$ depend on the axial, flexural and torsional stiffnesses of pile as well as its interaction with the surrounding soil media and the boundary conditions.

The elastic pile constants may be obtained directly from load test but three possible analytical formulations for the pile constants were offered by the author. They are:

1. The pile may be analyzed as a semi-infinite beam on an elastic foundation for flexure, as a modified compression block for axial deformation, and as a modified shaft in torsion.

2. The elastic pile constants in bending may be derived using as a model cantilever beam fixed at an arbitrary depth $L$.

3. Another possible alternative occurs especially in elevated platforms such as those used in offshore where the piling extend above soil level. The elastic pile constants for bending were derived by considering the pile as a cantilever beam of two regimes; one free of lateral restraint, and the second as a semi-infinite beam on spring foundation.

Applying the necessary transformations, the relationship of the pile's forces to its deflections in an orthogonal coordinate system (local) parallel to the chosen (global) axes of the foundation could be given by,

$$\{F'_i\}_i = [a]_i [b]_i [a]_i^T \{x'_i\}_i \ldots (2.63)$$
where $[a]_i$ is the transformation matrix of pile $i$ due to its rotated position with respect to the established coordinate system of the foundation.

Now assuming piling cap to be rigid and relating its deflection with piling, the relationship between the foundation loads and the pile cap deflections could be written as,

$$\{q\} = \sum_{i=1}^{n} [a]_i \{q\}_i = \sum [C]_i \{F\}_i = [S][\Delta] \quad (2.64)$$

in which $[C]_i$ is static matrix and $S_{ij}$ are stiffness influence coefficients for the foundation as a whole; $S_{ij}$ being the force in direction $i$ at the origin required for a unit deflection of the rigid cap in direction $j$. Once piling cap deflections are determined the loads in the pile also can be determined. Although matrix method of analysis together with electronic computation leads itself to obtain solutions, formulas for the stiffness influence coefficients and for obtaining pile forces were also given.

Once more, in all cases mentioned above soil was assumed to be a spring or Winckler material. Poulos (61) presented an elastic analysis, assuming soil as an elastic continuum, for the horizontal displacement and rotation of a laterally loaded pile group and the distribution of horizontal forces within the group. The interaction between two identical equally loaded piles were analysed first, and the increases in displacement and rotation of the piles, in relation to the single pile values, were expressed in terms of interaction factors.
The analysis was essentially similar to that for a single pile except that the influence of the adjacent pile on the soil displacements must be considered. The interaction factors for displacement and rotation, for various spacing and various values of departure angle between the piles, were established. Using these interaction factors, a method for calculating the displacement and rotation of a general pile group, based on the principle of superposition, was described.

2.7. PILE SUBJECTED TO TORQUE

Kaldjian (38) has carried out finite element analyses of short rigid piers subjected to torque. The method is very general with respect to geometry and material properties but with following restrictions.

(1) The geometry and material properties be axisymmetric.

(2) The loads be symmetric about a plane containing the axis of revolution.

Wilson's (87) axisymmetric element stiffness, involving the expansion of nodal point forces and nodal point displacements of a triangular ring element in Fourier series, was used directly by setting the harmonic n equal to zero.

The finite element model adopted by him is shown in figure 2.12. The plane of rotation was divided into 1, 152 triangles formed by 24 rows and 24 columns and 625 nodal points. The smaller elements were located where the stress gradient was expected to be high. The dimension of the elements varies from 1 in x 1 in to 170 in x 270 in.
The half space was approximated by a space of radius \( r = 900 \text{ inch} \) and depth \( Z = 1200 \text{ inch} \). A circular portion of radius \( r_0 = 100 \text{ inch} \) on the surface was rotated through an angle \( \theta = 0.001 \text{ rad} \).

The radial and vertical displacement can be shown to be zero in the torsion case throughout the half space and hence they were enforced as boundary condition. Moreover to avoid singularity at \( r = 0 \), the nodal circles at the axis of symmetry were set to a small value, and zero radial displacement conditions were enforced on them.

Numerical values of torsional spring constants with corresponding stress and displacement distribution were obtained for circular footings for various embedment. The presence of rigid discs to simulate thin rock layer and voids to simulate loose soil pockets below the surface were considered, and torsional stiffnesses were obtained. The stress distribution around a footing boundary was considered in the elastic range.

Broms and Silberman (14) carried out model tests on steel piles in sand. Stoll (78) suggested that, for friction piles in clay, it may be easier and cheaper to determine the ultimate axial load capacity of a pile from a torsional loading test rather than a conventional axial load test. In that manner, he has devised a simple field torque shear load test. The key feature of the test apparatus is that torque is applied by small capacity hydraulic jacks reacting horizontally against adjoining job piles, utilizing the large mechanical advantage available at the usual spacing. An early set up
includes dial gauge and reference beam to measure torque displacement at top of the test pile.

Poulos (64) described an analysis method for the determination of the response of a single cylindrical pile subjected to torque. In his analysis pile was circular, of length $L$, having a constant shaft diameter, $d$, a base diameter, $d_b$, and the shear modulus of the pile material was $G_p$. The pile was subjected to a torque, $T$, at the ground surface and was situated in a semi-infinite elastic soil mass having shear modulus $G_s$ and Poisson's ratio $\mu_s$ (figure 2.13a). The pile shaft was divided into $n$ equal cylindrical elements while the base comprises $m$ annular elements, each element being acted upon by an unknown uniform interaction stress.

Now considering a small elemental area at the pile-soil interface (figure 2.13c), the tangential load on this area is $dp = 0.5 \bar{T}_j \cdot d\theta \cdot \delta z$; in which $\bar{T}_j$ = interaction stress; $\delta \theta$ = included angle of area; $\delta z$ = length of elemental area. This load will cause a deflection, $d \rho$, tangential to the pile surface at any point in the soil mass, which could be expressed as,

$$d \rho = d \rho_{xx} \cos \theta + d \rho_{yx} \sin \theta \quad \ldots \quad (2.65)$$

in which $d \rho_{xx}$ = component of deflection parallel to direction of load $dp$; $d \rho_{yx}$ = component of deflection normal to $dp$; $\theta$ = angle defined in figure 2.13c. $d \rho_{xx}$ and $d \rho_{yx}$ was evaluated from the equations of Mindlin for a horizontal subsurface point load. By integration of equation 2.65 over
a cylindrical or annular element, the tangential deflection, and thus rotation, at any point, \( j \), was obtained. This double integration was carried out partly analytically and partly numerically. For all elements on the pile, the soil rotations was then expressed as

\[
\{ \varphi_s \} = \left[ \frac{I_s}{G_s} \right] \{ \tau \} \quad \ldots \quad (2.66)
\]

in which \( \left[ \frac{I_s}{G_s} \right] \) is a \((m+n) \times (m+n)\) matrix of soil rotation influence factors obtained from integration of Mindlin's equation; \( \{ \varphi_s \} \) = soil rotation vector; and \( \{ \tau \} \) = interaction stress vector.

By application of the torsion equation for a circular cylinder, the following expression was obtained for the pile rotation at the midpoints of the elements:

\[
\{ \varphi_p \} = \frac{Ld^3}{nG_pJ_p} \left[ AD \right] \left[ PEQ \right] \{ \tau \} + \varphi_b \{ 1 \} \quad \ldots \quad (2.67)
\]

in which

- \( \{ \varphi_p \} \) = pile rotation vector
- \( J_p \) = polar moment of inertia of pile
- \( \{ \tau \} \) = interaction stress vector
- \( \varphi_b \) = tip rotation of pile
- \( [AD] \) = \((m+n) \times (m+n)\) pile matrix in which \( AD_{ij} = 1 \) for \( j \neq i \), 0 for \( j = i \), and 0.5 for \( j = i \)
- \( [PEQ] \) = \((m+n) \times (m+n)\) pile matrix in which \( PEQ_{ij} = F_j \) for \( j \neq i \) and \( i \neq n \), 0.5 \( F_j \) for \( j = i \) and \( i \neq n \), and 0 for \( j \neq i \) or \( i \neq n \), \( F_j = A_jL_j/d^3 \)
- \( A_j \) = Surface area of element \( j \)
- \( L_j \) = Lever arm of interaction stress on element \( j \)
For elastic conditions at the pile-soil interface, the soil displacement is equal to pile displacement and thus formed following sets of equations

\[
\left[ \frac{L}{mdK_T} \mathbf{A} \mathbf{D} FEQ - \frac{I_s}{G_s} G_{sr} \right] \begin{bmatrix} \frac{r}{G_{sr}} \\ \end{bmatrix} + \varphi \{ \bar{1} \} = 0 \quad (2.68)
\]

in which \( G_{sr} \) is a reference value of soil shear modulus, and \( K_T \) = torsional stiffness factor in which

\[
K_T = \frac{G_P J_p}{G_{sr} d^4} \quad \ldots \quad (2.69)
\]

In addition to the \((m+n)\) simultaneous equations provided by equation 2.68, equilibrium of the pile yielded,

\[
\sum_{j=1}^{m+n} F_j \tau_j = \frac{T}{d^3} \quad \ldots \quad (2.70)
\]

Solution of these equations gave \( m+n \) unknown values of \( \tau \) and the value of \( \varphi \), from which the pile rotations was calculated through equation 2.67.

Parametric solution for the torque-rotation relationship for the pile head were presented for both uniform soil and a soil in which shear modulus and pile-soil adhesion increases linearly with depth. The method of analysis was extended to take into account for pile-soil slip. He also described results of a series of model tests on piles in clay, in which the axial and torsional responses of the piles were investigated. The tests indicated that both the working load and ultimate behaviour of a pile subjected to torsion may be reasonably predicted from the results of axial load tests or vice versa.
However the test carried out on model cylindrical piles embedded in soils with a constant shear modulus (clay) and shear modulii increasing linearly with depth (sand), by O'Neill and Dutt (47) indicated excellent agreement with Poulos analysis up to about 40% of the failure torque, whereafter greater rotation at corresponding levels of torque was evident in the measured data. The deviation in the high torque range is due to the fact that maximum interfacial stress that varies with rotation, could not be simulated by Poulos yield correction factors (F_0 and F_0'). However since the piles tested were quite flexible for each case considered, and since the approximations previously described are most significantly manifested in flexible piles, the deviation noted by O'Neill and Dutt may be considered as upper limit of anticipated deviations.

2.8. BUCKLING OF PILES

Many offshore and other structures are founded on long piles which extend for a considerable distance above the ground line. Such piles must be designed to be safe against the possibility of failure by buckling. The foundation soil provides neither an ideal fixed nor hinged support to the free standing part of the pile. Therefore, it is not immediately obvious what kind of fixity conditions exist at the ground or what buckling formula is appropriate to use for the pile above the ground line.

Earlier the possibility of a slender pile buckling under load when completely embedded in clay was ignored. No cases of
buckling in soft soils were known and the theory referred to (Granholm (31)) showed that buckling could not take place except in such soft material where the pile could well be designed as a free standing column.

There is much evidence in literature (Cummings (19), Casagrande (17), Spolford (76)) that pile in very soft clay could bear, in test, loads well in excess of their Euler loads. But at the same time there have been some examples of piles failing under loads including axial stresses below the yield point of the pile material. When these piles were extracted it could be seen that they had failed at a part of their length which was in a very soft stratum.

The theory usually quoted, due to Granholm (31), assumes an elastic clay and an elastic pile. Subsequent treatment (Cummings (19), Glick (30), Banerjee (4)) produced a design formula based on a maximum permissible stress in the pile and an initial deformation of the pile, and Glick modified Granholm's concept of coefficient of lateral resistance, while still basing it on classical elastic theory. All these theories assumed both pile and soil to be elastic.

Bergfelt (8) had shown that buckling of steel piles with a solid cross-section area is possible in soft clay. The writer has made a large number of model tests on piles of various materials (steel, brass, bronze and wood) of various shapes and dimensions. Tests have shown that the buckling load of steel piles can be given by,
\[ P = 8 \text{ to } 10 \left( \frac{T}{E_p I_p} \right)^{1/2} \quad \ldots \quad (2.71) \]

where \( T \) is the shear strength of the soil and \( E_p, I_p \) have usual meaning.

Davisson and Robinson (23) has developed an approximate procedure for treating the problem of buckling of partially embedded piles. The basic features of their analysis method are as follows,

The governing differential equation for pile bending in a soil may be given by,

\[ E_p I_p \left( \frac{d^4 y}{dx^4} \right) + P \left( \frac{d^2 y}{dx^2} \right) + K_x y = 0 \quad \ldots \quad (2.72) \]

where \( E_p I_p \) is the flexural stiffness of pile, \( P \) is the axial load, \( x \) and \( y \) are the co-ordinates (refer figure 2.14) and \( K_x \) is the subgrade modulus in units of force per length^2 and is applicable to full width of the pile. \( K_x \) is the function of \( x \) and from the top of the pile to the ground surface \( K = 0 \), below the ground surface \( K \) can be assigned values compatible with the soil profile. Two cases of \( K \) were considered, viz, \( K \) equals to constant and \( K \) equals to \( n_h x \) where \( n_h \) is the constant of horizontal subgrade reaction (force/length^3).

For \( K \) equal to constant a set of new variables were introduced such as

\[ R = \frac{4}{E_p I_p} \sqrt{K}, \quad L = x/R \quad \ldots \quad (2.73) \]

and \( U = \frac{PR^2}{E_p I_p} \quad \ldots \quad (2.74) \)

Then equation 2.72 yielded
\[ \frac{d^4 y}{dL^4} + U \left( \frac{d^2 y}{dL^2} \right) + y = 0 \quad \ldots (2.75) \]

It was then possible to express the dimensions on figure 2.14 in non-dimensional terms as indicated on figure 2.15, where

\[ L_{\text{max}} = L/R ; \quad S_R = L_S/R ; \quad J_R = L_u/R \quad \ldots (2.76) \]

The equivalent length of free-standing pile \( L_e \) is then equal to \( S_R + J_R \). For \( K \) equal to \( n_h \), the following variables were introduced

\[ T = \frac{5}{n_h} \left( \frac{E_P I_P}{n_h} \right) \quad ; \quad Z = x/T \quad \ldots (2.77) \]

and

\[ V = P T^2/E_P I_P \quad \ldots (2.78) \]

Then equation 2.72 yielded

\[ \frac{d^4 y}{dZ^4} + V \left( \frac{d^2 y}{dZ^2} \right) + ZY = 0 \quad \ldots (2.79) \]

It was then possible to express the dimensions on figure 2.14 in non-dimensional terms as indicated on figure 2.15, where

\[ Z_{\text{max}} = L/T ; \quad S_T = L_S/T ; \quad J_T = L_u/T \quad \ldots (2.80) \]

The equivalent length of free-standing pile \( L_e \) would be equal to \( S_T + J_T \).

For a pile with a free head and free tip equation 2.75 could easily be solved for the critical values of \( U \), denoted as \( U_{\text{cr}} \). The critical load \( P_{\text{cr}} \) then becomes

\[ P_{\text{cr}} = U_{\text{cr}} \left( \frac{E_P I_P}{R^2} \right) \quad \ldots (2.81) \]

For a pile with a free head and a fixed base, and having an equivalent length equal to \( S_R + J_R \), the critical load may be given by,
By combining equations 2.81 and 2.82, the relationship between $S_R$ and $J_R$ was found and was plotted for various end conditions in both the case of soil type. Thus a partially embedded pile can be transformed into an equivalent free standing pile with fixed base; which can then be solved for critical load by standard method.

Analysis of the stability behaviour of an axially loaded pile with continuous spring lateral support was considered by Toakley (81). He applied energy method, in the framework of elastic behaviour, for determining the buckling loads of elastic systems. His method of analysis had following features:

Under certain loadings, an elastic system may reach a condition of neutral equilibrium in which the straight undeflected form and adjacent deflected shapes are possible equilibrium positions. These adjacent deflected shapes correspond to the appropriate buckling mode and are associated with a definite buckling load. In going from the straight to the deflected form, there will be no change in the total potential, and any further displacement in this form must satisfy the equation 2.83.

$$\delta (U + V) = 0 \quad \cdots \quad (2.83)$$

in which $U$ is the potential energy of the external loads, $V$ denotes the strain energy of the system and $\delta (U+V)$ represents the incremental change in total potential caused by the variation in the displacement.
Choosing a displacement function of the form
\[ y = a_1 \phi_1 + a_2 \phi_2 + \ldots + a_n \phi_n \quad (2.84) \]
in which \( y \) is the lateral deflection of pile, \( a_1, a_2, \ldots, a_n \) are constants, \( \phi \) a set of function satisfying the same boundary conditions as the deflection \( y \), it could be shown that equation 2.83 is satisfied if the constants, \( a_n \), are so chosen that \((U+V)\) is stationary for variations in all values of \( a_n \), i.e.,
\[
\frac{\partial}{\partial a_n} (U + V) = 0 \quad \ldots \quad (2.85)
\]
For an elastic system \( U \) and \( V \) are quadratic functions of \( a_n \), and equation 2.85 yields a system of \( n \) homogeneous linear equations. This system of equations does not have solutions different from zero unless the determinant \( \Delta \) of its coefficient is zero. Thus
\[
\Delta = 0 \quad \ldots \quad (2.86)
\]
is a polynomial equation of degree \( n \) in the unknown critical load \( P_{cr} \). The smallest root of equation 2.86 gives the required buckling load.

Various Fourier series for various end conditions were assumed as displacement functions. Number of terms to be considered in the displacement function depends on the accuracy required in the solution. Two types of soil reaction was considered, viz, uniform and linear variation of the coefficient of lateral reaction along pile shaft. The nondimensional parameters used in plotting the various solutions enabled elastic
buckling loads to be determined for any combination of pile length, flexural rigidity and lateral restraints.

Load tests on 1/4 inch to 1/2 inch diameter piles in dry sand have been performed by Lee (40) for the purpose of investigating the validity of the theoretical equations published by Davission and Robinson (23) for predicting the critical buckling load of partly embedded pile. Various types of pile materials, viz, Aluminium tube, steel rods were considered while uniformly graded medium grained quartz sand with angular particles surved the purpose of foundation soil. The mean grain size was approximately 0.5mm and the void ratios at the minimum and maximum densities were 0.965 and 0.618 respectively. The results of the model pile tests, taken together with the results of a field test on a full size timber pile reported by Klohn and Hughes were all close to the appropriate predicted values, even though the range of pile stiffnesses varied over six orders of magnitude. The close agreement, which was obtained between the theory and the experimental data, indicates the accuracy of the theoretical formulation.

Based on the assumptions that the soil is represented by Winkler's model and that the axial force is invariant with depth, the study of buckling behaviour of partially and fully embedded piles by Davission (24), Toakley (81), and Davision and Robinson (23) bring out the influence of various parameters such as (1) End fixidity conditions;
(2) the variation of soil modulus along the embedded length of the pile; and (3) the length of the pile on buckling loads. Toakley also reports the influence of axial force variation on buckling loads for a fully embedded pile with pinned-pinned end conditions. However, the quantitative influence of the skin friction was relatively less understood in comparison to the other factors previously mentioned.

Reddy and V:lsangkar (67) adopted Rayleigh-Ritz method and assuming displacement function as beam vibration function, the effect of the axial load distribution with depth for different boundary conditions was studied for fully and partially embedded piles. For example, the displacement function adopted in case of a partially supported free-free pile for which the axial force variation and the variation of soil modulus is shown in figure 2.16a and 2.16b respectively, were

\[ y = A + B \left( \frac{x}{L} - \frac{1}{2} \right) + \sum_{n=1}^{\infty} a_n \phi_n(x) \ldots (2.87) \]

in which \( L = \) Length of the pile 
\( A, B, a_n = \) the coefficients 
\( \phi_n = \text{Cosh} \beta_n x + \text{Cos} \beta_n x - \alpha_n (\text{Sinh} \beta_n x + \text{Sin} \beta_n x) \ldots (2.88) \)

The values of characteristic roots \( \alpha_n \) and \( \beta_n L \) were evaluated from tables by Young and Felgar.

The strain energy of the buckled system is due to bending of the pile, \( V_1 \), and to the energy stored in the soil, \( V_2 \), where \( V_1 \) could be given by,

\[ V_1 = \frac{E_p I_p}{2} \int_{0}^{L} \left( \frac{d^2y}{dx^2} \right)^2 dx \ldots (2.89) \]
Substituting the value of $y$ and using integral tables given by Felgar (27), $V_1$ yielded as,

$$V_1 = \frac{E_p l P}{2} \sum_{n=1}^{\infty} a_n^2 \left( \frac{\theta}{n} \right) L \quad \ldots \quad (2.90)$$

The strain energy, $V_2$, stored in the soil was then given by,

$$V_2 = \frac{k_1}{2} \int_{nL}^{L} (x-nL)^2 dx = \frac{k_1}{2} \int_{nL}^{L} (x-nL) \left[ A + B \left( \frac{x}{L} - \frac{1}{2} \right) \right. \\
\left. + \sum_{n=1}^{\infty} a_n \phi_n(x) \right]^2 dx \quad \ldots \quad (2.91)$$

The integration was evaluated by using the integral tables of Felgar.

The potential energy, $U$, of load $P_{cr}$ could be given by,

$$U = - \frac{P_{cr}}{2} \int_{nL}^{L} \left[ 1 - \Psi \left( \frac{x}{L} - n \right) \right] \left( \frac{\partial v}{\partial x} \right)^2 dx \quad \ldots \quad (2.92)$$

From equation 2.85,

$$\frac{\partial}{\partial A} (V_1 + V_2 + U) = 0$$

$$\frac{\partial}{\partial B} (V_1 + V_2 + U) = 0 \quad \ldots \quad (2.93)$$

and

$$\frac{\partial}{\partial a_n} (V_1 + V_2 + U) = 0$$

Substituting for $V_1$, $V_2$ and $U$, equation 2.93 led to $n+2$ homogeneous equations in $n+2$ unknowns which have nontrival solutions, provided the determinant of the unknown coefficients is equal to zero. This led to a polynomial of order $n+2$ in unknown buckling load $P_{cr}$.

The analysis was carried out for two cases of soil variations, namely, uniform and linear and two cases of variation of axial force, namely, linear and parabolic.
Various pile boundary conditions were considered and numerical results were presented in non-dimensional form. It was shown that variation of axial force along the embedded length of the piles and boundary conditions have considerable influence on the buckling loads of piles. The results of the fully embedded piles had shown that there was considerable increase in buckling loads even when only 50% of the load transfer had taken place due to skin friction for some boundary conditions at top and tip of pile. However, when the axial force distribution was parabolic with depth, the increase in buckling loads was less than for a linearly decreasing axial force. Also the effect of skin friction was seen to be considerably dependent on the pile boundary condition.
FIG 2.1 STRESSES ASSOCIATED WITH PILE

(a) STRESSES IN SOIL
ADJACENT TO PILE

(b) STRESSES ACTING
ON PILE

(c) STRESSES IN PILE

FIG 2.2 STRESSES ASSOCIATED WITH END BEARING PILE

STRESS ON SOIL ADJACENT TO PILE

STRESSES ACTING ON PILE

FIG 2.3 DISCRETISATION OF THE LAYER AND THE PILE SOIL INTERFACE
Fig 2.4 A TWO LAYER IDEALIZATION

Fig 2.5 VARIABLE DEPTH $Z$ AT WHICH PILE IS INSTRUMENTED TO KNOW LOAD IN PILE AND DEPTH.

$Q_0$ - LOAD ON PILE TOP.
$\delta$ - TOP DISPLACEMENT.
$T$ - TIP LOAD.
$\gamma_T$ - TIP SETTLEMENT.
$Q_1, Q_2, Q_3$ - LOAD IN PILE.
$\gamma_{01}$ - ETC ARE MID POINT MOVEMENT OF PILE SEGMENTS.

Fig 2.6 PILE SHOWING FORCES ACTING ON SEGMENTS.
FIG. 2.7 LOAD IN PILE AT VARIOUS DEPTH

FIG. 2.8 COMPARISON OF SETTLEMENT RATIOS OF CAP BEARING PILE GROUPS WITH FREE STANDING PILE GROUPS

FIG. 2.9 LATERALLY LOADED PILE
FIG 2.10 PILE GROUPS PINNED AT BOTH ENDS

FIG 2.11 FLEXURAL DEFORMATION OF A PILE
**FIG 2.12** EMBEDDED CIRCULAR FOOTING AND SOIL

**FIG 2.13** PILE SUBJECTED TO TORQUE
FIG 2.14 PARTIALLY EMBEDDED PILE.

FIG 2.15 NON DIMENSIONAL REPRESENTATION OF PARTIALLY EMBEDDED PILE.

FIG 2.16 - VARIATION OF (a) AXIAL FORCE (b) SOIL MODULUS FOR PARTIALLY EMBEDDED PILE.