CHAPTER ONE

INTRODUCTION AND PRELIMINARIES

INTRODUCTION

In 1921, Emmy Noether proved that a commutative ring has the ascending chain condition on ideals if and only if all ideals are finitely generated. Such rings, now called commutative Noetherian rings, were extensively studied from the 1920s onwards because of their importance in algebraic geometry.

The noetherian condition is very natural in commutative ring theory, since it holds for the rings of integers in algebraic number fields and the co-ordinate rings crucial to algebraic geometry. The first important result in the theory of non-commutative Noetherian rings was Goldie's theorem (1958) which gives an analogue of the familiar result that every commutative domain can be embedded in its quotient field. Since then, Noetherian ring theory has steadily gathered strength, partly from its own impetus and partly through feedback from...
neighbouring areas in which Noetherian ideas found applications. By now, various methods and results from the theory of commutative Noetherian rings have been adapted to non-commutative Noetherian rings.

In commutative ring theory, we have the elementary but powerful technique of localisation at a prime ideal. If $R$ is a commutative ring and $P$ is a prime ideal in $R$, then the set $S = R \setminus P$ is multiplicatively closed, and the localisation of $R$ at $P$ is got by considering the set $R \times S$ and defining an equivalence relation $\sim$ on it by $(a,b) \sim (c,d)$ if and only if $(ad - bc) e = 0$ for some $e \in S$. This gives the ring of fractions $R_P$. This is the generalisation of the formulation of the field of fractions of a commutative integral domain (in that case, $S = R \setminus \{0\}$).

We can reduce questions on arbitrary rings and modules over such rings to the case of local rings via localisation at prime ideals. In many important instances, a result will be valid for a ring $R$, if it holds for every localised ring $R_P$ (where $P$ is a prime ideal in $R$). For a non-commutative ring, such a localisation is not, in general, possible, even at the zero ideal of an integral domain. Ore (1930) characterised those non-commutative domains which have right rings of
fractions that are division rings. For years, mathematicians worked to find a procedure which would enable one to localise non-commutative Noetherian rings at prime ideals. The standard procedure that emerged took the commutative situation and the situation in Goldie’s theorem as models and attempted to use Ore’s method to localise Noetherian rings at semiprime ideals.

In the 1970s and 1980s, Jategaonkar, Mueller and others worked on the problem of localisation at a prime ideal. They found that there exist "links" between prime ideals and that these links "obstruct" localisation. But in the case of Noetherian rings satisfying the "second layer condition", Jategaonkar has found that it is possible to describe localisation at a prime (or a collection of primes) under certain conditions.

Goodearl (1988) defined links between uniform injective right modules over a right Noetherian ring. He observed that links between "tame" injectives correspond to prime ideal links, while, there exist other injective module links which provide more obstructions to Ore localisations than prime ideal links do.
For a right Noetherian ring, there is a one-to-one correspondence between uniform injectives and prime torsion classes. Because of this connection, we have tried to study Ore localisation using the torsion-theoretic approach.

Before proceeding further, we take a look at the preliminary definitions and results required in the rest of the thesis.

PRELIMINARY DEFINITIONS AND RESULTS

Most of the material in this section is taken from [G4], [GW], [J], [MR] and [S1].

CONVENTIONS

All rings are assumed to be associative with 1 and all modules are unital. We denote the fact that \( M \) is a right \( R \)-module, by writing \( MR \). The set of all right \( R \)-modules is denoted by \( \text{Mod}-R \). We use the notations \( \leq, <, \notin \) for inclusions among submodules or ideals. In particular, if \( A \) is a module, the notation \( B \leq A \) means that \( B \) is a submodule of \( A \) and the notation \( B < A \) means that \( B \) is a proper submodule of \( A \). An ideal refers to a two-sided ideal. One sided ideals will be referred to as such. This convention applies to other one-sided properties also.
THE NOETHERIAN CONDITION

A collection \( \mathcal{A} \) of subsets of a set \( A \) satisfies the ascending chain condition (ACC) if there does not exist a properly ascending infinite chain \( A_1 < A_2 < \ldots \) of subsets from \( \mathcal{A} \). A set \( B \in \mathcal{A} \) is said to be maximal in \( \mathcal{A} \), if there does not exist a set in \( \mathcal{A} \) which properly contains \( B \).

PROPOSITION 1.1: Let \( R \) be a ring and \( A \) be a right \( R \)-module. The following conditions are equivalent:

a) \( A \) has ACC on submodules.

b) Every non-empty family of submodules of \( A \) has a maximal element.

c) Every submodule of \( A \) is finitely generated.

A right \( R \)-module \( A \) is said to be Noetherian if and only if the above equivalent conditions are satisfied. A ring \( R \) is said to be completely prime if \( R \) is a prime ring. A ring \( R \) is Noetherian if it is both right and left Noetherian.

If \( B \) is a submodule of \( A \), then \( A \) is Noetherian if and only if \( B \) and \( A/B \) are Noetherian. Any finite direct sum of
Noetherian modules is Noetherian. If $R$ is a Noetherian ring, then all finitely generated right $R$-modules are Noetherian.

**PRIME IDEALS**

A proper ideal $P$ in a commutative ring $R$ is said to be prime if whenever we have two elements $a$ and $b$ of $R$ such that $ab \in P$, it follows that $a \in P$ or $b \in P$, equivalently, $P$ is a prime ideal if and only if the factor ring $R/P$ is a domain. We need a non-commutative analogue of a prime ideal. An ideal $P$ in a ring $R$ is said to be completely prime if $R/P$ is an integral domain. Thus, if $R$ is commutative, $P$ is prime if and only if it is completely prime.

There are non-commutative rings, however, in which there are not many completely prime ideals, and sometimes none. For example, in a simple Artinian ring, the only proper ideal is the zero ideal. Also, we would like every maximal ideal to be prime. The following definition, proposed by Krull in 1928, satisfies this property, and reduces to the familiar one in commutative rings: $P$ is prime if for any ideals $I$ and $J$, $IJ \subseteq P \Rightarrow I \subseteq P$ or $J \subseteq P$. The set of prime ideals of $R$ is denoted by $\mathfrak{p} \text{ac} R$. If $O$ is a prime ideal, we say that $R$ is a prime ring. If $O$ is a completely prime ideal, $R$ is a domain.
PROPOSITION 1.2: For a proper ideal $P$ in a ring $R$, the following are equivalent:

a) $P$ is a prime ideal.

b) $R/P$ is a prime ring.

c) If $x, y \in R$ with $xRy \subseteq P$, either $x \in P$ or $y \in P$.

d) If $I$ and $J$ are any right ideals of $R$ such that $IJ \subseteq P$, either $I \subseteq P$ or $J \subseteq P$.

e) If $I$ and $J$ are any left ideals of $R$ such that $IJ \subseteq P$, either $I \subseteq P$ or $J \subseteq P$.

f) If $I, J$ are right ideals of $R$, such that $I \cap J \subseteq P$, then either $I \subseteq P$ or $J \subseteq P$. This is a lattice theoretic condition.

It immediately follows that if $P$ is a prime ideal in a ring $R$ and $J_1, \ldots, J_n$ are right ideals of $R$ such that $\prod_{i=1}^{n} J_i \subseteq P$, then some $J_i \subseteq P$.

By a maximal ideal in a ring, we mean an ideal which is a maximal element in the collection of proper ideals. Then, every maximal ideal $M$ of a ring $R$ is a prime ideal.

SEMIPRIME IDEALS

A semiprime ideal in a ring $R$ is any ideal of $R$ which is an intersection of prime ideals.
PROPOSITION 1.3: For an ideal $I$ in a ring $R$, the following are equivalent:

a) $I$ is a semiprime ideal.

b) If $J$ is any ideal of $R$ such that $J^2 \subseteq I$, then $J \subseteq I$.

c) If $x \in R$ with $xRx \subseteq I$, then $x \in I$.

A semiprime ring is any ring in which $0$ is a semiprime ideal. The prime radical of a ring $R$ is the intersection of all the prime ideals of $R$. A ring $R$ is semiprime if and only if its prime radical is zero. In any ring $R$, the prime radical equals the intersection of the minimal prime ideals of $R$.

ANNIHILATORS

If $M$ is a right $R$-module, the annihilator of $M$, written $\text{ann}_R M$ (or $\text{ann}_R M^0$) is the set $\{r \in R : mr = 0 \text{ for all } m \in M\}$. If $M$ is a right $R$-module and $S$ is a subset of $R$, then the annihilator of $S$ in $M$, written $\text{ann}_R S$ is $\{x \in M : xS = 0\}$. If $S$ is a left ideal of $R$, then $\text{ann}_R S$ is a submodule of $M$. If $N$ is any subset of $M$, the annihilator of $N$ is $\text{ann} N = \{r \in R : Nr = 0\}$. $\text{Ann} N$ is a right ideal of $R$, and if $N$ is a submodule of $M$, then $\text{ann} N$ is a two-sided ideal. In particular, this defines the right annihilator $r\text{-ann} S$ of a subset $S$ of $R$:

$r\text{-ann} S = \{r \in R : sr = 0 \ \forall s \in S\}$. The left annihilator $l\text{-ann} S$ of $S$ is defined similarly.
A right $R$-module $M$ is said to be faithful if $\text{ann} M = 0$. $M$ is fully faithful if $\text{ann} N = 0$ for every non-zero submodule $N$ of $M$.

**Essential Submodules**

A submodule $M'$ of $M$ is said to be essential in $M$, denoted $M' \leq M$, if $N \neq 0 \Rightarrow N \cap M' \neq 0$ for any submodule $N$ of $M$. If $M' \leq M$, then $M$ is called an essential extension of $M'$. If $R$ is considered as a right (or left) $R$-module, we obtain essential right (or left) ideals. A module $M$ is uniform if all its non-zero submodules are essential.

A ring $R$ is right bounded if every essential right ideal of $R$ contains an ideal which is essential as a right ideal. A ring $R$ is right fully bounded if every prime factor ring of $R$ is right bounded. A right (left) FBN ring is any right (left) fully bounded right (left) Noetherian ring. An FBN ring is any right and left FBN ring.

**Assassinators and Primary Modules**

Let $R$ be a right Noetherian ring, and let $V$ be a uniform right $R$-module. Then the set of the annihilator ideals of non-zero submodules of $V$ has a unique largest member, say $P$. 
Then $P$ is a prime ideal of $R$ and is called the assassinator of $V$, denoted $\text{ass } V$. For any non-zero submodule $W$ of $V$, we have $\text{ass } W = P$. Moreover, setting $W = \text{ann}_V P$, we have $W = 0$ and $\text{ass } W = \text{ann}_V W = P$.

For an arbitrary right $R$-module, the set 

$$\{ \text{ass } V : V \text{ is a uniform submodule of } M \}$$

is called the assassinator of $M$, and is denoted as $\text{ass } M$. The members of $\text{ass } M$ are often referred to as the assassinator prime ideals of $M$.

A non-zero right module $M$ over a right Noetherian ring is called a primary module if $\text{ass } M$ is a singleton set. If $P$ is the sole member of $\text{ass } M$, the module $M$ is called a $P$-primary module. For any prime ideal $P$ in a right Noetherian ring $R$, the class of all $P$-primary modules is closed under non-zero submodules, essential extensions and arbitrary direct sums.

Let $S$ be a semiprime ideal in a right Noetherian ring $R$. A right $R$-module $M$ is called an $S$-primary module if $\text{ass } M \subseteq \text{ass}(R/S)$.

INJECTIVE MODULES

A right $R$-module $A$ is injective provided that for any right
R-module $B$ and any submodule $C$ of $B$, all homomorphisms $C \to A$ extend to homomorphisms $B \to A$. Given $M \in \text{Mod-}R$ is called an injective envelope (injective hull) of $M$, if $E$ is a minimal injective module containing $M$. Alternatively, an injective hull for $M$ turns out to be a maximal essential extension of $A$.

A right $R$-module $A$ is said to be simple if $A$ has no proper submodules. A ring $R$ is simple if it has no proper ideals.

A right $R$-module $A$ is the sum of all simple submodules of $A$ and is denoted by $\text{soc} \, A$. This is the direct sum of all simple submodules of $A$. A is semisimple if $A$ is the direct sum of all simple submodules of any module $A$.

PROPOSITION 1.4:

i) Every module has an injective envelope, unique up to isomorphism and denoted by $\text{EC}(M)$.

ii) A right $R$-module $M$ is injective if and only if $M = \text{EC}(M)$.

iii) If $M \leq N$, then $\text{EC}(M) = \text{EC}(N)$.

iv) If $M$ is injective and $M \leq N$, then $M$ is a direct summand of $N$.

v) If $e \in \text{EC}(M)$ is injective, (for instance, if $A$ is finite),
then $\text{EC}(e \alpha) = \text{EC}(\alpha e \text{EC}(M))$.

vi) Direct products and direct summands of injective modules are injective.

vii) A non-zero module $M$ is uniform if and only if $\text{EC}(M)$ is indecomposable.

viii) If $E$ is an indecomposable injective module, then $E$ is the injective hull of every non-zero submodule of $E$.
If $M, M'$ are right $R$-modules such that $E(M) = E(M')$, we say that $M$ and $M'$ are similar.

SIMPLE AND SEMISIMPLE MODULES

A right $R$-module $A$ is said to be simple if $A$ has no proper submodules. A ring $R$ is simple if it has no proper ideals. The socle of a right $R$-module $A$ is the sum of all simple submodules of $A$ and is denoted by $\text{soc } A$. This is the direct sum of some simple submodules of $A$. $A$ is semisimple if $A = \text{soc } A$ if and only if $A$ is a direct summand of any module containing it.

ARTINIAN MODULES

A module $A$ is Artinian if $A$ satisfies the descending chain condition (DCC) on submodules, i.e., there does not exist a properly descending infinite chain of submodules of $A$. A ring $R$ is called right (left) Artinian if the right $R$-module $R$ (left $R$-module $R$) is Artinian. If both conditions hold, $R$ is called an Artinian ring. A right $R$-module $A$ is Artinian if and only if $A/B$ and $B$ are Artinian where $B$ is a submodule of $A$. Any finite direct sum of Artinian modules is Artinian. If $R$ is a right Artinian ring, all finitely generated right
R-modules are Artinian. If R is a right Artinian ring, then R is also right Noetherian. If R is a non-zero right or left Artinian ring, then all prime ideals in R are maximal.

**SEMI SIMPLE ARTINIAN RINGS**

In a ring R, the following sets coincide:

a) The intersection of all maximal right ideals.

b) The intersection of all maximal left ideals.

This intersection is called the *Jacobson radical* $J(R)$ of R.

**PROPOSITION 1.5:** For any ring R, the following conditions are equivalent:

a) R is right Artinian and semiprime.

b) R is left Artinian and semiprime.

c) All right R-modules are semisimple.

d) All left R-modules are semisimple.

e) $R_R$ is semisimple.

f) $R_R$ is semisimple.

g) R is right Artinian and $J(R) = 0$.

h) R is left Artinian and $J(R) = 0$.

i) All right R-modules are injective.

j) All left R-modules are injective.

k) $R = M_{1}^{n_1} (D_1) \times M_{2}^{n_2} (D_2) \times \ldots \times M_{k}^{n_k} (D_k)$ for some positive integers $n_1, n_2, \ldots, n_k$ and division rings $D_1, \ldots, D_k$.

Although the product of non-zero elements is a multiplicative set
A ring satisfying the above conditions is called a semisimple Artinian ring.

PROPOSITION 1.6: For a ring $R$, the following conditions are equivalent:

a) $R$ is prime and right Artinian.

b) $R$ is prime and left Artinian.

c) $R$ is simple and right Artinian.

d) $R$ is simple and left Artinian.

e) $R$ is simple and semisimple Artinian.

f) $R \cong M_n(D)$ for some positive integer $n$ and some division ring $D$.

The rings characterised above are referred to as simple Artinian rings.

RINGS OF FRACTIONS

In the theory of commutative rings, localisation plays a very important role. Most basic is the idea of a quotient field, without which one cannot imagine studying integral domains. Next comes the idea of localisation at a prime ideal, which reduces many problems to the study of local rings and their maximal ideals.

However, this is not the case with non-commutative rings. Although the set of non-zero elements is a multiplicative set...
in any domain, we have examples of domains which do not possess a division ring of quotients. It was in 1930, that Ore characterised those non-commutative domains which possess division rings of fractions. In 1952, Gabriel gave the necessary condition for a multiplicative set in a ring to have a right (left) ring of fractions.

A subset $C$ of a ring $R$ is a multiplicatively closed set if $1 \in C$ and $c_1, c_2 \in C \Rightarrow c_1 c_2 \in C$. A multiplicatively closed subset $C$ of $R$ is a right (left) Ore set if, given $r \in R$, $c \in C$, there exist $s \in R$ and $d \in C$ such that $rd = cs$ $(dr = sc)$. If $C$ is a right and left Ore set, it is called an Ore set. $C$ is a right reversible set if $r \in R$, $c \in C$ with $cr = 0$ in $R$ implies $rd = 0$ for some $d \in C$. A right Ore, right reversible set is called a right denominator set. In a right Noetherian ring, every right Ore set is right reversible.

Let $C$ be a multiplicative set in a ring $R$. A right quotient ring (or a right ring of fractions or right Ore localisation) of $R$ relative to $C$ is a pair $(Q, f)$, where $Q$ is a ring and $f$ is a ring homomorphism from $R$ to $Q$ such that

a) $f(C)$ is a unit of $Q$ for all $c \in C$.

b) Each element of $Q$ has the form $f(r)f(c)^{-1}$ for some $r \in R$, $c \in C$.

c) $\ker f = \{ r \in R : rc = 0$ for some $c \in C \}$. 
By abuse of notation, we usually refer to $Q$ as the right ring of fractions and we write elements of $Q$ in the form $rc^{-1}$ for $r \in R$, $c \in C$.

**Theorem 1.7:** Let $C$ be a multiplicative set in a ring $R$. Then there exists a right ring of fractions for $R$ with respect to $C$ if and only if $C$ is a right denominator set.

If $C$ is the set of regular elements of $R$ and if the right quotient ring $QR$ of $R$ relative to $C$ exists, we say that $R$ is a right order in $QR$.

A ring $R$ is a domain if it has no zero divisors. The non-zero elements in a domain form a multiplicative set and if $C = R \setminus \{0\}$, then we have the following corollary to the above theorem:

**Corollary 1.8:** A domain $R$ has a right division ring of fractions if and only if $C$ is a right Ore set if and only if the intersection of any two non-zero right ideals is non-zero.

A domain which satisfies this condition is called a right Ore domain.
GOLDIE'S THEOREMS

A very useful technique in commutative ring theory is to pass from a commutative ring $R$ to a prime factor ring $R/P$. In the non-commutative case we could ask whether it is possible to pass to a factor ring from which a division ring may be built from fractions. Since non-commutative Noetherian rings need not have any factor rings which are domains, this is rather restrictive. Instead we look for factor rings from which simple artinian rings can be built using fractions. The main result is Goldie's theorem which says that if $P$ is a prime ideal in a noetherian ring, then the factor $R/P$ has a ring of fractions. It turns out to be no extra work to investigate rings from which semisimple rings of fractions can be built.

A regular element in a ring $R$ is any non-zero-divisor, i.e., any element $x \in R$ such that $r\text{-ann}(x) = 0$ and $l\text{-ann}(x) = 0$.

A right(left) annihilator in a ring $R$ is any right(left) ideal of $R$ which equals the right(left) annihilator of some subset of $R$.

We say that a right $R$-module $M$ has finite Goldie dimension if $M$ does not contain a direct sum of an infinite number of
non-zero submodules. A ring $R$ is said to have finite right Goldie dimension if $R$ has finite Goldie dimension as a right $R$-module.

PROPOSITION 1.9: If $M$ has finite Goldie dimension, then there is a largest positive integer $r$ such that $M$ contains a direct sum of $n$ non-zero submodules. This is called the Goldie dimension of $M$.

A right Goldie ring is any ring $R$ that has finite right Goldie dimension and ACC on annihilators. For example, every right Noetherian ring is right Goldie.

PROPOSITION 1.10 (Goldie): Let $R$ be a semiprime right Goldie ring, and let $I$ be a right ideal of $R$. Then $I$ is an essential right ideal if and only if $I$ contains a regular element.

THEOREM 1.11 (Goldie): A ring $R$ is a right order in a semisimple ring if and only if $R$ is a semiprime right Goldie ring.

THEOREM 1.12 (Goldie, Lesieur-Croisot): A ring $R$ is a right order in a simple artinian ring if and only if $R$ is a prime right Goldie ring.
Let $R$ be a semiprime right Goldie ring. Any semisimple ring $Q$ in which $R$ is a right order is called a right Goldie quotient ring of $R$. An important property of $Q_R$ is that it is an injective hull of $R_R$.

**TORSION CLASSES**

It is often convenient to think of localisation in the broader context of torsion classes. We can characterise the right Ore condition on a multiplicative set in terms of the associated torsion class. In this subsection we define right torsion classes and other torsion theoretic terms which we use later.

A right torsion class $\sigma$ for a ring $R$ is a non-empty class of right $R$-modules satisfying the following two conditions:

i) The direct sum of any family of modules in $\sigma$ is also in $\sigma$.

ii) For any exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ of right $R$-modules, $M$ belongs to $\sigma$ if and only if $M'$ and $M''$ both belong to $\sigma$.

It follows that a torsion class is closed under submodules and homomorphic images. The set of all torsion classes over $R$ is denoted by $\mathcal{T}_{\text{tors}}R$. Over a commutative domain,
the modules which are torsion in the usual sense form a right
torsion class.

We define the notions that are usually associated with
'torsion'. Let \( \alpha \) be a right torsion class for a ring \( R \). For
any right \( R \)-module \( H \), the unique largest submodule of \( H \)
belonging to \( \alpha \) is called the \( \alpha \)-torsion submodule of \( H \) and is
denoted as \( \alpha H \). \( H \) is called \( \alpha \)-torsion if \( \alpha H = H \)
and a \( \alpha \)-torsion-free module if \( \alpha H = 0 \). The class of
\( \alpha \)-torsion-free modules is closed under submodules, injective
hulls, direct products and isomorphic copies. Let \( N \) be a
submodule of \( H \). Then \( N \) is said to be \( \alpha \)-dense in \( H \) if \( H/N \) is
\( \alpha \)-torsion and \( \alpha \)-closed in \( H \) if \( H/N \) is \( \alpha \)-torsion-free. A
\( \alpha \)-dense (\( \alpha \)-closed) submodule of \( R_R \) is called a \( \alpha \)-dense
(\( \alpha \)-closed) right ideal of \( R \).

A module \( B \in \text{Mod-} R \) is \( \alpha \)-torsion-free if and only if
\( \text{Hom}(A,B) = 0 \) or every \( \alpha \)-torsion module \( A \in \text{Mod-} R \). A module
\( A \in \text{Mod-} R \) is \( \alpha \)-torsion if and only if \( \text{Hom}(A,B) = 0 \)
for every \( \alpha \)-torsion-free module \( B \in \text{Mod-} R \).

The set \( \mathcal{J}_\text{tor-} R \) is partially ordered under inclusion. Under
this partial order, \( \mathcal{J}_\text{tor-} R \) is a complete lattice in which
meet and join of any collection of torsion classes exist.
Given $A \in \text{Mod-}R$, the torsion class $\chi(A)$ is the greatest torsion class $\sigma$ such that $A$ is $\sigma$-torsion-free.

**Proposition 1.13:**

(a) A right $R$-module $B$ is $\chi(A)$-torsion if and only if $\text{Hom}(C, A) = 0$ for all submodules $C$ of $B$ if and only if $\text{Hom}(B, \text{E}(A)) = 0$, where $\text{E}(A)$ is the injective envelope of $A$.

(b) $\chi(0)$ is the largest element of $\text{Tor}_0-R$.

(c) If $A \in \text{Mod-}R$, and $B \leq A$, then $\chi(A) = \chi(B)$.

(d) For every $\sigma \in \text{Tor}_0-R$, there is an injective module $E$ with $\sigma = \chi(E)$.

(e) If $E, E'$ are injective right $R$-modules, then $\chi(E) \geq \chi(E')$ if and only if $E'$ can be embedded in a product of copies of $E$.

Corresponding to the notion of a prime ideal in the lattice of two-sided ideals of a ring (a prime ideal is $\pi$-irreducible by proposition 1.2), Simmons [S1] has defined a prime element in the lattice of torsion classes.

A point of $\text{Tor}_0-R$ is a $\lambda$-irreducible element, i.e., an element $\pi \in \text{Tor}_0-R$ such that $\pi \neq \chi(0)$ and $\sigma \land \tau \leq \pi$ implies $\sigma \leq \pi$ or $\tau \leq \pi$ for each $\sigma, \tau \in \text{Tor}_0-R$. $\text{PI-}R$ denotes the set of points of $\text{Tor}_0-R$. 

EXAMPLE 1.14: \( \chi(A) \) is a point for each uniform module \( A \) over \( R \), and \( \chi(R/P) \) is a point for each prime ideal \( P \) of \( R \).

Let \( R \) be a right Noetherian ring. If \( \tau \in \text{Spec} \) and \( \tau \in \text{Spec} \) for some critical right \( R \)-module \( N \), then we say that \( \tau \) is prime \([G3]\). For example, if \( M \) is a simple right \( R \)-module, then \( \chi(M) \) is prime. The set of all prime torsion classes of \( \text{Spec} \) is denoted by \( \text{Spec} \). Every prime torsion class is a point. In a right Noetherian ring \( R \), every point is a prime and hence \( \text{Spec} = \text{Spec} \). The map \( \phi : \text{Spec} \rightarrow \text{Spec} \) is an injection, where \( \phi(P) = \chi(R/P) \) for \( P \in \text{Spec} \).

If \( R \) is a commutative Noetherian ring or an FBN ring, then \( \text{Spec} = \text{Spec} \) for every \( P \in \text{Spec} \).

PROPOSITION 1.15:

(i) If \( \sigma \in \text{Spec} \), then there is a uniform injective right \( R \)-module \( E \) such that \( \sigma = \chi(E) \).
(ii) If \( \sigma \in \text{on-}R \) and \( M, M' \) are \( \sigma \)-critical uniform injectives, then \( \text{EC}(M) = \text{EC}(M') \).

(iii) Let \( R \) be a right Noetherian ring. If \( \tau \in \text{Two-}R \) and \( \tau \neq \chi(O) \), then
\[
\tau = \bigwedge \{ \chi(M) : M \text{ is a } \tau \text{-critical right } R\text{-module} \}.
\]

A point \( \pi \) is a principal point if there is an ideal \( Q \) such that if \( I \) is a two-sided ideal of \( R \), then \( I \) is \( \pi \)-dense if and only if \( I \notin Q \). Then \( Q \) is the union of all the ideals of \( R \) that are not \( \pi \)-dense, and is \( Q \) is prime. We write \( Q = \wp(\pi) \) and say that \( \pi \) is \( Q \)-principal. Every prime torsion class is a principal point.

**Proposition 1.16:**

(i) If \( E \) is a uniform injective right \( R \)-module, then,
\[
\wp(\chi(E)) = \text{ass } E.
\]

(ii) If \( \pi \) is a principal point and \( I \) is a two-sided ideal of \( R \), then \( R/I \) is \( \pi \)-torsion if and only if \( I \notin \wp(\pi) \).

C-TORSION AND C-TORSION-FREE MODULES

Given a multiplicative set \( C \) in a ring \( R \), there is a torsion class \( \rho_C \) associated with it: A right \( R \)-module \( M \) is said to be \( \rho_C \)-torsion (or \( C \)-torsion) if, for every \( m \in M \), there is \( c \in C \)
such that \( mc = 0 \). \( H \) is \( \rho_C \)-torsion-free if \( \rho_C(M) = 0 \), where \( \rho_C(M) \) is the \( \rho_C \)-torsion submodule of \( H \). If \( C \) is a right Ore set, then \( H \) is \( \rho_C \)-torsion-free if and only if, given \( m \in H \), there is no \( c \in C \) such that \( mc = 0 \).

The right Ore condition on \( C \) can be characterised in terms of \( C \), as follows:

**Proposition 1.17:** A multiplicative set \( C \) in a ring \( R \) is right Ore if and only if \( R/cR \) is a \( \rho_C \)-torsion module for every \( c \in C \) if and only if, for any \( m \in H \), there is no \( c \in C \) such that \( mc = 0 \) for some \( c \in C \).

If \( C \) is the set of regular elements of \( R \), we use the term 'torsion' for '\( C \)-torsion' and 'torsion-free' for '\( C \)-torsion-free'.

For any ideal \( I \) of \( R \), we denote by \( \mathfrak{C}(I) \), the multiplicative set of elements of \( R \) that are regular modulo \( I \), i.e.,

\[
\mathfrak{C}(I) = \{ r \in R : r+I \text{ is regular in } R/I \}.
\]

**Proposition 1.18 (LM):** If \( R \) is a right Noetherian ring, then \( \rho_S(S) = \mathfrak{C}(R/S) \) for any semiprime ideal \( S \) of \( R \).
THE UNIFORM INJECTIVE MODULE $E_P$

Let $P$ be a prime ideal in a right Noetherian ring $R$. We use the notation $E_P$ to denote the right $R$-injective hull of a uniform right ideal of $R/P$. Up to an $R$-isomorphism, the indecomposable right $R$-injective module $E_P$ is uniquely determined by $P$. If $n$ denotes the Goldie dimension of $R/P$, then $E(R/P) \cong E_P^n$. Then $\text{ass}(E_P) = \text{ass}(E(R/P)) = P$ and $\chi(R/P) = \chi(E(R/P)) = \chi(E_P)$.

TAME MODULES AND WILD MODULES

Let $V$ be a uniform right module over a right Noetherian ring $R$. Set $P = \text{ass } V$, $W = \text{ann}_R P$, and $R' = R/P$. Then $P$ is a prime ideal of $R$, and the uniform right $R'$-module $W$ has no non-zero unfaithful submodules. Moreover, as a module over the prime right Noetherian ring $R'$, $W$ is either a torsion module or a torsion-free module but not both.

If the $R'$-module $W$ is torsion then we call the $R$-module $V$ a wild module or a $P$-wild module, if we wish to convey that $P$ is the assassinator of $V$. If the $R'$-module $W$ is torsion-free then we call the $R$-module $V$ a tame module or a $P$-tame module. $W$ is torsion-free over $R' \leftrightarrow E(W)_R$ is a direct summand of
Hence, a uniform right $R$-module $V$ over a right Noetherian ring is $P$-tame if and only if $E(V) \cong E_P$. Thus a $P$-tame uniform module is uniquely determined by $P$ up to similarity.

**EXAMPLE 1.19:** Uniform modules over commutative Noetherian rings and over right Artinian rings are tame. A uniform module over a simple Noetherian ring is tame if and only if it is torsion-free.

**A SUMMARY OF THE THESIS**

In this thesis, we study Ore localisation and related ideas from the point of view of torsion classes. Hence we have tried to get torsion-theoretic versions of various definitions and results of Jategaonkar, Goodearl etc.. In the case of commutative rings, for a prime ideal $P$, the set $R \setminus P$ is a right Ore multiplicative set. The localisation of $R$ at $P$, which is the localisation of $R$ at the set $R \setminus P$, always exists. If $R$ is not commutative, then $R \setminus P$ is not necessarily a multiplicative set. The counterpart of $R \setminus P$ in this case is

$$\mathcal{V}(P) = \{ r \in R : r+P \in R/P \text{ is regular} \},$$

which is a multiplicative set and is equal to $R \setminus P$ if $R$ is
commutative. The localisation of $R$ at $\mathfrak{m}(P)$, called the
localisation of $R$ at $P$ exists if and only if $\mathfrak{m}(P)$ is a right
denominator set. Hence it is important to find when $\mathfrak{m}(P)$ is
right Ore. In [64] Goodearl considers, for a right module $E$
over a right Noetherian ring $R$, the multiplicative set
\[ \mathfrak{m}(E) = \{ r \in R : \text{ann}_E r = 0 \}. \]
By [J, proposition 3.1.4], if $R$ is a prime ideal in a right
Noetherian ring $R$, then $\mathfrak{m}(P)$ is right Ore if and only if
$\mathfrak{m}(P) \subseteq \mathfrak{m}(E(P))$. In Chapter 2, we get a generalisation
of this result for an arbitrary multiplicative set $C$, by
defining, for a torsion class $\tau \in \mathfrak{m}(R)$, a multiplicative
set
\[ C_\tau = \{ r \in R : R/\tau R \text{ is } \tau\text{-torsion} \}. \]
Then $C$ is right Ore if and only if $C \subseteq C_\tau$. We see that if $E$
is an injective right $R$-module, then $C_\mathfrak{m}(E) = \mathfrak{m}(E)$. Using
torsion classes, we get some situations when $\mathfrak{m}(P)$ is right
Ore, for $P \in \mathfrak{m}(R)$.

Given a multiplicative set $C$ in a ring $R$, it is known that
there is a right Ore set contained in $C$, which contains all
right Ore sets contained in $C$. Using torsion classes, we
construct this largest right Ore subset.

Let $R$ be a right Noetherian ring. To study the regularity of
an element of $R$ at different prime ideals, it is convenient
to put a topology on \( \text{Spec} \, R \). Two such topologies are the Zariski topology and the patch topology. In [Sl], Simmons has generalised the Zariski topology to prime torsion theories. In the case of prime ideals, the patch topology and the "generic regularity condition" are important in the study of localisation. Hence we find it appropriate to get a torsion-theoretic version of these concepts. We discuss some properties of the patch topology and see that if \( R \) is Artinian, then the patch topology on \( \text{on-R} \) is the discrete topology. We also see some collections of prime torsion classes that satisfy the generic regularity condition.

As we have already mentioned, Jategaonkar has defined links between prime ideals and Goodearl has generalised these links between uniform injective right modules over a right Noetherian ring. In chapter four, we define links between prime torsion classes in such a way that an injective (Goodearl) link between two uniform injectives implies a generalised injective link between the prime torsion classes cogenerated by them. An example shows that these links provide more obstructions to Ore localisation than injective links do. We also see some sets that are "right stable" under these links.
The construction of the largest right Ore subset of a multiplicative set has motivated us to define new links between prime torsion classes. In chapter five, we define these links (Ore links) and observe that they provide obstructions to Ore localisations in the following sense: If a multiplicative set $C$ in $R$ is a right Ore set, then, whenever $C \subseteq C_\tau$, we should have $C \subseteq C_\sigma$ for prime torsion classes $\sigma$ and $\tau$ such that $\sigma$ is Ore-linked to $\tau$.

The following result of Jategaonkar is important in characterising localisable sets of prime ideals: Let $X$ be a non-empty set of prime ideals in a right Noetherian ring $R$. If $X$ is "right stable" and satisfies the "right second layer condition" and the "right intersection condition", then $\mathcal{X} \mathcal{O} = \cap_{P \in \mathcal{X}} \mathcal{M}(P)$ is a right Ore set.

We define a intersection condition for a set of uniform injectives (or, equivalently, the prime torsion classes cogenerated by them), analogous to Jategaonkar's condition, using $\mathcal{K}(E)$ instead of $\mathcal{M}(P)$ and obtain a version of the above result, for Ore links, without assuming the right second layer condition.

We also discuss the behaviour of Ore links in various cases.
and obtain some situations when the set
\[ \Lambda \left\{ C_\tau : \tau \in \text{"rt cl o"} \right\} \text{ is right Ore.} \]

We conclude by discussing the scope for further work and by mentioning certain problems that arose in the torsion-theoretic study of Ore localisation and links.

Propositions 2.1, 2.4, 2.6, 4.12, 4.14 and example 4.16 were included in my M.Phil dissertation. They are mentioned here for the sake of completeness.