ON THE CATEGORY TS OF ALL TOPOLOGICAL SEMIGROUPS

Introduction

J.H. Carruth, J.A. Hildebrant and R.J. Koch \([C-H-K_2]\) interpret several categorical concepts in various categories of topological semigroups like category of compact semilattices and category of compact Lawson semilattices. In 1973, Crawley \([CR]\) made an extensive study in this direction.

In topology, in the category of Hausdorff spaces, the epimorphisms are the mappings with dense range \([W]\). But in the category of topological semigroups, every epimorphism need not be of this form \([C-H-K_2]\). In 1973, Herrlich and Strecker \([H-S]\) showed that group epimorphisms are surjective. In 1975, Hofmann and Mislove \([HO-M]\) established that discrete-semilattice epimorphisms are surjective. In 1966, Husain \([HUS]\) proved that in the category of Topological abelian groups (locally compact abelian groups) each epimorphism is dense. The compact abelian group epimorphisms are surjective follows from the result of Section 5 of chapter 1 \([C-H-K_1]\) and showed that
Abelian group, Topological abelian group, Locally compact abelian group epimorphisms are dense. In 1966, Hofmann and Mostert [HO-M1] gave an example to show that compact semigroup epimorphisms are not necessarily surjective. In 1975, Lamatrin [L] showed that epimorphisms in the category of Hausdorff [abelian] K-groups need not be dense. However, question remains unanswered in various other categories of topological semigroups.

In this chapter, in Section 5.2, we discuss epimorphisms in the category of all topological semigroups. In Section 5.3, we define weak extremal monomorphism and prove that if the images are ideals the weak extremal monomorphisms in the category of all topological semigroups are the closed embeddings.

5.1 Preliminaries

In the theory of (topological) semigroups morphisms are simply (continuous) homomorphisms except that in the monoid categories morphisms are required to be identity-preserving. The rule of composition in each category is ordinary composition of functions. Isomorphisms are precisely the topological isomorphisms [C -H-K2].
Definition 5.1.1.

A morphism \( e: A \rightarrow B \) is an epimorphism if for every pair of morphisms the equality \( foe = goe \) implies that \( f = g \).

\[
\begin{array}{ccc}
A & \xrightarrow{e} & B \\
\downarrow{f} & & \downarrow{g} \\
\end{array}
\]

Definition 5.1.2.

A morphism \( f \) is a monomorphism if for every pair of morphisms the equality \( fog = foh \) implies that \( f = g \).

\[
\begin{array}{ccc}
A & \xleftarrow{f} & B \\
& \xleftarrow{g} & \leftarrow C \\
& \downarrow{h} &
\end{array}
\]

Note:

Monomorphisms in the category of all topological semigroups are precisely injective homomorphisms [C-H-K2].

Definition 5.1.3.

A functor \( r \) from a category \( C^* \) to a subcategory \( R^* \) of \( C^* \) is a **reflective functor** if there is a unique morphism \( \eta_c : c \rightarrow rc \) and if every morphism from \( c \) to any object \( Y \) of \( R^* \) factors uniquely through \( rc \) via \( \eta_c \) so that
the following diagram commutes [W].

If \( r: C^* \rightarrow R^* \) is a reflective functor, the subcategory \( R^* \) is called a reflective subcategory. The object \( rc \) is called the reflection of \( c \) in \( R^* \). [W]

Definition 5.1.4

A reflective functor \( r \) is said to be **epi-reflective** if the morphism

\[
\eta_X : X \longrightarrow rX
\]

is an epimorphism. [W]

5.2 **Epimorphisms in the category TS**

The epimorphisms in the category of Hausdorff spaces are the mappings with dense range. [W]

In the case of topological semigroups also, continuous homomorphisms with dense range are epimorphisms. But the converse need not be true. For example, let \( S \) be a semigroup of non-negative integers under addition
with discrete topology and let \( \emptyset: S \rightarrow Z \) be the inclusion homomorphism. Then \( \emptyset \) is not dense in \( Z \) but \( \emptyset \) is an epimorphism. [C-H-K_2].

In this situation we study when will the converse hold. As a result, we have the following propositions.

**Proposition 5.2.1.**

Let \( f:X \rightarrow Y \) be a continuous homomorphism such that \( \langle f(X) \rangle \) is dense in \( Y \). If \( g,h \) agree on \( \langle f(X) \rangle \), then \( g=h \) (where \( \langle f(X) \rangle = f(X) \cup Yf(X) \cup f(X) Y \cup Yf(X)Y \), the ideal generated by \( f(X) \)).

**Proof.**

Let \( f:X \rightarrow Y \) be a continuous homomorphism such that \( \langle f(X) \rangle = Y \) and \( g,h \) agree on \( \langle f(X) \rangle \).

**Claim.**

\( f \) is an epimorphism.

For this show that \( g(x)=h(x) \) for all \( x \in Y \)

\[
\begin{array}{ccc}
X & f & Y \\
\hline
& g & \rightarrow Z \\
\end{array}
\]

If not, assume that \( g(x) \neq h(x) \) for at least one \( x \in Y \setminus \langle f(X) \rangle \). Since \( Z \) is a Hausdorff space, there exists disjoint open sets say \( U \) and \( U' \) containing \( g(x) \)
and \( h(x) \) respectively, \( U \cap U' = \emptyset \). Choose a neighbourhood \( V \) of \( x \) such that \( g(V) \subseteq U \), \( h(V) \subseteq U' \). This is possible, since \( g, h \) are continuous. Since \( x \in Y = \langle f(X) \rangle \), \( V \) intersects \( \langle f(X) \rangle \) in some point say \( 'y' \) other than \( x \).

Then \( g(y) \in U \), \( h(y) \in U' \)

but \( g(y) = h(y) \) ( \( \because y \in \langle f(X) \rangle \) )

i.e., \( U \cap U' \neq \emptyset \)

This contradicts the fact that \( U \) and \( U' \) are disjoint.

\( \therefore \ g(x) = h(x) \) for all \( x \in Y \)

\( \therefore \ f \) is an epimorphism.

Proposition 5.2.2 (converse of 5.2.1)

Suppose \( f: X \longrightarrow Y \) be an epimorphism then \( \langle f(X) \rangle \) is dense in \( Y \).

Proof

For this assume that \( \langle f(X) \rangle \neq Y \), then we will show that \( f \) cannot be an epimorphism by constructing a topological semigroup \( Z \) and two continuous homomorphisms \( L_1 \) and \( L_2 \) from \( Y \) into \( Z \) which agree on \( f(X) \), but which are not equal.

For, \( \quad \text{let } Y_1 = Y \times \{1\}, \quad Y_2 = Y \times \{2\} \)
$Y_1$ and $Y_2$ are topological semigroups with product topology and multiplication defined by $(x,i) (y,i) = (xy,i)$, for each $i=1,2$. Let $h_i : Y_i \longrightarrow Y_i$ be defined by $h_i(y) = (y,i)$, for each $i = 1,2$. Then $h_i$, $i = 1,2$, are topological isomorphisms.

$$Y_1 \cup Y_2 = \{ (x,i) : (x,i) \in Y_1 \text{ or } Y_2 \text{ for each } i = 1,2 \}$$

The disjoint topological sum $Y_1 \cup Y_2$ is a topological semigroup with multiplication defined by

$$(x,i) (y,j) = (xy, \min \{i,j\})$$

Multiplication is well-defined.

For,

if $(x,i) = (x',j)$ then $x = x'$, $i = j$

and if

$$(y,j) = (y',i) \text{ then } y = y', i = j$$

\[ \therefore \quad xy = x'y', \quad i = j \]

i.e., $(x,i)(y,j) = (xy, \min \{i,j\})$

$= (x'y', \min \{i,j\})$

$= (x',j) (y',i)$
Clearly multiplication is associative and continuous.

Let \( i_1 : Y_1 \longrightarrow Y_1 \cup Y_2 \)
\[ i_2 : Y_2 \longrightarrow Y_1 \cup Y_2 \]

be the inclusion maps.

\( i_1 \circ h_1 < f(X) > \cup i_2 \circ h_2 < f(X) > \) is the set of copies of \( < f(X) > \) contained in \( Y_1 \cup Y_2 \). Let \( Z \) be the image of the quotient map 'q' obtained by identifying
\[ i_1 \circ h_1 (y) = i_1(y,1) \quad \text{and} \quad i_2 \circ h_2 (y) = i_2(y,2) \quad \text{if} \quad y \in < f(X) > \]

Define \( q(x,i) \quad q(y,j) = q(xy, \min \{i,j\} ) \).

This multiplication is well defined. For,

if \( q(x,i) = q(x',j) \) then either \( x = x' \) and \( i = j \)
or \( i \neq j \) and \( x = x' \in < f(X) > \) and if \( q(y,j) = q(y',i) \)
then either \( y = y' \) and \( i = j \) or \( i \neq j \) and \( y = y' \in < f(X) > \).

Then there are four cases.

1. \( i = j \) and \( x = x' \)
\[ i = j \quad \text{and} \quad y = y' \]
i.e., \( xy = x'y' \), \( i = j \)
i.e., \( q(xy,i) = q(x'y',j) = q(xy,j) \)
\[ = q(x'y',i) \]
1. \[q(xy, \min \{i,j\}) = q(x'y', \min \{i,j\})\]

i.e., \[q(x,i) q(y,j) = q(x',j) q(y',i)\]

2. \(i \neq j\) and \(x = x' \in \langle f(X) \rangle\)

\(i \neq j\) and \(y = y' \in \langle f(X) \rangle\)

Then \(xy = x'y' \in \langle f(X) \rangle\) (\(\therefore\) \(\langle f(X) \rangle\) is a subsemigroup)

i.e., \[q(xy,i) = q(x'y',j)\]

\(\therefore xy = x'y' \in \langle f(X) \rangle\)

\[= q(xy,j) = q(x'y',i)\]

3. \(i = j\) and \(x = x'\)

\(i \neq j\) and \(y = y' \in \langle f(X) \rangle\)

Then \(xy = x'y' \in \langle f(X) \rangle\) (\(\therefore\) \(\langle f(X) \rangle\) is an ideal)

i.e., \[q(xy,i) = q(x'y',i) = q(x'y',j) = q(xy,j)\]

i.e., \[q(xy, \min \{i,j\}) = q(x'y', \min \{i,j\})\]

\(\therefore q(x,i) q(y,j) = q(x',j) q(y',i)\)

4. \(i \neq j\) and \(x = x' \in \langle f(X) \rangle\)

\(i = j\) and \(y = y'\)

Then \(xy = x'y' \in \langle f(X) \rangle\)
and similarly we have
\[ q(x,i) q(y,j) = q(x',j) q(y',i) \]

Clearly multiplication is associative. Thus \( Z \) is a semigroup with multiplication continuous and \( q \) is a homomorphism.

\[ \therefore L_1 = qo_{i_1}oh_1 \quad \text{and} \quad L_2 = qo_{i_2}oh_2 \]

are continuous homomorphisms from \( Y \) into \( Z \).

Now if \( x \) is a point of \( X \), then the maps \( i_1oh_1 \) and \( i_2oh_2 \) split the point \( f(x) \) into two and is joined again by

\[ q : Y_1 \cup Y_2 \rightarrow Z \]

Thus we see that

\[ ((qo_{i_1}oh_1)of)(x) = ((qo_{i_2}oh_2)of)(x) \]

Hence

\[ (qo_{i_1}oh_1)of = (qo_{i_2}oh_2)of \]

However, any point lying outside of \( \langle f(X) \rangle \) in \( Y \) is split by \( i_1oh_1 \) and \( i_2oh_2 \), but is not joined again by \( q \).

Hence

\[ qo_{i_1}oh_1 \neq qo_{i_2}oh_2 \]
This would show that $f$ cannot be an epimorphism if we show that $Z$ is a topological semigroup. So it remains to show that the quotient space $Z$ is Hausdorff.

Let $p$ and $r$ be two distinct points of $Z$. We have to find two disjoint open sets containing $p$ and $r$ respectively. Then we have six cases.

**Case-1.**

$p, r \in \text{qoi}_{1,0} \left( Y \setminus <f(X)> \right)$.

Since $<f(X)>$ is closed, $Y \setminus <f(X)>$ is open

$\left( \text{qoi}_{1,0} \right)^{-1}(p), \left( \text{qoi}_{1,0} \right)^{-1}(r) \in Y \setminus <f(X)>$, there exists open sets $U_p$ and $U_r$ in $Y \setminus <f(X)>$ such that

$\left( \text{qoi}_{1,0} \right)^{-1}(p) \in U_p \subset Y \setminus <f(X)>$

$\left( \text{qoi}_{1,0} \right)^{-1}(r) \in U_r \subset Y \setminus <f(X)>$

Again since $Y$ is Hausdorff we get disjoint neighbourhoods $V_p$ and $V_r$ of $\left( \text{qoi}_{1,0} \right)^{-1}(p)$ and $\left( \text{qoi}_{1,0} \right)^{-1}(r)$ respectively. Thus the required neighbourhoods are

$\text{qoi}_{1,0}(U_p) \cap \text{qoi}_{1,0}(V_p)$ and

$\text{qoi}_{1,0}(U_r) \cap \text{qoi}_{1,0}(V_r)$.
Case-2

\( p, r \in \text{goi}_2 \text{oh}_2 (Y \setminus < f(X) >) \). This is the same as Case-1 with suffix changed.

Case-3

\( p \in \text{goi}_1 \text{oh}_1 (Y \setminus < f(X) >) \) and
\( r \in \text{goi}_2 \text{oh}_2 (Y \setminus < f(X) >) \)

Here the two given sets \( \text{goi}_1 \text{oh}_1 (Y \setminus < f(X) >) \), \( \text{goi}_2 \text{oh}_2 (Y \setminus < f(X) >) \) containing the points are already disjoint.

Case-4

\( p \in \text{goi}_1 \text{oh}_1 (Y \setminus < f(X) >) \) and
\( r = \text{goi}_1 \text{oh}_1 (y) \) for some \( y \in < f(X) > \)

Since \( Y \setminus < f(X) > \) is open, there exists open set \( U \) such that \( (\text{goi}_1 \text{oh}_1)^{-1}(p) \subset U \subset Y \setminus < f(X) > \).

Since \( Y \) is Hausdorff there exists disjoint open sets \( U \) and \( V \) with \( y \in V \),
\( p \in \text{goi}_1 \text{oh}_1 (U) \) and
\( r \in q[i_1 \text{oh}_1 (V) \cup i_2 \text{oh}_2 (V)] \)

are disjoint and open.
Case-5

If \( p \in qoi_{2}oh_{2}(Y \setminus \langle f(X) \rangle) \) and
\[ r = qoi_{1}oh_{1}(y) \text{ for some } y \in \langle f(X) \rangle \]
same as that of case-4 with suffix changed.

Case-6

\( p = qoi_{1}oh_{1}(x) \) and \( r = qoi_{1}oh_{1}(y) \)
where, \( x \neq y \in \langle f(X) \rangle \). Since \( Y \) is Hausdorff there exists disjoint neighbourhoods \( U \) and \( V \) such that \( x \in U, y \in V \). Then disjoint neighbourhoods of \( p \) and \( r \) in \( Z \) is given by

\[ q \left[ i_{1}oh_{1}(U) \cup i_{2}oh_{2}(U) \right] \]
and
\[ q \left[ i_{1}oh_{1}(V) \cup i_{2}oh_{2}(V) \right] \]

Hence \( Z \) is Hausdorff

\[ . \cdot \cdot \]
\( Z \) is a topological semigroup.

Notation.

TS- denotes the category of all topological semigroups.

From propositions (5.2.1) and (5.2.2) we obtain the following proposition as a particular case.
Proposition 5.2.3

If the images are ideals, the epimorphisms in the category TS are morphisms with dense range.

Proof.

Let \( f: X \to Y \) be a continuous homomorphism with \( f(X) \) an ideal and \( f(X) = Y \), then \( f \) is an epimorphism (proof is same as that of (5.2.1), since \( f(X) = \langle f(X) \rangle \) an ideal).

Conversely, let \( f: X \to Y \) be an epimorphism with \( f(X) \) an ideal, then \( f(X) = Y \) (proof is same as that of (5.2.2), since \( f(X) = \langle f(X) \rangle \) an ideal).

Note.

Proofs of proposition (5.2.1) and (5.2.2) are on the same lines as those of the corresponding results in the category of all Hausdorff spaces given by R.C. Walker [W].

5.3 Weak extremal monomorphisms in the Category TS

When a mapping is factored through the closure of its image, the second factor is a closed embedding. These maps also have a categorical characterization in the category of all Hausdorff spaces [W].
Definition 5.3.1

A monomorphism \( m \) is an extremal monomorphism if whenever \( m \) can be factored as illustrated

\[
\begin{array}{c}
Z \\
\downarrow^e \\
X \\
\downarrow^m \\
\end{array} \quad \begin{array}{c}
\downarrow^h \\
Y \\
\end{array}
\]

so that \( e \) is an epimorphism, then \( e \) is an isomorphism. In the diagram, the object \( X \) is said to be an extremal subobject of \( Y \). \[W\]

It is known that the extremal monomorphisms in the category of Hausdorff spaces are the closed embeddings and thus, the extremal subobjects are the closed subspaces. \[W\]

Next we define weak extremal monomorphism in the category \( TS \) of all topological semigroups.

Definition 5.3.2.

A monomorphism \( m^* \) is a weak extremal monomorphism if whenever \( m^* \) can be factored as illustrated so that \( e(X) \) is
an ideal and \( e \) is an epimorphism, then \( e \) is a topological isomorphism.

\[
\begin{array}{c}
Z \\
e \\
\downarrow \\
X \\
m^* \\
Y
\end{array}
\]

The object \( X \) is said to be a weak extremal subobject of \( Y \). We will show that if the images are ideals, the weak extremal monomorphisms in the category \( TS \) are the closed embeddings.

Proposition 5.3.3.

If the images are ideals, the weak extremal monomorphisms in the category \( TS \) are the closed embeddings.

Proof

We first show that a weak extremal monomorphism \( m^* : X \rightarrow Y \) with \( m^*[X] \) is an ideal is a closed embedding.

We can factor \( m^* : X \rightarrow Y \) through the closure of its image. Since \( e(X) \) is an ideal, and the range of \( e \) is dense, \( e \) is an epimorphism (5.2.1).
Then $e$ must be a topological isomorphism, since $m^*$ is a weak extremal monomorphism.

i.e., $X \xrightarrow{m^*} m^*(X)$ is a topological isomorphism, where $m^*(X)$ is a closed ideal of $Y$.

$\therefore m^*$ is a closed embedding.

On the other hand, let $m^* : X \rightarrow Y$ be a closed embedding. Assume that $m^* = hoe$ is a factorization of $m^*$, where $e$ is an epimorphism and $e(X)$ an ideal.

Claim: 1) $m^* : X \rightarrow Y$ is a monomorphism

2) $e$ is a topological isomorphism

Clearly $m^*$ is a monomorphism, because it is one-one homomorphism. Thus it remains to show that $'e'$ is a topological isomorphism.
We can also factor $m^*$ through its image thus obtaining the diagram.

We will show that the epimorphism $e$ is a topological isomorphism by obtaining a left inverse for $e$. Since $e(X)$ an ideal, and $e$ is an epimorphism $e$ has dense range. i.e., $\overline{e(X)} = C$ [5.2.2] and $m^*[X]$ is closed in $Y$ (given).

$$h[C] = h[\overline{e(X)}] \subseteq h[e(X)] = \overline{m^*[X]} = m^*[X]$$

$\therefore$ $h(C)$ is contained in $m^*[X]$.

Thus if we define $h' = C \longrightarrow m^*[X]$ by $h'(x) = h(x)$, we have that

$$h = ioh'$$

But then we also have

$$ioh'oe = hoe = ioa,$$

where $i$ is a monomorphism (since $i$ is a one-one homomorphism)

$\therefore$ $h'oe = a$
Since a is a topological isomorphism we have

\[ l_X = (a^{-1} \circ h') \circ e \]

Thus e is an epimorphism with a left inverse and is therefore a topological isomorphism.

Proposition 5.3.4

If the images are ideals, epireflective subcategories are closed under weak extremal subobjects.

Proof.

Let \( R^* \) be an epi-reflective subcategory of \( C^* \). Let \( Y \) belong to \( R^* \) and let \( m^* : X \to Y \) be a weak extremal monomorphism.

Since \( \eta_X \) is an epi-reflective functor, there exists \( f : rX \to Y \) such that

\[ f \eta_X = m^* \]

where \( \eta_X \) is an epimorphism \([5.1.4]\) and \( \eta_X(X) \) is an ideal, then \( \eta_X \) is a topological isomorphism (since \( m^* \) is a weak extremal monomorphism).

i.e. weak extremal subobject \( X \in R^* \)
Note

Proofs of propositions (5.3.3) and (5.3.4) are on the same lines as those of the corresponding results in the category of all Hausdorff spaces given by R.C.Walker[W].
BIBLIOGRAPHY

[A-H] L.W. Anderson and R.P. Hunter,
The H-equivalence in compact semigroups,
Bull. de l'A Soc. Math. de Belg.,
14 (1962a), 274-296.

[A-H₁] L.W. Anderson and R.P. Hunter,
On the compactification of certain semigroups,
Proc. Intn. Symp. on Extension Theory of

[B-B] J.W. Baker and R.J. Butcher,
The Stone–Čech Compactification of a

[BE] J.F. Berglund, On extending almost periodic

[BE-H] J.F. Berglund and K.H. Hofmann,
Compact Semitopological Semigroups and
weakly almost periodic functions,
Lecture Notes, Springer Verlag Series,
New York (1967).

Universal mapping properties of semigroup
compactifications, Semigroup Forum,
15(1978a), 375-386.
[BE-M]  J.F. Berglund and P. Milnes,  
Algebras of functions on semitopological  
left-groups, Trans. Amer. Math. Soc.,  
222(1976), 157-178.

Providence, (1967).

[BR-F₁]  D.R. Brown and M. Friedberg,  
Representation theorems for uniquely  
divisible semigroups, Duke Math. J.,  
35(1968), 341-352.

[BR-F₂]  D.R. Brown and M. Friedberg,  
A Survey of compact divisible commutative  
semigroups, Semigroup Forum,  
1(1970a), 143-161.

The Theory of Topological Semigroups-I,  

The theory of Topological Semigroups-II,  

[C-L]  J.H. Carruth and J.D. Lawson,  
On the existence of one-parameter semigroups,  


K. Deleeuw and I. Glicksberg,

K. Deleeuw and I. Glicksberg, Almost periodic functions on semigroups,


M. Friedberg, On compactifying semigroups, Semigroup Forum, 10(1975), 39-54.


[L] W. LaMatrin, On the foundations of K-group theory, Louisiana State Univ. at New Orleans manuscript.


