CHAPTER-2

GENERALIZED NONPARAMETRIC TESTS FOR ONE SAMPLE LOCATION PROBLEM

2.1 Introduction

One of the important and fundamental problems extensively considered in the nonparametric inference is one-sample location problem. The problem is to test for the location parameter (the median) of a distribution when the samples are drawn from a continuous symmetric distribution.

Let $X_1, \ldots, X_n$ be a random sample of size $n$ from an absolutely continuous symmetric distribution with cumulative distribution function.

$$F_\theta(x) = P[X_i < x] = G(x - \theta), i = 1, 2, \ldots, n,$$  \hspace{1cm} (2.1.1)

where $G$ admits density $g$ satisfying $g(x) = g(-x)$ for all $-\infty < x < \infty$, $\theta$ is the location parameter and is the median of the distribution $F$. The problem of interest is to test the hypothesis,

$$H_0: \theta = 0 \quad against \quad H_1: \theta \neq 0,$$  \hspace{1cm} (2.1.2)

Some of the well known nonparametric tests in the literature to test the above hypothesis are Sign and Wilcoxon signed-rank tests and their generalizations.

Madhava Rao (1990) proposed a test by using the sub-samples median. Mehra et al. (1990) proposed a class of tests $T_\alpha$ by using a kernel that depends on an arbitrary constant ‘$\alpha$’. For different $G(.)$’s they have obtained the optimal values of ‘$\alpha$’ by maximizing the efficacy. Recently Pandit and Math (2011) have used the test statistic $T_\alpha$
to develop nonparametric control chart for location. Shetty and Pandit (2000) have generalized the test proposed by Mehra et al. (1990) by considering relative positions of two symmetric order statistics taken from a sub-sample of size \( k \). They denoted the statistic as \( U_a(k, r) \). In particular, they studied the performance (in Pitman sense) of \( U_a(4, 2) \) and \( U_a(5, 2) \) with their optimal values and have shown that the first one performs better. Bandyopadhyay and Datta (2007) have proposed an adaptive nonparametric tests for a single sample location problem. Larocque et al. (2008) have developed one-sample location tests for multilevel data.

It is well known that sign test performs better when the underlying distributions are heavy tailed. When the tails are moderate, Wilcoxon signed-rank test performs better (see Randels and Wolfe (1979)). Even though the problem of location in univariate case seems to be pretty old, all the time researchers are finding some scope to improve the earlier proposed tests.

In this chapter, we propose a class of distribution-free tests based on U-Statistics, which is the modification of the test proposed by Shetty and Pandit (2000). The kernel is based on the \( r^{th}, (k-r+1)^{th} \) order statistics together with the median of a sub-sample of size \( k \) taken from a random sample of size \( n \). General expressions for Expected value and asymptotic variance of the proposed test statistics are derived for arbitrary value of \( (k, r) \). The optimal value of constant ‘\( a \)’ is obtained for different values of \( (k, r) \) by maximizing the efficacy. Note that, the proposed class of tests includes \( T_a \) and \( U_a(4, 2) \) as a special cases. The performance of the test is evaluated by means of asymptotic relative efficiency (ARE) relative to sign test, Wilcoxon signed-rank test and other competitors, efficacy of these tests are reported in Table 2.1. Empirical power study is carried out for some standard symmetric distributions and for a typical class of heavy tailed distributions. Some useful results to compute expected value and asymptotic variance are also given.

2.2 Proposed class of tests

Let \( X = (X_1, X_2, ..., X_n) \) be a random sample from an absolutely continuous distribution function \( F_\theta(x) = G(x - \theta) \), where \( G(y) + G(-y) = 1 \). The proposed statistic to test the hypothesis \( H_0: \theta = 0 \) against the alternative \( H_1: \theta \neq 0 \) is a U-statistic,
\[ V_a(k, r) = \frac{\sum_s \varphi(x_1, x_2, \ldots, x_k)}{\binom{n}{k}} \]  

(2.2.1)

where the summation is over all the \( \binom{n}{k} \) combinations of the integers \{1, 2, \ldots, k\}, \( r \) is fixed such that \( r < k - r + 1 \) and

\[
\varphi(x_1, x_2, \ldots, x_k) = \begin{cases} 
1 & \text{if } x_{(r)} > 0 \\
 a(-a) & \text{if } x_{(r)} x_{(k-r+1)} < 0, x_{(r)} + x_{(k-r+1)} > (>)0, \\
 \text{Med}(x_1, x_2, \ldots, x_k) & \text{if } x_{(K-r+1)} > (<=)0 \\
-1 & \text{if } x_{(k-r+1)} < 0 \\
0 & \text{otherwise} 
\end{cases}
\]  

(2.2.2)

where \( x_{(i)} \) is the \( i^{th} \) order statistic and \( \text{Med}(x_1, x_2, \ldots, x_k) \) is the median of a sub-sample of size \( k \). The constant \( a \), is arbitrary and one can choose it suitably. Reject \( H_0 \) when \( |V_a(k, r)| > c \) and \( c \) is chosen so that the size \( \alpha \) of the test is satisfied. When \( k \) is even \( (k = 2m, \text{say}) \), then median of \( X_1, X_2, \ldots, X_k \) is any number in between \( X_{(m)} \) and \( X_{(m+1)} \), however for definiteness one may define it to be, \( (X_{(m)} + X_{(m+1)})/2 \). If \( k \) is odd \( (k = 2m - 1, \text{say}) \), then median of \( X_1, X_2, \ldots, X_k \) is \( X_{(m)} \).

2.3 Asymptotic distribution of \( V_a(k, r) \)

In the following we derive general expressions for expectation and asymptotic variance for \( V_a(k, r) \).

2.3.1 Expectation of \( V_a(k, r) \)

To obtain \( E[V_a(k, r)] = E(\varphi(X_1, X_2, \ldots, X_k)) \), it is enough to compute the probabilities of the following events.

\[
E_1 = \{ x_{(r)} > 0 \}, \\
E_2 = \{ x_{(r)} x_{(k-r+1)} < 0, x_{(r)} + x_{(k-r+1)} > 0, \text{Med}(x_1, x_2, \ldots, x_k) > 0 \}, \\
E_3 = \{ x_{(r)} x_{(k-r+1)} < 0, x_{(r)} + x_{(k-r+1)} < 0, \text{Med}(x_1, x_2, \ldots, x_k) < 0 \}, \\
E_4 = \{ x_{(k-r+1)} < 0 \}.
\]

The probabilities of the above events are given by,
\[ P_\theta[E_1] = \sum_{i=k-r+1}^{k}(\binom{k}{i}) (1 - G(-\theta))^i G^{k-i}(-\theta), \]
\[ = 1 - B(G(-\theta); r, k - r + 1). \tag{2.3.1} \]

\[ P_\theta[E_4] = \sum_{i=0}^{r-1}(\binom{k}{i}) G^i(-\theta) (1 - G(-\theta))^{k-i}, \]
\[ = B(G(-\theta); k - r + 1, r). \tag{2.3.2} \]

where \( B(\cdot; p, q) \) is the beta distribution function of first kind with parameters \( p, q \).

In further derivations we consider \( k \) is odd. We obtain \( P_\theta[E_2] \) and \( P_\theta[E_3] \) by considering the joint density of \( X_1 \), \( X_{(k+1)/2} \) and \( X_{(k-r+1)} \).

\[ P_\theta[E_2] = \int_{-\infty}^{0} \int_{-\infty}^{w} f(u, v, w) dv du, \]

where \( f(u, v, w) \) is the joint density of \( X_1 \), \( X_{(k+1)/2} \) and \( X_{(k-r+1)} \). Further,

\[ P_\theta[E_2] = C \int_{0}^{\infty} \int_{-\infty}^{\infty} f(u, v, w) \left[ F(u) - F(w) \right] \left[ F(v) - F(u) \right] \left[ F(w) - F(v) \right]^{k-2r-1} \frac{k!}{(r-1)!} dF(v) dF(w) dF(u), \]

where \( C = \frac{k!}{(r-1)! \left( \frac{k-2r-1}{2} \right)!} \)

Using the relation \( F_\theta(x) = G(x - \theta) \), we get,

\[ P_\theta[E_2] = C \int_{0}^{\infty} \int_{-\infty}^{\infty} \left[ G(u - \theta) (1 - G(w - \theta)) \right] \left[ G(v - \theta) - G(u - \theta) \right] \left[ G(w - \theta) - G(v - \theta) \right] \left[ G(w - \theta) - G(v - \theta) \right]^{k-2r-1} \frac{k!}{(r-1)!} dG(v - \theta) dG(w - \theta) dG(u - \theta), \tag{2.3.3} \]

Similarly,

\[ P_\theta[E_3] = C \int_{-\infty}^{0} \int_{-\infty}^{w} \int_{0}^{u} \left[ G(u - \theta) (1 - G(w - \theta)) \right] \left[ G(v - \theta) - G(u - \theta) \right] \left[ G(w - \theta) - G(v - \theta) \right] \left[ G(w - \theta) - G(v - \theta) \right]^{k-2r-1} \frac{k!}{(r-1)!} dG(v - \theta) dG(w - \theta) dG(u - \theta), \tag{2.3.4} \]

\[ \mu(\theta; k, r) = E[V_\theta(k, r)] = P_\theta[E_1] + \alpha \{ P_\theta[E_2] - P_\theta[E_3] \} - P_\theta[E_4]. \tag{2.3.5} \]
Next, we prove the following relations between probabilities of the above events.

**Lemma 2.3.1:** Following are the relations between probabilities of the events $E_i, i = 1,2,3,4$.

(i) $P_{\theta}[E_1] = P_{-\theta}[E_4]$.

(ii) $P_{\theta}[E_2] = P_{-\theta}[E_3]$.

**Proof:** (i) From (2.3.1), we have,

$$P_{\theta}[E_1] = 1 - \int_0^G (x)^{r-1} \frac{1-x}{x} \frac{(1-x)^{k-r}}{\beta(r,k-r+1)} dx,$$

Putting $1-x = y$, we have,

$$P_{\theta}[E_1] = \int_0^G (y)^{k-r} \frac{1-y}{y} \frac{(1-y)^{r-1}}{\beta(k-r+1,r)} dy$$

$$\Rightarrow P_{\theta}[E_1] = P_{-\theta}[E_4].$$

(ii) This can be established similarly by transforming $u,v,w$ to $-x,-y,-z$ respectively.

Hence the proof. □

Note that under $H_0: \theta = 0$, $P_0[E_1] = P_0[E_4]$ and $P_0[E_2] = P_0[E_3]$. Hence,

$$E_{H_0}[V_a(k,r)] = 0.$$

### 2.3.2 Asymptotic variance of $V_a(k,r)$

Since $V_a(k,r)$ is a one-sample U-statistic, from theorem 1.2, the asymptotic distribution of $V_a(k,r)$ follows normal with mean 0 and variance $k^2 \zeta_1(k,r)$, where,

$$\zeta_1(k,r) = \text{Cov}(\varphi(X_1,X_2,...,X_k),\varphi(X_1,X_{k+1},X_{k+2},...,X_{2k-1}))$$

$$= \text{Var}_{H_0}[E_{H_0}\varphi(X_1,X_2,...,X_k)|X_1 = x].$$

To obtain the general expression for $\text{Var}[E_{H_0}\varphi(X_1,X_2,...,X_k)|X_1 = x]$, we consider the following cases.
**Case 1:** $x > 0$: Let $(X_1, X_2, \ldots, X_k)$ be denoted by $(X_1, Y_1, \ldots, Y_{k-1}) = (X_1, Y)$, where $Y = (Y_1, \ldots, Y_{k-1})$. Let $y_{(1)} < y_{(2)} < \cdots < y_{(k-1)}$ be the ordered values of $y \in R^{k-1}$ (with $y_{(0)} = -\infty$, $y_{(k)} = \infty$). For given $x, \varphi(x, y)$ takes the respective values $1, a, -a, -1$ and 0 on the sets,

$$E_1(x) = \{ y \in R^{k-1}: y_{(r)} > 0 \}.$$

$$E_2(x) = \left\{ y \in R^{k-1}: \left\{ 0 < y_{(k-r)} < x \right\}, \left\{ -\min\{x, y_{(k-r+1)}\} < y_{(r)} < 0 \right\} \text{ and } \left\{ 0 < y_M < y_{(k-r)} \right\} \right\} \text{ or } \left\{ \left\{ x < y_{(k-r)} < \infty \right\}, \left\{ -y_{(k-r)} < y_{(r)} < 0 \right\} \text{ and } \left\{ 0 < y_M < y_{(k-r)} \right\} \right\}.$$

$$E_3(x) = \left\{ y \in R^{k-1}: \left\{ -\infty < y_{(k-r)} < x \right\}, \left\{ y_{(k-r+1)} > \max\{0, y_{(k-r)}\} \right\} \left\{ -\infty < y_{(r)} < -\min\{y_{(k-r)}, 0\} \right\} \text{ or } \left\{ \left\{ y_{(k-r)} > x \right\}, \left\{ -\infty < y_{(r)} < -y_{(k-r)} \right\} \text{ and } \left\{ y_{(r)} < y_M < 0 \right\} \right\}.$$

$$E_4(x) = \{ y \in R^{k-1}: y_{(k-r+1)} < 0 \}.$$

$$E_5(x) = R^{k-1} - E_1(x) \cup E_2(x) \cup E_3(x) \cup E_4(x).$$

where $y_M = Med(x, Y_1, \ldots, Y_{k-1})$.

To obtain $E_{H_0}(\varphi(X_1, X_2, \ldots, X_k)|X_1 = x)$, it is enough to compute the probabilities for the above sets (events) under null by considering the joint distribution of the concerned order statistics from $Y_1, \ldots, Y_{k-1}$. The null probabilities are,

$$P[E_1(x)] = 2^{-(k-1)} \sum_{i=0}^{r-1} \binom{k-1}{i}.$$

(2.3.6)

(since $x > 0$. For $E_1(x)$ there can be at most $(r - 1)$ of $Y$‘s negative and under $H_0$, $P[Y_i > 0] = 1/2$).

$$P[E_4(x)] = 2^{-(k-1)} \sum_{i=k-r+1}^{k-1} \binom{k-1}{i}.$$

(2.3.7)
(since \( x > 0 \). For \( E_4(x) \) there can be at least \((k - r - 1)\) of \( Y \)'s negative).

To obtain \( P[E_2(x)] \), the concerned order statistics and their appropriate domains of integrations are indicated below. Let \((u, v, w, z) = \( (y_{(r)}, y_{(r)}, y_{(k-r)}, y_{(k-r+1)}) \).

**Case 1.1: \(0 < w < x\):** domain of integration for the first term in the right hand side of equation (2.3.8).

\[
\begin{align*}
P[E_2(x)] = & \int_0^x \int_{-\min\{x, z\}}^0 \int_{-\min\{x, z\}}^w f(u, v, w, z) dv du dz dw + \int_x^\infty \int_0^w \int_{-\min\{x, z\}}^w f(u, v, w) dv du dz dw \\
& \quad \text{(2.3.8)}
\end{align*}
\]

where \( f(u, v, w, z) \) is the density function of \( Y_{(r)}, Y_M = Med(x, Y_1, ..., Y_{k-1}), Y_{(k-r)} \) and \( Y_{(k-r+1)} \) and \( f_1(u, v, w) \) is the density function of \( Y_{(r)}, Y_M = Med(x, Y_1, ..., Y_{k-1}), Y_{(k-r)} \) in a random sample of \( Y_1, ..., Y_{k-1} \) from \( G(\cdot) \).

Similarly, the \( P[E_3(x)] \) is,

\[
\begin{align*}
P[E_3(x)] = & \int_{-\infty}^x f_1(u, v, w, z) dw du dz dw + \int_{-\infty}^0 f_1(u, v, w, z) dw du dz dw \\
& \quad \text{(2.3.9)}
\end{align*}
\]

where \( f_1(u, v, w, z) \) and \( f_1(u, v, w) \) are defined as in (2.3.8).
Lemma 2.3.2: Under $H_0$, when $x > 0$ and for the events $E_i(x)$, $i = 1, 2, 3, 4$, we have

(i) $P[E_2(x)] = P[E_3(-x)]$,

(ii) $P[E_1(x)] = P[E_4(-x)]$.

Proof: (i) For simplicity, we give proof when $x < 0$ and $k = 2m - 1$, as a consequence of this, one can prove for $x > 0$.

We have,

$$E_2(x) = \{ \{y_{(m-1)} > 0\} : y_{(r-1)} < 0, y_{(k-r)} > 0 : y_{(r)} < x \} \cap \{y_{(k-r)} > -y_{(r)}\} \cap \{y_{(k-r)} \cap \{y_{(r)} < -\max \{y_{(r-1)}, x\}\} ,$$

By using the fact that, $Y$ and $-Y$ have same distribution, transforming $-Y_{(r)}$ to $U_{(k-r)}$, putting $t = -x$, we get,

$$E_2(-t) = \{ \{u_{(m)} < 0\} : u_{(k-r+1)} > 0, u_{(r)} < 0 : u_{(k-r)} > t \} \cap \{u_{(r)} < -u_{(k-r)}\} \cap \{u_{(r)} < -\min\{u_{(k-r+1)}, t\}\} = E_3(t).$$

This implies,

$$P[E_2(x)] = P[E_3(-x)]$$

when $x < 0$.

(ii) One can establish this similar to the above.

Hence the proof.

Case 2: $x < 0$: From Lemma 2.3.2. We have,

$E_1(x) = E_4(-x)$ and $E_2(x) = E_3(-x)$ for $x < 0$.

$\Rightarrow P[E_1(x)] = P[E_4(-x)]$ and $P[E_2(x)] = P[E_3(-x)]$.

Therefore,

$$E_{H_0}(\varphi(X_1, X_2, ..., X_k) | X_1 = x)$$

$$= \begin{cases} P[E_1(x)] - P[E_4(x)] + a(P[E_2(x)] - P[E_3(x)]) & \text{if } x \geq 0 \\ P[E_4(-x)] - P[E_1(-x)] + a(P[E_3(-x)] - P[E_2(-x)]) & \text{if } x \leq 0 \end{cases}$$

(2.3.10)

Finally,
\[
\zeta_1(k,r) = \text{Var}_{H_0}\left[ E_{H_0} \varphi(X_1, X_2, \ldots, X_k) | X_1 = x \right] \\
= \int_0^\infty \left[ P[E_1(x)] - P[E_4(x)] + a(P[E_2(x)] - P[E_3(x)]) \right]^2 dG(x) + \\
\int_{-\infty}^0 \left[ P[E_4(-x)] - P[E_1(-x)] + a(P[E_3(-x)] - P[E_2(-x)]) \right]^2 dG(x).
\] 

(2.3.11)

2.3.3 Expectation and asymptotic variance for particular cases

In this section, we obtain expectation and asymptotic variance for particular cases.

Case 1: \((k, r) = (3, 1)\): when \(k=3\) and \(r=1\), from (2.3.1) and (2.3.2), we have,

\[
\begin{align*}
P_\theta[E_1] &= (1 - G(-\theta))^3 \\
P_\theta[E_4] &= G^3(-\theta)
\end{align*}
\]

(2.3.12) (2.3.13)

and from (2.3.3), (2.3.4), we have,

\[
P_\theta[E_2] = 3! \int_{-\infty}^{\infty} \int_{-u}^{u} dG(v - \theta)dG(w - \theta)dG(u - \theta),
\]

upon simplification we have,

\[
P_\theta[E_2] = 3 \left\{ (1 - 2G(-\theta))G(-\theta) - \int_{-\infty}^{\theta} G(-t - 2\theta)G(-t - 2\theta) - 2G(-\theta)\right\}.
\]

(2.3.14)

Similarly,

\[
P_\theta[E_3] = 3 \left\{ 2 \int_{-\infty}^{\theta} G(-t - 2\theta)G(-\theta) - G(t) \right\}.
\]

(2.3.15)

Now, substituting (2.3.12)-(2.3.14) in (2.3.5) yields,

\[
\mu(\theta; 3,1) = (1 - G(-\theta))^3 - G^3(-\theta) + 3a\left\{ (1 - 2G(-\theta))G(-\theta) + G^3(-\theta) - \int_{-\infty}^{\theta} G(-t - 2\theta)G(-t - 2\theta) - 2G(t) \right\}.
\]

(2.3.16)

Next, we compute \(P[E_i(x)], i = 1,2,3,4\) under null hypothesis when \(x > 0\). We have,

\[
P[E_1(x)] = 1/4 \text{ and } P[E_4(x)] = 0.
\]

(2.3.17)

To obtain \(P[E_2(x)], P[E_3(x)]\), substituting the joint density of \(Y_1\) and \(Y_2\) in (2.3.8), we get,

\[
P[E_2(x)] = 2 \left\{ \int_0^x \int_{-x}^0 dG(u)dG(v) + \int_x^\infty \int_{-v}^0 dG(u)dG(v) \right\},
\]

upon simplification, we get,
\[ P[E_2(x)] = G^2(x) - G(x) + \frac{1}{2}. \tag{2.3.18} \]

and
\[ P[E_3(x)] = 2\int_{x}^{\infty} \int_{x}^{-\infty} dG(u)dG(v) + \int_{-\infty}^{x} \int_{x}^{\infty} dG(u)dG(v). \]

i.e. \[ P[E_3(x)] = G(x) - G^2(x). \tag{2.3.19} \]

Similarly, for \( x < 0 \), from Lemma 2.3.2, if we replace \( x \) by \( -x \) in the
above, we get,
\[ P[E_1(-x)] = 0 \text{ and } P[E_4(-x)] = 1/4. \]
\[ P[E_2(-x)] = G(x) - G^2(x), \]
\[ P[E_3(-x)] = G^2(x) - G(x) + \frac{1}{2}. \]

Thus, putting \( P[E_i(x)] \) and \( P[E_i(-x)] \), \( i = 1, 2, 3, 4 \) in (2.3.10), we get,
\[ E_{H_0}(\varphi(X_1, X_2, X_3)|X_1 = x) = \begin{cases} a \left[ 2G(x) - 2G^2(x) - \frac{1}{2} \right] - \frac{1}{4} & \text{if } x \geq 0 \\ \frac{1}{4} - a \left[ \frac{1}{2} + 2G^2(x) - 2G(x) \right] & \text{if } x \leq 0 \end{cases}. \]

Hence,
\[ \zeta_1(3, 1) = Var_{H_0}[E_{H_0}\varphi(X_1, X_2, X_3)|X_1 = x] = \left( \frac{1}{16} + \frac{a}{12} + \frac{a^2}{20} \right). \tag{2.3.20} \]

Therefore,
\[ Var_{H_0}(V_a(3, 1)) = 3^2 \zeta_1(3, 1) = 9 \left( \frac{1}{16} + \frac{a}{12} + \frac{a^2}{20} \right). \tag{2.3.21} \]

**Case 2: \((k, r) = (5, 1)\):** From (2.3.1) and (2.3.2), we have,
\[ P_\theta[E_1] = (1 - G(-\theta))^5, \]
\[ P_\theta[E_4] = G^5(-\theta). \]

and from (2.3.3), (2.3.4), we have,
\[ P_\theta[E_2] = 5! \int_{-\infty}^{0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{[G(v - \theta) - G(u - \theta)][G(w - \theta) - G(v - \theta)][G(w - \theta) - G(u - \theta)]\} dG(v - \theta)dG(w - \theta)dG(u - \theta), \]

upon simplification, we have,
\[ P_\theta[E_2] = 10G^4(-\theta) - 10G^2(-\theta) + 5G(-\theta) - \int_{-\infty}^{\theta} [5G^4(-t - 2\theta) - 20G(t)G^3(-t - \theta - \theta) - (G^2(-\theta) - 2G(t)G(-\theta))G^2(-t - \theta) + 20(2G^3(-\theta) - 3G(t)G^2(-\theta))G(-t - \theta)]dG(t), \tag{2.3.22} \]

Similarly,
\[ P_\theta[E_3] = 5G^5(-\theta) + \int_{-\infty}^{-\theta} 30(G^2(-\theta) - 2G(t)G(-\theta)) + G^2(-t - \theta) - \\
20(2G^3(-\theta) - 3G(t)G^2(-\theta) + G^3(t))G(-t - \theta) dG(t). \] (2.3.23)

Now, substituting the above probabilities in (2.3.5), we get,
\[ \mu(\theta; 5, 1) = (1 - G(-\theta))^5 - 5G(-\theta) + 5a \left\{ (1 - 2G(-\theta))G(-\theta) + \\
(2 - G(-\theta))G^4(-\theta) + \int_{-\infty}^{-\theta} G(-t - 2\theta) \{ 4G^3(t) - 6G^2(t)G(-t - 2\theta) + \\
4G(t)G^2(-t - 2\theta) - G^3(-t - 2\theta) \} dG(t) \right\}. \] (2.3.24)

Next, we compute \( P[E_i(x)], i = 1, 2, 3, 4 \) under null hypothesis.
\[ P[E_1(x)] = 1/16 \text{ and } P[E_4(x)] = 0. \]

To obtain \( P[E_2(x)], P[E_3(x)] \), substituting the joint density of \( Y_1, Y_3 \) and \( Y_4 \) obtained from a random sample of \( Y_1, Y_2, Y_3, Y_4 \) in (2.3.8), we get,
\[ P[E_2(x)] = 4! \left\{ \int_0^x \int_{-w}^0 \int_0^u [G(v) - G(u)] dG(v) dG(u) dG(w) + \\
\int_x^0 \int_{-w}^0 \int_0^u [G(v) - G(u)] dG(v) dG(u) dG(w) \right\}, \]
\[ = 4G^4(x) - 8G^3(x) + 6G^2(x) - 2G(x) + \left( \frac{5}{8} \right). \] (2.3.25)

and
\[ P[E_3(x)] = 4! \left\{ \int_0^x \int_{-w}^0 \int_0^u [G(v) - G(u)] dG(v) dG(u) dG(w) + \\
\int_x^0 \int_{-w}^0 \int_0^u [G(v) - G(u)] dG(v) dG(u) dG(w) + \\
\int_0^x \int_{-w}^0 \int_0^u [G(v) - G(u)] dG(v) dG(u) dG(w) + \\
\int_{-\infty}^{-x} \int_{-w}^0 \int_0^u [G(v) - G(u)] dG(v) dG(u) dG(w) \right\}, \]
\[ = -4G^4(x) + 8G^3(x) - 6G^2(x) + 2G(x). \] (2.3.26)

Similarly, for \( x < 0 \), from Lemma 2.3.2., if we replace \( x \) by \( -x \) in the above, we get,
\[ P[E_1(-x)] = 0 \text{ and } P[E_4(-x)] = 1/16. \]
\[ P[E_2(-x)] = -4G^4(x) + 8G^3(x) - 6G^2(x) + 2G(x), \]
\[ P[E_3(-x)] = 4G^4(x) - 8G^3(x) + 6G^2(x) - 2G(x) + \left( \frac{5}{8} \right). \]

Thus, putting \( P[E_i(x)] \) and \( P[E_i(-x)], i = 1, 2, 3, 4 \) in (2.3.11) we get,
\[ \zeta_1(5,1) \]
\[ = \int_0^\infty \left[ \left( \frac{1}{16} \right) + a \left\{ 8G^4(x) - 16G^3(x) + 12G^2(x) - 4G(x) + \left( \frac{5}{8} \right) \right\} \right]^2 dG(x) \]
\[ + \int_0^\infty \left[ a \left\{ -8G^4(x) + 16G^3(x) - 12G^2(x) + 4G(x) - \left( \frac{5}{8} \right) \right\} - \left( \frac{1}{16} \right) \right]^2 dG(x). \]

Upon simplification, we get,

\[ \zeta_1(5,1) = \frac{1}{256} + \frac{9a}{320} + \frac{197a^2}{2880}. \]  \hspace{1cm} (2.3.27)

Therefore,

\[ Var_{H_0}(V_a(5,1)) = 5^2 \zeta_1(5,1) = 25 \left( \frac{1}{256} + \frac{9a}{320} + \frac{197a^2}{2880} \right). \]  \hspace{1cm} (2.3.28)

2.4 Performance of the tests based on Efficacy

Let \( T_n \) be a sequence of test statistics for testing the hypothesis that \( H_0: \theta = 0 \) against the suitable alternative. Let \( E(T_n) = \mu_n(\theta) \) and \( Var(T_n) = \sigma_n^2(\theta) \). Under certain regularity conditions (may refer to Randles & Wolfe (1979), pp. 147-149) the efficacy of \( T \) is given by,

\[ eff[T] = \lim_{n \to \infty} \frac{\mu_n(\theta)}{\sqrt{n} \sigma_n(\theta)}. \]  \hspace{1cm} (2.4.1)

By considering \( T = V_a(k,r) \), \( \mu_n(\theta) = \mu(\theta; k, r) \), we will have,

\[ eff^2[V_a(k,r)] = \frac{[\mu'(0;k,r)]^2}{k^2 \zeta_1(kr)}. \]  \hspace{1cm} (2.4.2)

Note that, (2.4.2) depends on the \( G(.) \) and the constant ‘\( a \)’. For given \( G(.) \), the optimal value \( a_{*}(k,r) \) of ‘\( a \)’ is obtained by solving \( (d/da) \ eff^2[V_a(k,r)] = 0 \) and verifying

\( (d^2/da^2) \ eff^2[V_a(k,r)] < 0 \) at the solution obtained.

Efficacies of the tests for various models:

Case I: \( (k, r) = (3,1) \): From (2.3.15) we have,
\[ \mu'(0; 3,1) = \frac{3}{2} g(0) + 12a_1, \quad (2.4.3) \]

where \( I_1 = \int_{-\infty}^{0} (1 - 2G(t)) g^2(t) dt \), hence from (2.4.2),

\[ e f f^2[V_0(3,1)] = \frac{60[2g(0)+8a_1]^2}{(15+20a+12a^2)}, \quad (2.4.4) \]

The optimal \( a^*_{(3,1)} \) is,

\[ a^*_{(3,1)} = \left( \frac{60I_1 - 5g(0)}{6g(0) - 40I_1} \right), \quad (2.4.5) \]

Similarly, efficacy of \( V_{a}(5,1) \) is given by,

\[ e f f^2[V_a(3,1)] = \frac{45[2g(0)+a(5g(0)+384I_2-1)]^2}{(45+324a+788a^2)}, \quad (2.4.6) \]

as

\[ \mu'(0,5,1) = \frac{5}{8} g(0) + \frac{5}{16} a\{5g(0) + 384I_2 - 1\}, \quad (2.4.7) \]

where \( I_2 = (1/3) \int_{-\infty}^{0} (1 - 2G(t))^3 g^2(t) dt \).

The optimal \( a^*_{(5,1)} \) is,

\[ a^*_{(5,1)} = \left( \frac{45+99g(0)-17325I_2}{1576-7718g(0)-606760I_1} \right). \quad (2.4.8) \]

In Table 2.1, optimal values \( a^*_{(3,1)} \), \( a^*_{(5,1)} \) together with the corresponding efficacies of \( V_{a^*_{(3,1)}}(3,1) \), \( V_{a^*_{(5,1)}}(5,1) \) and other competitors for various models are given below.
Table 2.1: Efficacy of Sign test (S), Wilcoxon Signed-rank test(W), $T_a^*$, $U_a^*(4,2), V_{a_{(3,1)}}^*(3,1)$ and $V_{a_{(5,1)}}^*(5,1)$ for few Standard models.

<table>
<thead>
<tr>
<th>Density</th>
<th>$a_{(3,1)}^*$</th>
<th>$a_{(5,1)}^*$</th>
<th>$eff^2(S)$</th>
<th>$eff^2(W)$</th>
<th>$eff^2(T_a^*)$</th>
<th>$eff^2(U_a^*(4,2))$</th>
<th>$eff^2(V_{a_{(3,1)}}^*(3,1))$</th>
<th>$eff^2(V_{a_{(5,1)}}^*(5,1))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cauchy</td>
<td>-0.1762</td>
<td>-0.0230</td>
<td>0.4053</td>
<td>0.3040</td>
<td>0.4053</td>
<td>0.4252</td>
<td>0.6411</td>
<td>0.4369</td>
</tr>
<tr>
<td>Laplace</td>
<td>0</td>
<td>0</td>
<td>2.0000</td>
<td>1.5000</td>
<td>2.0000</td>
<td>2.0000</td>
<td>2.0000</td>
<td>2.0000</td>
</tr>
<tr>
<td>Logistic</td>
<td>2.5088</td>
<td>0.0033</td>
<td>0.8225</td>
<td>1.0966</td>
<td>1.0966</td>
<td>1.1364</td>
<td>1.0799</td>
<td>0.8247</td>
</tr>
<tr>
<td>Normal</td>
<td>7.2960</td>
<td>0.0051</td>
<td>0.6366</td>
<td>0.9549</td>
<td>0.9643</td>
<td>0.9503</td>
<td>0.9868</td>
<td>0.6414</td>
</tr>
<tr>
<td>Parabolic</td>
<td>-6.1008</td>
<td>-0.0149</td>
<td>0.4500</td>
<td>0.8460</td>
<td>0.9360</td>
<td>0.8125</td>
<td>1.0829</td>
<td>1.5956</td>
</tr>
<tr>
<td>Triangular</td>
<td>4.4766</td>
<td>0.0059</td>
<td>0.6667</td>
<td>0.8889</td>
<td>0.8889</td>
<td>0.7965</td>
<td>0.9657</td>
<td>0.6735</td>
</tr>
<tr>
<td>Uniform</td>
<td>-2.4991</td>
<td>0.0180</td>
<td>0.3333</td>
<td>1.3333</td>
<td>1.3333</td>
<td>0.6869</td>
<td>2.0021</td>
<td>0.4041</td>
</tr>
<tr>
<td>$\frac{1}{2} \log</td>
<td>y</td>
<td>/ \mu$</td>
<td>-1.5000</td>
<td>0.0286</td>
<td>0.0000</td>
<td>2.6667</td>
<td>10.6600</td>
<td>16.4563</td>
</tr>
</tbody>
</table>

From the Table 2.1, we note that the efficacy of $V_{a_{(3,1)}}^*(3,1)$ performs better than other competitors for almost all the models. For logistic model efficacy of $V_{a_{(3,1)}}^*(3,1)$ is closer to Wilcoxon signed-rank test, the one known to be locally most powerful for this model.

Under $H_0$ it is only known that $G(.)$ is symmetric about zero. Hence it is desirable to choose the constant ‘$a$’ irrespective of a particular density. Whateoeve the symmetric model, we recommend $a = 2.5088$, the optimal value corresponding to the logistic model. Mehra et al. (1990); Shetty and Pandit (2000) have recommended the values corresponding to normal model which are respectively, 1.2426 and 2.0933.
Table 2.2: Efficacy of $T_{1.2426}$, $U_{2.0938}(4,2)$,$V_{2.5088}(3,1)$ for few Standard models.

<table>
<thead>
<tr>
<th>Density</th>
<th>$eff^2(T_{1.2426})$</th>
<th>$eff^2(U_{2.0938}(4,2))$</th>
<th>$eff^2(V_{2.5088}(3,1))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cauchy</td>
<td>0.3676</td>
<td>0.1323</td>
<td>0.2688</td>
</tr>
<tr>
<td>Laplace</td>
<td>1.3204</td>
<td>1.8604</td>
<td>1.5227</td>
</tr>
<tr>
<td>Logistic</td>
<td>1.0859</td>
<td>0.3572</td>
<td>1.0799</td>
</tr>
<tr>
<td>Normal</td>
<td>0.9643</td>
<td>0.9503</td>
<td>0.9707</td>
</tr>
<tr>
<td>Parabolic</td>
<td>0.9052</td>
<td>0.1894</td>
<td>0.9486</td>
</tr>
<tr>
<td>Triangular</td>
<td>0.8803</td>
<td>0.1916</td>
<td>0.9595</td>
</tr>
<tr>
<td>Uniform</td>
<td>1.1068</td>
<td>0.1771</td>
<td>1.2878</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{n}}</td>
<td>\left</td>
<td>I_{y} \right</td>
<td>_{\left</td>
</tr>
</tbody>
</table>

From the Table 2.2, we note that still the efficacy of $V_{2.5088}(3,1)$ continues to be better sign test and other competitors.

In next section, we perform empirical power study to assess the performances of the proposed tests $T_a$, $U_a(4,2)$,$V_a(3,1)$ with t-test, sign and Wilcoxon signed-rank tests.

2.5 Performance of the tests based on Empirical power

Under $H_0$, the statistic $V_a(3,1)$ is asymptotically normal with mean 0 and variance given by (2.3.15). Thus the criterion to test $H_0$ versus $H_1$ at level $\alpha$ is,

reject $H_0$ if $\frac{\sqrt{n} \left| V_{a(3,1)} \right|}{3\sqrt{\chi^2_{1}(3,1)}} \geq Z_{\alpha/2}$.

(2.5.1)

Where $Z_{\alpha/2}$ is the upper ($\alpha/2$)th percentile of standard normal distribution. Similarly, the criterion for the test statistics $T_a$ and $U_a(k, r)$ proposed by Mehraet al. (1990); Shetty and Pandit (2000) are also defined.

An empirical study was carried out for moderate sample size $n = 25$, using the samples from first seven standard models given in Table 2.1 and two models from a family of heavy tailed distributions with density $f(x, p)$,
The pdf \( f(x, p) \) is defined as,

\[
f(x, p) = \frac{s\sin(\frac{\pi x}{p})}{\pi} \times \frac{1}{1 + |x|^p}, \quad -\infty < x < \infty \text{ and } p > 1.
\]

Note that, Cauchy distribution is a member of \( f(x, p) \). For the above family of distributions, it appears obtaining optimal value of \( 'a' \) that maximizes the efficacy is difficult, as it involves solving the complicated integrals when \( p \neq 2 \). We propose the test using optimal value of \( 'a' \) corresponding to the Cauchy model. In Table 2.3, empirical powers of the tests for various models are given.

We have also studied the performances of \( V_a(3,1) \) based on empirical power using optimal \( 'a' \) corresponding to the logistic model, i.e. \( a = 2.5088 \). The results are tabulated in the Table 2.4.

**Table 2.3:** Empirical Power of the Tests for various models with \( \alpha = 0.05, \ n = 25 \) and number of Monte Carlo simulations are 10000.

<table>
<thead>
<tr>
<th>Density</th>
<th>Tests</th>
<th>( \theta = 0 )</th>
<th>( \theta = 0.2 )</th>
<th>( \theta = 0.4 )</th>
<th>( \theta = 0.6 )</th>
<th>( \theta = 0.8 )</th>
<th>( \theta = 1.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cauchy</td>
<td>( t )</td>
<td>0.019</td>
<td>0.026</td>
<td>0.052</td>
<td>0.091</td>
<td>0.145</td>
<td>0.202</td>
</tr>
<tr>
<td></td>
<td>( S  )</td>
<td>0.042</td>
<td>0.083</td>
<td>0.206</td>
<td>0.386</td>
<td>0.583</td>
<td>0.735</td>
</tr>
<tr>
<td></td>
<td>( W  )</td>
<td>0.046</td>
<td>0.083</td>
<td>0.175</td>
<td>0.321</td>
<td>0.476</td>
<td>0.618</td>
</tr>
<tr>
<td></td>
<td>( T_{a^*} )</td>
<td>0.042</td>
<td>0.083</td>
<td>0.206</td>
<td>0.386</td>
<td>0.584</td>
<td>0.735</td>
</tr>
<tr>
<td></td>
<td>( U_{a^*} )</td>
<td>0.046</td>
<td>0.092</td>
<td>0.220</td>
<td>0.406</td>
<td>0.604</td>
<td>0.754</td>
</tr>
<tr>
<td></td>
<td>( V_{a(3,1)} )</td>
<td>0.049</td>
<td>0.095</td>
<td>0.228</td>
<td>0.412</td>
<td>0.606</td>
<td>0.755</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Laplace</td>
<td>( t )</td>
<td>0.045</td>
<td>0.077</td>
<td>0.171</td>
<td>0.325</td>
<td>0.513</td>
<td>0.683</td>
</tr>
<tr>
<td></td>
<td>( S  )</td>
<td>0.044</td>
<td>0.094</td>
<td>0.211</td>
<td>0.389</td>
<td>0.579</td>
<td>0.736</td>
</tr>
<tr>
<td></td>
<td>( W  )</td>
<td>0.045</td>
<td>0.091</td>
<td>0.212</td>
<td>0.392</td>
<td>0.594</td>
<td>0.752</td>
</tr>
<tr>
<td></td>
<td>( T_{a^*} )</td>
<td>0.044</td>
<td>0.094</td>
<td>0.211</td>
<td>0.389</td>
<td>0.578</td>
<td>0.736</td>
</tr>
<tr>
<td></td>
<td>( U_{\text{a}^*} )</td>
<td>0.044</td>
<td>0.094</td>
<td>0.211</td>
<td>0.389</td>
<td>0.578</td>
<td>0.736</td>
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<tr>
<td></td>
<td>( V_{a_{(3,1)}^*} )</td>
<td>0.044</td>
<td>0.094</td>
<td>0.211</td>
<td>0.389</td>
<td>0.578</td>
<td>0.736</td>
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</tr>
<tr>
<td></td>
<td>( t )</td>
<td>0.046</td>
<td>0.155</td>
<td>0.499</td>
<td>0.825</td>
<td>0.965</td>
<td>0.996</td>
</tr>
<tr>
<td></td>
<td>( S )</td>
<td>0.042</td>
<td>0.125</td>
<td>0.409</td>
<td>0.722</td>
<td>0.914</td>
<td>0.982</td>
</tr>
<tr>
<td></td>
<td>( W )</td>
<td>0.046</td>
<td>0.160</td>
<td>0.517</td>
<td>0.840</td>
<td>0.971</td>
<td>0.997</td>
</tr>
<tr>
<td></td>
<td>( T_{a^*} )</td>
<td>0.050</td>
<td>0.169</td>
<td>0.533</td>
<td>0.848</td>
<td>0.972</td>
<td>0.998</td>
</tr>
<tr>
<td></td>
<td>( U_{a^*} )</td>
<td>0.062</td>
<td>0.144</td>
<td>0.523</td>
<td>0.829</td>
<td>0.967</td>
<td>0.996</td>
</tr>
<tr>
<td></td>
<td>( V_{a_{(3,1)}^*} )</td>
<td>0.046</td>
<td>0.154</td>
<td>0.500</td>
<td>0.822</td>
<td>0.959</td>
<td>0.993</td>
</tr>
<tr>
<td>----</td>
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</tr>
<tr>
<td></td>
<td>( t )</td>
<td>0.053</td>
<td>0.165</td>
<td>0.488</td>
<td>0.814</td>
<td>0.969</td>
<td>0.998</td>
</tr>
<tr>
<td></td>
<td>( S )</td>
<td>0.048</td>
<td>0.113</td>
<td>0.329</td>
<td>0.623</td>
<td>0.859</td>
<td>0.965</td>
</tr>
<tr>
<td></td>
<td>( W )</td>
<td>0.052</td>
<td>0.151</td>
<td>0.467</td>
<td>0.796</td>
<td>0.963</td>
<td>0.997</td>
</tr>
<tr>
<td></td>
<td>( T_{a^*} )</td>
<td>0.054</td>
<td>0.161</td>
<td>0.478</td>
<td>0.807</td>
<td>0.963</td>
<td>0.997</td>
</tr>
<tr>
<td></td>
<td>( U_{a^*} )</td>
<td>0.058</td>
<td>0.170</td>
<td>0.462</td>
<td>0.765</td>
<td>0.942</td>
<td>0.992</td>
</tr>
<tr>
<td></td>
<td>( V_{a_{(3,1)}^*} )</td>
<td>0.053</td>
<td>0.153</td>
<td>0.445</td>
<td>0.762</td>
<td>0.901</td>
<td>0.980</td>
</tr>
<tr>
<td>----</td>
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</tr>
<tr>
<td></td>
<td>( t )</td>
<td>0.051</td>
<td>0.499</td>
<td>0.984</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>( S )</td>
<td>0.042</td>
<td>0.167</td>
<td>0.628</td>
<td>0.981</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>( W )</td>
<td>0.046</td>
<td>0.451</td>
<td>0.944</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>( T_{a^*} )</td>
<td>0.051</td>
<td>0.590</td>
<td>0.976</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>( U_{a^*} )</td>
<td>0.055</td>
<td>0.398</td>
<td>0.898</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>( V_{a_{(3,1)}^*} )</td>
<td>0.049</td>
<td>0.622</td>
<td>0.977</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>----</td>
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<td>-------</td>
<td>-------</td>
<td>-------</td>
</tr>
<tr>
<td></td>
<td>( t )</td>
<td>0.051</td>
<td>0.156</td>
<td>0.477</td>
<td>0.826</td>
<td>0.973</td>
<td>0.999</td>
</tr>
<tr>
<td></td>
<td>( S )</td>
<td>0.044</td>
<td>0.110</td>
<td>0.301</td>
<td>0.577</td>
<td>0.816</td>
<td>0.946</td>
</tr>
<tr>
<td></td>
<td>( W )</td>
<td>0.049</td>
<td>0.145</td>
<td>0.436</td>
<td>0.780</td>
<td>0.956</td>
<td>0.997</td>
</tr>
<tr>
<td></td>
<td>( T_{a^*} )</td>
<td>0.053</td>
<td>0.154</td>
<td>0.453</td>
<td>0.796</td>
<td>0.962</td>
<td>0.997</td>
</tr>
<tr>
<td></td>
<td>( U_{a^*} )</td>
<td>0.058</td>
<td>0.143</td>
<td>0.405</td>
<td>0.732</td>
<td>0.931</td>
<td>0.991</td>
</tr>
<tr>
<td>( V_{a_{(3,1)}} )</td>
<td>0.048 0.149 0.457 0.810 0.961 0.998</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>----------------</td>
<td>----------------------------------</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( t )</td>
<td>0.050 0.154 0.470 0.823 0.976 0.998</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( S )</td>
<td>0.042 0.080 0.191 0.392 0.659 0.863</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( W )</td>
<td>0.046 0.143 0.417 0.743 0.939 0.992</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( T_{a^*} )</td>
<td>0.053 0.183 0.500 0.816 0.962 0.996</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( U_{a^*} )</td>
<td>0.051 0.154 0.357 0.644 0.863 0.976</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( V_{a_{(3,1)}} )</td>
<td>0.046 0.173 0.497 0.816 0.961 0.997</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Uniform**

<table>
<thead>
<tr>
<th>( f(x, 1.4) )</th>
<th>( t ) 0.048 0.093 0.223 0.445 0.690 0.866</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S )</td>
<td>0.042 0.092 0.220 0.404 0.605 0.748</td>
</tr>
<tr>
<td>( W )</td>
<td>0.046 0.096 0.228 0.441 0.672 0.842</td>
</tr>
<tr>
<td>( T_{a^*} )</td>
<td>0.042 0.092 0.220 0.404 0.605 0.748</td>
</tr>
<tr>
<td>( U_{a^*} )</td>
<td>0.046 0.099 0.238 0.451 0.669 0.824</td>
</tr>
<tr>
<td>( V_{a_{(3,1)}} )</td>
<td>0.049 0.100 0.230 0.410 0.607 0.758</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( f(x, 2.2) )</th>
<th>( t ) 0.049 0.546 0.992 1.000 1.000 1.000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S )</td>
<td>0.045 0.260 0.796 0.994 1.000 1.000</td>
</tr>
<tr>
<td>( W )</td>
<td>0.047 0.481 0.974 1.000 1.000 1.000</td>
</tr>
<tr>
<td>( T_{a^*} )</td>
<td>0.045 0.260 0.796 0.994 1.000 1.000</td>
</tr>
<tr>
<td>( U_{a^*} )</td>
<td>0.049 0.342 0.898 0.995 1.000 1.000</td>
</tr>
<tr>
<td>( V_{a_{(3,1)}} )</td>
<td>0.051 0.261 0.784 0.992 1.000 1.000</td>
</tr>
</tbody>
</table>

From Table 2.3, we observe that the empirical power of \( V_{a_{(3,1)}} \) (3, 1) is higher than the Sign test, which is known to perform better for heavy tailed models. Though for other models, the performance of \( V_{a_{(3,1)}} \) (3, 1) is not the best, its performance is pretty close to the superior ones from the class of tests considered.
Table 2.4: Empirical Power of $T_{(1,2426)}, U_{2.0938}, V_{2.5088}$ with $\alpha = 0.05$, $n = 25$ and number of Monte Carlo simulations are 10000.

<table>
<thead>
<tr>
<th>Density</th>
<th>Tests</th>
<th>$\theta = 0$</th>
<th>$\theta = 0.2$</th>
<th>$\theta = 0.4$</th>
<th>$\theta = 0.6$</th>
<th>$\theta = 0.8$</th>
<th>$\theta = 1.0$</th>
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</thead>
<tbody>
<tr>
<td>Cauchy</td>
<td>$T_{(1,2426)}$</td>
<td>0.050</td>
<td>0.082</td>
<td>0.162</td>
<td>0.292</td>
<td>0.431</td>
<td>0.553</td>
</tr>
<tr>
<td></td>
<td>$U_{2.0938}$</td>
<td>0.054</td>
<td>0.105</td>
<td>0.209</td>
<td>0.367</td>
<td>0.530</td>
<td>0.654</td>
</tr>
<tr>
<td></td>
<td>$V_{2.5088}$</td>
<td>0.046</td>
<td>0.074</td>
<td>0.141</td>
<td>0.245</td>
<td>0.353</td>
<td>0.449</td>
</tr>
<tr>
<td>Laplace</td>
<td>$T_{(1,2426)}$</td>
<td>0.056</td>
<td>0.088</td>
<td>0.214</td>
<td>0.373</td>
<td>0.571</td>
<td>0.733</td>
</tr>
<tr>
<td></td>
<td>$U_{2.0938}$</td>
<td>0.058</td>
<td>0.082</td>
<td>0.183</td>
<td>0.322</td>
<td>0.477</td>
<td>0.596</td>
</tr>
<tr>
<td></td>
<td>$V_{2.5088}$</td>
<td>0.051</td>
<td>0.084</td>
<td>0.200</td>
<td>0.342</td>
<td>0.528</td>
<td>0.684</td>
</tr>
<tr>
<td>Parabolic</td>
<td>$T_{(1,2426)}$</td>
<td>0.052</td>
<td>0.590</td>
<td>0.976</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>$U_{2.0938}$</td>
<td>0.058</td>
<td>0.398</td>
<td>0.898</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>$V_{2.5088}$</td>
<td>0.046</td>
<td>0.548</td>
<td>0.967</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>Triangular</td>
<td>$T_{(1,2426)}$</td>
<td>0.048</td>
<td>0.149</td>
<td>0.448</td>
<td>0.798</td>
<td>0.961</td>
<td>0.998</td>
</tr>
<tr>
<td></td>
<td>$U_{2.0938}$</td>
<td>0.053</td>
<td>0.118</td>
<td>0.322</td>
<td>0.575</td>
<td>0.733</td>
<td>0.784</td>
</tr>
<tr>
<td></td>
<td>$V_{2.5088}$</td>
<td>0.046</td>
<td>0.149</td>
<td>0.451</td>
<td>0.808</td>
<td>0.966</td>
<td>0.998</td>
</tr>
<tr>
<td>Uniform</td>
<td>$T_{(1,2426)}$</td>
<td>0.049</td>
<td>0.165</td>
<td>0.474</td>
<td>0.798</td>
<td>0.958</td>
<td>0.996</td>
</tr>
<tr>
<td></td>
<td>$U_{2.0938}$</td>
<td>0.053</td>
<td>0.139</td>
<td>0.353</td>
<td>0.656</td>
<td>0.887</td>
<td>0.976</td>
</tr>
<tr>
<td></td>
<td>$V_{2.5088}$</td>
<td>0.046</td>
<td>0.174</td>
<td>0.497</td>
<td>0.817</td>
<td>0.961</td>
<td>0.997</td>
</tr>
<tr>
<td>$f(x,1.4)$</td>
<td>$T_{(1,2426)}$</td>
<td>0.051</td>
<td>0.080</td>
<td>0.218</td>
<td>0.438</td>
<td>0.606</td>
<td>0.764</td>
</tr>
<tr>
<td></td>
<td>$U_{2.0938}$</td>
<td>0.050</td>
<td>0.105</td>
<td>0.238</td>
<td>0.442</td>
<td>0.673</td>
<td>0.824</td>
</tr>
<tr>
<td></td>
<td>$V_{2.5088}$</td>
<td>0.050</td>
<td>0.091</td>
<td>0.248</td>
<td>0.455</td>
<td>0.687</td>
<td>0.850</td>
</tr>
<tr>
<td>$f(x,2.2)$</td>
<td>$T_{(1,2426)}$</td>
<td>0.052</td>
<td>0.524</td>
<td>0.982</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>$U_{2.0938}$</td>
<td>0.053</td>
<td>0.342</td>
<td>0.764</td>
<td>0.803</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>$V_{2.5088}$</td>
<td>0.049</td>
<td>0.370</td>
<td>0.899</td>
<td>0.998</td>
<td>1.000</td>
<td>1.000</td>
</tr>
</tbody>
</table>

From Table 2.4, we observe that, in practice one can safely use $V_{2.5088}(3,1)$. 

31
2.6 Conclusion

In this chapter, we have proposed a class of distribution-free tests for one-sample location problem, which includes test statistics $T_{\alpha}$ and $U_{\alpha}(4,2)$ proposed by Mehra et al.(1990); Shetty and Pandit (2000) respectively. The proposed test statistics are of U-Statistics type, depend on a constant '$\alpha$' and $r^h$, $(k-r+1)^{th}$ order statistics taken from sub-sample of size $k$ together with its median. Expressions for expected value and asymptotic variance are obtained for arbitrary value of $(k, r)$.

The optimal value of '$\alpha$' is obtained by maximizing the efficacy (in Pitman sense) of the test. Though the optimal '$\alpha$' depends on $(k, r)$ and $G(\cdot)$, from practical point of view one can safely use the test $V_{(2.5088)}(3,1)$.

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