CHAPTER TWO
TORISON CLASSES AND MULTIPLICATIVE SETS

INTRODUCTION

For a ring $R$, there is a bi injection of $\text{Mod}^\tau R$ into the collection of all isomorphic classes of uniform injective right $R$-modules, given by

$$\tau \mapsto \left\{ M \in \text{Mod}^\tau R : E \cong \text{ECM}_E \text{ for some } \tau\text{-critical right } R\text{-module } M \right\}$$

This map is well-defined, since, if $M, M'$ are $\tau$-critical, then $\text{ECM}_M = \text{ECM}_{M'}$. If $R$ is right Noetherian, this map is a bijection [Gl]. This fact induces us to study localisation from the point of view of torsion theories. Since this approach seems to be promising, we have tried to generalise various results of Jategaonkar, Goodearl etc. to torsion classes.

In this chapter, we define, for a torsion theory $\tau$, a corresponding multiplicative set $C_\tau$ as the set of elements of
that generate $\tau$-dense right ideals. We see its connection with the multiplicative set $\mathcal{S}(P)$ of elements regular modulo a prime ideal $P$ and the set $\mathcal{N}(E)$ of elements that act regularly on an injective right $R$-module $E$. We obtain some results concerning the right Ore condition for these sets. We also see a new proof of the fact that every multiplicative set $S$ has a largest right Ore subset (i.e., one that contains every right Ore subset of $S$). Several of these results were published in [SC].

**The Multiplicative Set $C_T$**

**Proposition 2.1:** Let $\tau \in \mathcal{T} = \mathcal{R}-R$. Then the set

$$C_T = \{r \in R : R/rR \text{ is } \tau \text{-torsion in } R\}$$

is multiplicatively closed.

**Proof:** Clearly, $1 \in C_T$. If $r_1, r_2 \in C_T$, then $r_1 r_2 r_1 r_2 R$ is a homomorphic image of $R/r_2 R$, i.e., $r_1 r_2$ is $\tau$-dense in $r_1 R$. So $r_1 r_2 R$ is $\tau$-dense in $R$, i.e., $r_1 r_2 \in C_T$.

**Note:**

(i) If $\tau = \langle (0) \rangle$, the smallest torsion class, then $C_T = \langle 1 \rangle$.

(ii) $C_T = R$ if and only if $\tau = \mathcal{N}(R)$, the largest torsion class.
NOTE 2.2: Let $\mathcal{M}$ be the class of all multiplicative sets in $R$. Define $f: \mathcal{M} \to \text{Fro}-R$ with $f(C) = \rho_C$ for $C \in \mathcal{M}$ and $g: \text{Fro}-R \to M$ with $g(\tau) = C_\tau$ for $\tau \in \text{Fro}-R$. Then both $f$ and $g$ are order-preserving.

PROPOSITION 2.3: If $\tau \in \text{Fro}-R$, then $\rho_{C_\tau}$ is order-preserving.

PROOF: If $M$ is a right $R$-module which is $\rho_{C_\tau}$ torsion, then for every $x \in M$, there is $c \in R$ such that $R/cR$ is $\tau$-torsion and $xc = 0$. So $xR$ is a homomorphic image of $R/cR$. Hence, $xR$ is $\tau$-torsion for all $x \in M$, i.e., $M$ is $\tau$-torsion.

PROPOSITION 2.4: If $C$ is a multiplicative set, then $C \leq C_{\rho_C}$ if and only if $C$ is a right Ore set.

PROOF: By proposition 1.17 we know that $C$ is a right Ore set if and only if $R/cR$ is $\rho_C$-torsion for every $c \in C$, i.e., if and only if $c \in C$ for every $ce C$.

COROLLARY 2.5: If $C_\tau$ is a right Ore set for some $\tau \in \text{Fro}-R$, then $C_\tau = C_{\rho_C}$. 
PROOF: By proposition 2.4, $C_T \subseteq C_{\rho C_T}$. By proposition 2.3 and note 2.2, $C_{\rho C_T} \subseteq C_T$.

PROPOSITION 2.6: If $C$ is a right Ore set, then $C_{\rho C}$ is a right Ore set but the converse is not true.

PROOF: We have

$C_{\rho C} = \{ r \in R : R/rR \text{ is } \rho C \text{-torsion} \}$

$\quad = \{ r \in R : \text{given } s \in R, \text{ there is } c \in C \text{ such that } sc \in rR \}$

By proposition 2.4, $C \subseteq C_{\rho C}$. Hence, if $r \in C_{\rho C}$ and $s \in R$, there is $c \in C_{\rho C}$ such that $sc \in rR$.

To see that the converse is not true, let $k$ be a field, and let $R$ be the ring of $2 \times 2$ upper triangular matrices over $k$. Then $R$ is an Artinian ring with two prime ideals $P$ and $Q$, where

\[
R = \begin{bmatrix}
  k & k \\
  0 & k \\
\end{bmatrix}, \quad P = \begin{bmatrix}
  k & k \\
  0 & 0 \\
\end{bmatrix}, \quad \text{and } Q = \begin{bmatrix}
  0 & k \\
  0 & 0 \\
\end{bmatrix}
\]

Then $\mathfrak{S}(P) = R \setminus P$ and $\mathfrak{S}(Q) = R \setminus Q$.

We compute $C_{\rho \mathfrak{S}(P)}$.

1) For $a, b \in k, \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \rho \mathfrak{S}(P)$, for, taking $\begin{bmatrix} d & e \\ 0 & 1 \end{bmatrix} \in R$.,
if there is \([c_1 \ c_2] \in R\) and \([g_1 \ g_2] \in g(P) (g_3 \neq 0)\) such that

\[
\begin{bmatrix}
a & b \\
0 & c_g
\end{bmatrix}
\begin{bmatrix}
0 \\
c_g
\end{bmatrix} =
\begin{bmatrix}
d & e \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
g_1 \\
g_2
\end{bmatrix},
\]

\[
\begin{bmatrix}
ac_1 + bc_g \\
0
\end{bmatrix} =
\begin{bmatrix}
dg_1 \\
dg_2 + eg_g
\end{bmatrix}, \text{ i.e., } g_g = 0.
\]

which is a contradiction.

ii) If \(\alpha \in k\), then \(\begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix} \in g(C(P))\), for if \(\alpha = 0\), then clear.

If \(\alpha \neq 0\), take \(\begin{bmatrix} 0 & 1 \\ 0 & f \end{bmatrix} \in R\). If \(\begin{bmatrix} c_1 & c_2 \\ 0 & c_g \end{bmatrix} \in R\) and

\[
\begin{bmatrix}
g_1 \\
g_2
\end{bmatrix} \in g(P) (g_3 \neq 0)\) such that

\[
\begin{bmatrix}
0 & 0 \\
0 & \alpha
\end{bmatrix}
\begin{bmatrix}
c_1 & c_2 \\
0 & c_g
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
0 & f
\end{bmatrix}
\begin{bmatrix}
g_1 \\
g_2
\end{bmatrix},
\]

then \(\begin{bmatrix} 0 & 0 \\ 0 & ac_g \end{bmatrix} = \begin{bmatrix} 0 & g_g \\ 0 & fg_g \end{bmatrix}\), i.e., \(g_g = 0\), which is false.
iii) If $a, b \in k$, then $\begin{bmatrix} a \\ b \end{bmatrix} \not\in \mathcal{R}_{\mu(C(P))}$, for, if $a = 0$, then we have the proof by case (ii). If $a \neq 0$, consider

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in R.$$ If there is $\begin{bmatrix} c_1 & c_2 \\ 0 & c_3 \end{bmatrix} \in R$ and $\begin{bmatrix} \xi_1 & \xi_2 \\ 0 & \xi_3 \end{bmatrix} \in \mathcal{R}(P)$ $(\xi_3 \neq 0)$, such that

$$\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \begin{bmatrix} c_1 & c_2 \\ 0 & c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 & \xi_2 \\ 0 & \xi_3 \end{bmatrix},$$

then

$$\begin{bmatrix} 0 & ac_3 \\ 0 & bc_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \xi_3 \end{bmatrix}.$$ Thus $ac_3 = 0$ and $bc_3 = \xi_3$. Since $a \neq 0$, we have $c_3 = 0$. So $\xi_3 = 0$, which is a contradiction.

iv) If $a, b, c \in k$ such that $a \neq 0, c \neq 0$, then

$$\begin{bmatrix} a & b \\ c & 1 \end{bmatrix} \in \mathcal{C}_{\mu(C(P))},$$ for, given $\begin{bmatrix} e & f \\ 0 & g \end{bmatrix} \in R$, we have

$$\begin{bmatrix} a^{-1}e & a^{-1}(e+f-bc^{-1}g) \\ 0 & c^{-1}g \end{bmatrix} \in R$$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \mathcal{R}(P)$ such that

$$\begin{bmatrix} a & b \\ c & 1 \end{bmatrix} \begin{bmatrix} a^{-1}e & a^{-1}(e+f-bc^{-1}g) \\ 0 & c^{-1}g \end{bmatrix} = \begin{bmatrix} e & f \\ 0 & g \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

---(AD)
Cases (i), (ii), (iii), (iv) together cover all the elements of \( R \) and hence we get

\[
C_{\rho_{B(P)}} = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in k, a \neq 0, c \neq 0. \right\}
\]

\[
= B(P) \cap C(Q).
\]

Now by case (iv), \( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in C_{\rho_{B(P)}} \) and hence by equation (A), we see that \( C_{\rho_{B(P)}} \) is right Ore. But \( B(P) \) is not right Ore, since.

\[
\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in B(P) \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in R \quad \text{such that if} \quad \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in R
\]

and \( \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} \in B(P) \) (\( f \neq 0 \)), then

\[
\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}
\]

\[
\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = \begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix}
\]

whereas \( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = \begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix} \) and these two cannot be equal for \( f \neq 0 \).
NOTE: Following Stenstrom [S2], we say that a multiplicative set $C$ in a ring $R$ satisfies property SO if, for $a, b \in R$, $ab \in R$ implies $a \in C$.

PROPOSITION 2.7: If a multiplicative set $C$ in a ring $R$ satisfies SO, then $C_{PC}$.

PROOF: Let $r \in C$. Since $1 \in R$, there are $c \in C, d \in R$ such that $1.c = r.d$, i.e., $rd \in C$. By property SO, $r \in C$.

NOTE 2.8: If $R/P$ is a right Goldie ring, then $\mathfrak{C}(P)$ satisfies SO, for, let $ab \in \mathfrak{C}(P)$. If $\tilde{a} = a+P \in R/P$, then $\tilde{a} \tilde{b} \in \mathfrak{C}(O) \leq R/P$. Thus $\tilde{a} \tilde{b}$ is invertible in $\mathfrak{C}(R/P)$, i.e., $\tilde{a}\tilde{b}\mathfrak{C}(R/P) = \mathfrak{C}(R/P)$. Also $\tilde{a}\tilde{b}\mathfrak{C}(R/P) \leq \mathfrak{C}(R/P)$. Thus, $\tilde{a}\mathfrak{C}(R/P) = \mathfrak{C}(R/P)$, i.e., $\tilde{a}$ is invertible in $\mathfrak{C}(R/P)$, i.e., $a \in \mathfrak{C}(P)$.

NOTE 2.9: If $E$ is a right $R$-module, then the set $\mathcal{M}(E)$ defined as $\mathcal{M}(E)=\{r \in R: \text{ann}_Er = 0\}$ is a multiplicative set [G4]. The set $\mathcal{M}(E)$ satisfies SO, for, if $ab \in \mathcal{M}(E)$, then, for $x \in E$, if $xab = 0$, then $x = 0$. Now, if $xa = 0$, then $xab = 0$ and so $x = 0$, i.e., $a \in \mathcal{M}(E)$.

PROPOSITION 2.11: Let $C$ be a multiplicative set in a ring $R$. If $E$ is an injective right $R$-module, then $C\mathcal{M}(E) = \mathcal{M}(E)$. 
PROPOSITION 2.10: Let \( \pi = \chi(E) \), where \( E \) is an injective right \( R \)-module. Then

\[
\mathcal{N}(E) = \bigcap \{ R \setminus I : I \text{ is a right ideal of } R, I \neq R \text{ and } R/I \text{ is } n\text{-torsion-free} \}.
\]

PROOF: Denote the left hand side of the above expression by \( Y \). Let \( c \in \mathcal{N}(E) \), and let \( I \) be a right ideal of \( R, I \neq R \), such that \( R/I \) is \( n\)-torsion-free. Then \( \chi(E(R/I)) \geq \pi = \chi(E) \), i.e., \( E(R/I) \) can be embedded in say, \( E' \), a product of copies of \( E \). Then \( \mathcal{N}(E(R/I)) \geq \mathcal{N}(E') = \mathcal{N}(E) \). Hence \( c \in \mathcal{N}(E(R/I)) \). Now if \( c \in I \), then there is \( 1+I \in E(R/I) \) such that \( (1+I)c = c+I = 0 \). But since we have \( c \in \mathcal{N}(E(R/I)) \), this means \( 1+I = 0 \), which is false. So \( c \in R \setminus I \). Hence \( \mathcal{N}(E) \subseteq Y \).

Next, let \( c \in Y \), and \( 0 \neq x \in E \). Since \( xR \) is a submodule of \( E \), \( xR \) is \( n\)-torsion-free. So \( \text{ann } x \) is one of the \( I \)'s in the definition of \( Y \). Hence \( c \in R \setminus \text{ann } x \), i.e., \( xc \neq 0 \). Thus \( c \in \mathcal{N}(E) \), i.e., \( Y \subseteq \mathcal{N}(E) \). This completes the proof.

NOTE: By the above proposition, it is clear that if \( E, E' \) are injective right \( R \)-modules, with \( \chi(E) = \chi(E') \), then \( \mathcal{N}(E) = \mathcal{N}(E') \).

PROPOSITION 2.11: Let \( C \) be a multiplicative set in a ring \( R \). If \( E \) is an injective right \( R \)-module, then \( C\chi(E) = \mathcal{N}(E) \).
PROOF: Let \( r \in \chi(E) \), i.e., \( R/rR \) is \( \chi(E) \)-torsion. Let \( I \) be a proper right ideal of \( R \) such that \( R/I \) is \( \chi(E) \)-torsion-free. If \( r \in I \), then \( rR \) is a submodule of \( I \) and hence \( R/I \) is \( \chi(E) \)-torsion. Thus, \( R/I \) is \( \chi(E) \)-torsion-free and \( \chi(E) \)-torsion, which is false, since \( I \neq R \). Hence \( r \in R \setminus I \). By Proposition 8, \( r \in \mathcal{M}(E) \), i.e., \( \chi(E) \subseteq \mathcal{M}(E) \).

Next, let \( c \in \mathcal{M}(E) \). Suppose there is a homomorphism \( f: R/cR \to E \) such that \( f(1+cR) = \chi \) (say). Then \( \chi c = f(1+cR)c = 0 \). Since \( c \in \mathcal{M}(E) \), we get \( x = 0 \), i.e., \( f(1+cR) = 0 \). Hence \( f = 0 \). Thus \( R/cR \) is \( \chi(E) \)-torsion. Thus \( \mathcal{M}(E) \subseteq \chi(E) \).

NOTE 2.12: If \( P \) is a prime ideal in a ring \( R \), then \( \chi(R/P) \) is the largest of all \( P \)-principal points, for, let \( \pi \) be a \( P \)-principal point. Then, \( \chi(\pi) = P \). Now, for a two-sided ideal \( I \) of \( R \), \( R/I \) is \( \pi \)-torsion if and only if \( I \notin \chi(\pi) \). Hence \( R/P \) is not \( \pi \)-torsion. By [J, proposition 5.4.2], \( R/P \) is \( \pi \)-torsion-free, i.e., \( \chi(R/P) \geq \pi \).

PROPOSITION 2.13: If \( E \) is an injective right \( R \)-module over a right Noetherian ring \( R \), such that \( \chi(E) \) is a \( P \)-principal point for some prime ideal \( P \) of \( R \), then \( \mathcal{M}(E) \subseteq \chi(P) \).
PROOF: By note 2.12, \( \chi(R/P) \geq \chi(E) \). Hence \( \chi(E) \leq \chi(R/P) \).
So, \( \mathcal{M}(E) = \mathcal{M}(E(R/P)) = \chi(R/P) = \chi(R/P) \leq \mathfrak{m}(P) \), by proposition 2.11, proposition 2.7 and note 2.8.

COROLLARY 2.14: If \( R \) is a right Noetherian ring and \( E \) is a uniform injective with \( \text{ass } E = P \), then \( \mathcal{M}(E) \leq \mathfrak{m}(P) \).

PROOF: Since \( E \) is a uniform injective right \( R \)-module, we have \( \gamma(\chi(E)) = \text{ass } E \) by proposition 1.16.

COROLLARY 2.15: If \( R \) is a right Noetherian ring and \( P \) is a prime ideal in \( R \), then \( \mathcal{M}(E(R/P)) = \mathfrak{m}(P) \) if and only if \( \mathfrak{m}(P) \) is right Ore.

PROOF: Follows from propositions 2.4 and 2.13.

THE RIGHT ORE CONDITION ON \( \mathfrak{m}(P) \)

In the next few propositions, we see some situations where \( \mathfrak{m}(P) \) is right Ore (for a prime ideal \( P \)), using torsion classes.

PROPOSITION 2.16: If \( R \) is a right duo ring (i.e., a ring in which every right ideal is two sided), then \( \mathfrak{m}(P) \) is right Ore for every prime ideal \( P \) in \( R \).
PROOF: By assumption, if \( r \in R \), then \( rR = RrR \). Now if \( r \in C \), then and only if \( R/rR \) is \( \mathfrak{M}(P) \)-torsion if and only if \( R''rR \) is \( \mathfrak{M}(P) \)-torsion if and only if \( R''rR \) is \( \mathfrak{M}(P) \)-torsion if and only if \( R''rR \subseteq P \) if and only if \( r \in R \setminus P \). Since \( \mathfrak{M}(P) \subseteq R \setminus P \), we have \( \mathfrak{M}(P) \subseteq C \). So, by proposition 2.4, \( \mathfrak{M}(P) \) is right Ore.

**Proposition 2.17:** Let \( R \) be a right Noetherian ring and \( P \) be a prime ideal of \( R \). If \( (R/P) \) is injective, then \( \mathfrak{M}(P) \) is right Ore.

**Proof:** Since \( (R/P) \) is injective, we have

\[
\mathfrak{M}(R/P) = \mathfrak{M}(R/P) = \{ r \in R : \text{ann}_{R/P} r = 0 \}
\]

\[
= \{ r \in R : xr = 0 \Rightarrow x = 0 \text{ for any } x \in R/P \}
\]

\[
= \{ r \in R : sr \subseteq P \Rightarrow s \subseteq P \text{ for any } s \in R \}
\]

Thus, by proposition 2.11, \( \mathfrak{M}(P) \subseteq C \). By proposition 2.4, \( \mathfrak{M}(P) \) is right Ore.

**Corollary 2.18:** If \( R \) is semisimple Artinian and \( P \) is any prime ideal of \( R \), then \( \mathfrak{M}(P) \) is right Ore.

**Proof:** Over a semisimple Artinian ring, any module is injective.
COROLLARY 2.19: If \( R \) is semisimple Artinian, and \( E \) is any simple right \( R \)-module, then \( \mathcal{M}(E) = \mathfrak{m}(P) \), where \( P = \text{ass } E \).

PROOF: Since \( R \) is semisimple Artinian, \( E \) is tame and so \( E = E_P \). By corollary 2.18, \( \mathfrak{m}(P) \) is right Ore and so by corollary 2.15, we have \( \mathcal{M}(E) = \mathcal{M}(E_{R/P}) = \mathcal{M}(E) \).

NOTE 2.20: In a general right Noetherian ring \( R \), if \( E \) and \( E' \) are uniform injectives with \( \text{ass } E = \text{ass } E' \), then \( \mathcal{M}(E) \) need not be equal to \( \mathcal{M}(E') \), for, let \( E \) be a uniform injective right module over a simple right Noetherian ring \( R \). Then \( \mathfrak{m}(O) \) is right Ore and so by corollary 2.15, \( \mathfrak{m}(O) = \mathcal{M}(E) \).

Now, suppose \( c \in \mathfrak{m}(O) \) such that \( cR = R \). Then \( cR \) is an essential right ideal and so \( R/cR \) is torsion. Let \( E \) be a uniform submodule of \( R/cR \). Then \( E \) is torsion. Hence, given \( x \in E \), there is \( r \in \mathfrak{m}(O) \) such that \( xr = 0 \), i.e., \( r \in \mathcal{M}(E) \). So \( \mathfrak{m}(O) \notin \mathcal{M}(E) \).

THE LARGEST RIGHT ORE SUBSET OF A MULTIPLICATIVE SET

THEOREM 2.22: Let \( C \) be a multiplicative set in a ring \( R \).

So far, we have seen many situations when multiplicative sets of interest to us are right Ore. But we know that there are cases when sets are not right Ore. Now, given a
multiplicative set $C$ we give a new proof that there exists a right Ore set contained in $C$ which contains all right Ore subsets of $C$.

This fact has been known for a long time. A proof is given in [GW, Exercise 9F]. However, our proof will lead to a characterisation of this subset as an intersection of right cliques, as conjectured in [G41, in the case $C = CCP7$.

To prove the next theorem, we define, for any multiplicative subset $C$ of $R$, a sequence of subsets $C_\alpha$, for every ordinal $\alpha$. Let $C_0 = C$, $C_1 = C \cap C$, and for any successor ordinal $\alpha$, let $C_{\alpha+1} = (C_\alpha)^\rho_C$. For a limit ordinal $\alpha$, let $C_\alpha = \cap_{\beta < \alpha} C_\beta$.

Then the $C_\alpha$'s form a descending chain of multiplicative sets in $R$.

**Lemma 2.21:** If $T$ is a right Ore subset of a multiplicative set $C$, then $T \subseteq C_1$.

**Theorem 2.22:** Let $C$ be a multiplicative set in a ring $R$. Then $C$ has a right Ore subset which contains every right Ore subset of $C$.

**Proof:** The map $\alpha \mapsto C_\alpha$, from the class of ordinals to the power set of $C$, cannot be one-one since the ordinals do not
form a set. Hence for some $\alpha$, $C_\alpha = C_{\alpha+1} = \cap C_\beta$ is an ordinal number. By proposition 2.4, $C_\alpha$ is a right Ore set. By lemma 2.21, it contains every right Ore subset of $C$.

INTRODUCTION

Let $R$ be a right Noetherian ring and $\text{Spec } R$ denote the set of prime ideals of $R$. To consider regularity of an element of $R$ at different prime ideals, it is convenient to put a topology on $\text{Spec } R$. One such topology is the Patch topology introduced by Hochster in 1969. In 1999, Goodall defined the generic regularity condition for subsets of $\text{Spec } R$. and this helps us to clarify the discussion of various continuity results on $\text{Spec } R$.

In this chapter, we give an analogue of the Patch topology for prime torsion classes and discuss its properties. We also define the generic regularity condition for prime torsion classes. In the case of prime ideals this condition has an important role in the study of localisation. Though we study Patch topology and generic regularity condition on prime torsion classes for their own sake, we hope that they can be used in the torsion theoretic approach to localisation.