CHAPTER 5

FRACTIONAL INTEGRATION OPERATORS
OF CERTAIN SPECIAL FUNCTIONS

The results established in this chapter have been published/accepted/communicated for publication as detailed below:

Marichev-Saigo-Maeda fraction integration operators of certain special functions (Communicated for publication).
5.1 INTRODUCTION

The fractional integral operators, involving various special functions with them have found significant importance and applications in science and engineering. During the last four decades fractional calculus has been applied almost to every field of science, engineering and mathematics. Many applications of fractional calculus can be found in fluid dynamics, Stochastic dynamical system, non-linear control theory and astrophysics. A number of workers like Love [16], McBride [18], Kalla [10,11], Kalla and Saxena [12,13], Saigo [22,23,24], Saigo and Maeda [25], Kiryakova [14], etc. have studied in depth, the properties, applications and different extensions of various hypergeometric operators of fractional integration. A detailed account of such operators along with their properties and applications can be found in the research monographs by Miller and Ross [19], Kiryakova [14,15] and Debnath and Bhatta [3] etc.

A useful generalization of the hypergeometric fractional integrals, including the Saigo operator [22,23,24] has been introduced by Marichev [17] (see details in Samko et al. [26, p.194, (10.47) and whole section 10.3] and later extended and studied by Saigo and Maeda [25, p.393, eqn. (4.12) and (4.13)] in terms of any complex order with Appell’s function $F_3(.)$ in the kernel.

This chapter consists of two sections ‘A’ and ‘B’. The section A is devoted to an introduction to generalized fractional calculus operators involving $\overline{H}$-function and a generalized polynomial set.
In section B, we apply generalized operators of fractional integration involving Appell’s function $F_3(.)$ due to Marichev-Saigo-Maeda, to the $\bar{H}$-function and a generalized polynomial set. The results are expressed generalized Wright hypergeometric function, Mittag-Leffler function and Riemann-Zeta function are presented to enhance the utility and importance of our main results.
5.2 GENERALIZED FRACTIONAL CALCULUS OPERATORS

Let $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{C}$ and $x > 0$ then the generalized fractional calculus operators (the Marichev-Saigo-Maeda operators) involving the Appell’s function [28] are defined as by the following equations:

$$
(I_{0+}^{\alpha, \alpha', \beta, \beta'; \gamma} f)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_{0}^{x} (x-t)^{\gamma-1} t^{-\alpha'} F_{3}\left(\alpha, \alpha', \beta, \beta'; \gamma; 1-\frac{t}{x}, 1-\frac{x}{t} \right) f(t) \, dt,
$$

(R(\gamma) > 0) \quad \cdots \text{(5.2.1)}

and

$$
(I_{0-}^{\alpha, \alpha', \beta, \beta'; \gamma} f)(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)} \int_{x}^{\infty} (t-x)^{\gamma-1} t^{-\alpha} F_{3}\left(\alpha, \alpha', \beta, \beta'; \gamma; 1-\frac{x}{t}, 1-\frac{t}{x} \right) f(t) \, dt,
$$

(R(\gamma) > 0) \quad \cdots \text{(5.2.2)}

The left-hand sided and right-hand sided generalized integration of the type (5.2.1) and (5.2.2) for a power function are given by:

$$
(I_{0+}^{\alpha, \alpha', \beta, \beta'; \gamma} x^{\rho-1})(x) = \Gamma \left[ \frac{\rho+\gamma-\alpha-\alpha'-\beta, \rho+\beta'-\alpha'}{\rho+\beta, \rho+\gamma-\alpha'-\alpha, \rho+\gamma-\alpha'-\beta} \right] x^{\rho-\alpha-\alpha'+\gamma-1}, \quad \cdots \text{(5.2.3)}
$$

where $\text{Re}(\gamma) > 0, \text{Re}(\rho) > \max \{0, \text{Re}(\alpha + \alpha' + \beta - \gamma), \text{Re}(\alpha'-\beta')\}$ and

$$
(I_{0-}^{\alpha, \alpha', \beta, \beta'; \gamma} x^{\rho-1})(x) = \Gamma \left[ \frac{1-\rho-\gamma+\alpha+\alpha', 1-\rho+\alpha+\beta'-\gamma, 1-\rho-\beta}{1-\rho, 1-\rho+\alpha+\alpha'+\beta'-\gamma, 1-\rho+\alpha+\beta} \right] x^{\rho-\alpha-\alpha'+\gamma-1}, \quad \cdots \text{(5.2.4)}
$$

where $\text{Re}(\gamma) > 0, \text{Re}(\rho) < 1+\min \{\text{Re}(-\beta), \text{Re}(\alpha + \alpha' - \gamma), \text{Re}(\alpha + \beta' - \gamma)\}$. The symbol occurring in (5.2.3) and (5.2.4) is given by
\[
\Gamma\left[\begin{array}{c}
\alpha, \\
\beta, \\
\gamma, \\
\delta, \\
\epsilon, \\
\zeta
\end{array}\right] = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\delta)\Gamma(\epsilon)\Gamma(\zeta)}
\]

**5.3 \(H\)-FUNCTION**

The \(H\)-function, which is a generalization of the Fox \(H\)-function was introduced by Inayat-Hussain [8,9] and studied by Buschman and Srivastava [1] and others, is defined and represented in the following manner

\[
\mathcal{H}_{P,Q}^{M,N}[z] = \mathcal{H}_{P,Q}^{M,N}\left[ z^{\left(\begin{array}{c}
(c_j^+, e_j^+) \\
(c_j^-, e_j^-)
\end{array}\right)_1,N \left(\begin{array}{c}
(c_j^+, e_j^-)
\end{array}\right)_{N+1,P}} \right]
\]

\[
= \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \phi(s) z^s \, ds, \quad \text{...(5.3.1)}
\]

where

\[
\phi(s) = \prod_{j=1}^{M} \Gamma(f_j s) \prod_{j=1}^{N} \left\{ \Gamma(1-e_j s) \right\}^{a_j^*} \prod_{j=M+1}^{Q} \left\{ \Gamma(1-f_j s) \right\}^{b_j^*} \prod_{j=N+1}^{P} \Gamma(e_j s)
\]

which contains fractional powers of some of the gamma functions. Here \(e_j\) (\(j = 1,\ldots,P\)) and \(f_j\) (\(j = 1,\ldots,Q\)) are complex parameters, \(E_j \geq 0\) (\(j = 1,\ldots,P\)), \(F_j \geq 0\) (\(j = 1,\ldots,Q\)) (not all zero simultaneously) and the exponents \(a_j^*\) (\(j = 1,\ldots,N\)) and \(b_j^*\) (\(j = M+1,\ldots,Q\)) can take non-integer values.

The contour in (5.3.1) is imaginary axis \(\text{Re}(s) = 0\). It is suitably intended in order to avoid the singularities of the gamma functions and to keep those
singularities on appropriate sides. Again, for $\alpha_j^* (j = 1, \ldots, N)$ not an integer, the pole of the gamma function of the numerator in (5.3.2) are converted to branch points. However, as long as there is no coincidence of poles from any $\Gamma(f_j - F_j s)(j = 1, \ldots, M)$ and $\Gamma(1 - e_j^* + E_j s)(j = 1, \ldots, N)$ pair, the branch cuts can be chosen so that the path of integration can be distorted in the usual manner.

The following sufficient conditions for the absolute convergence of the defining integral for the $\bar{H}$-function given by equation (5.3.1) have been given by Buschman and Srivastava [1],

$$T = \sum_{j=1}^{M} F_j + \sum_{j=1}^{N} |\alpha_j^* E_j| - \sum_{j=M+1}^{Q} |\beta_j^* F_j| - \sum_{j=N+1}^{P} E_j > 0, \quad \ldots(5.3.3)$$

and

$$|\arg(z)| < \frac{1}{2} \pi T. \quad \ldots(5.3.4)$$

The behaviour of the $\bar{H}$-function for small values of $|z|$ follows easily due to Inayat-Hussain ([9], pp.4119-4128, see also [8])

$$\bar{H}_{p,q}^{M,N}[z] = O(|z|^{\alpha^*}),$$

$$\alpha^* = \min_{1 \leq j \leq M} \left[ \text{Re} \left( \frac{f_j}{F_j} \right) \right], \quad |z| \to 0. \quad \ldots(5.3.5)$$

Again, due to Inayat-Hussain ([9], pp.4119-4128) for large values of $z$, we have

$$\bar{H}_{p,q}^{M,N}[z] = O(|z|^{\beta^*}),$$
\[ \beta^* = \min_{1 \leq j \leq N} \left[ \text{Re} \left( \frac{\alpha_j^* (e_j - 1)}{E_j} \right) \right]. \]  

...(5.3.6)

5.4 A GENERALIZED POLYNOMIAL SET

Raizada has introduced and studied a generalized polynomial set and is defined by the following Rodrigues type formula [21, p.64, eq. (2.1.8)]

\[ S_{n}^{A, B, r}[x; r, h, q, A', B', m, k, \ell] = (A'x + B')^{-A} (1 - \tau x^r)^{-\frac{B}{\tau}} \]

\[ \times T_{k, \ell}^{m+n} \left( (A'x + B')^{A+qn} (1 - \tau x^r)^{\frac{B+hn}{\tau}} \right), \]  

...(5.4.1)

where the differential operator \( T_{k, \ell} \) being defined as

\[ T_{k, \ell} = x^{\ell} \left( k + x \frac{d}{dx} \right). \]  

...(5.4.2)

The explicit form of this generalized polynomial set [21, p.71, eq. (2.3.4)] is given by

\[ S_{n}^{A, B, r}[x; r, h, q, A', B', m, k, \ell] = B^{qn} x^{\ell(m+n)} (1 - \tau x^r)^{hn} \ell^{(m+n)} \]

\[ \times \sum_{p=0}^{m+n} \sum_{c=0}^{m+n} \sum_{\delta=0}^{m+n} \sum_{i=0}^{\delta} \frac{(-1)^{\delta} (1 - \delta)_i (A)_\delta (-p)_c (-A - qn)_i}{p! \delta! i! e!} \frac{(-B - hn)}{(1 - A - \delta)!} \]  

\[ \times \left( \frac{i + k + re}{\ell} \right)_{m+n} \left( \frac{-\tau x^r}{1 - \tau x^r} \right)^p \left( \frac{A'x}{B'} \right)^i. \]  

...(5.4.3)

It is to be noted that the polynomial set defined by (5.4.1) is very general in nature and it unifies and extends a number of classical polynomials introduced
and studied by various research workers such as Chatterjea [2], Gould-Hopper [6], Singh and Srivastava [27] etc. Some of special cases of (5.4.1) are given by Raizada in a tabular form [21]. We shall require the following explicit form of (5.4.1) which will be obtained by taking $A' = 1, B' = 0$ and let $\tau \to 0$ in (5.4.1) and use the well known confluence principle

$$\lim_{|b| \to \infty} \left( \frac{x}{b} \right)^n = x^n,$$

we arrive at the following polynomial set

$$S_n^{A,B,0}[x] = S_n^{A,B,0}[x; r, q, 1, 0, m, k, \ell] = x^{qn+r(m+n)} \ell^{m+n} \sum_{p=0}^{m+n} \sum_{e=0}^{p} \frac{(-p)^e}{p!e!} \times \left( \frac{A + qn + k + re}{\ell} \right)_m^n (Bx^r)^p. \quad \ldots(5.4.5)$$
SECTION B

5.5 MAIN RESULTS

This section starts with the assumption of two theorems on the product of the \( H \)-function and generalized polynomial set associated with Saigo-Maeda fractional integral operator (5.2.1) and (5.2.2). These theorems can be used to establish image formulas for the \( H \)-function in terms of the various special functions.

**Theorem 1.** Let \( \alpha, \alpha', \beta, \beta', \gamma, \rho, \epsilon \in \mathbb{C}, x > 0, T > 0 \) and \(|\arg(z)| < \frac{1}{2} \pi T\) be such that

\[
\Re(\gamma) > 0, \Re[\rho + \lambda q n + \lambda \ell (m + n) + \lambda \rho p] > \max \{0, \Re(\alpha + \alpha' + \beta - \gamma, \Re(\alpha' - \beta')\},
\]

then there hold the formula

\[
\left(\frac{1}{2^{\gamma}}\right) S_{n}^{\lambda, \beta, \gamma} \left[ z^{\lambda} : r, q, 1, 0, m, k, \ell \right] H_{n}^{M, N}_{p, q, r, s} = \frac{z^{\lambda}}{(2\pi i)^{\ell}} \sum_{\rho=0}^{m+n} \sum_{e=0}^{p} \left( \frac{-p}{p!} \right) e^{\left( \frac{A + q n + k + \Re}{\ell} \right)} (B)^{p}
\]

\[
\times H_{n+1}^{M, N+1}_{p+3, Q+3} \left[ x^{\xi} \right] \left( (1-\Delta, \xi), (1-\Delta-\gamma+\alpha+\beta, \xi), (1-\Delta-\beta'+\alpha', \xi), (1-\Delta-\beta'+\alpha', \xi), (1-\Delta-\gamma+\alpha+\beta, \xi), (1-\Delta-\gamma+\alpha+\beta, \xi) \right],
\]

where

\[
\Delta = \rho + \lambda q n + \lambda \ell (m + n) + \lambda \rho p
\]
Proof

In order to prove (5.5.1) first, we express the generalized polynomial set occurring on the left-hand side of (5.5.1) in the series form given by (5.4.5), and replace the $H$-function in well-known Mellin-Barnes contour integral with the help of (5.3.1), and also using (5.2.1), then interchanging the order of summations and integration, which is permissible under the conditions stated with Theorem 1, it take the following form after a little simplification

$$
\left\{ H_{0,+}^{\alpha,\beta,\gamma} z^{p-1} S_n A,B,0 \left[ z^{\gamma}; r,q,1,0, m,k, \ell \right] \left( \frac{1}{z^5} \right) \right. \\
= \lim_{m+n \to 0} \sum_{p=0}^{m+n} \sum_{e=0}^{p} \frac{(-p)_e}{p!e!} \left( A+qn+k+re \right) \left( \frac{\ell}{m+n} \right) (B)^p \\
\left. \right\} (x) \frac{1}{2\pi i} \int_L \phi(s) \left\{ I_{0,+}^{\alpha,\beta,\gamma} z^{p+\lambda qn+\lambda/(m+n)+\lambda re+\xi s} \right\} (x) ds.
$$

Finally, applying the known result (5.2.3) with $\rho$ replaced by $\rho+\lambda qn+\lambda/(m+n)+\lambda re+\xi s$, we arrive at the result (5.5.1).

**Theorem 2.** Let $\alpha, \alpha', \beta, \beta', \gamma, \rho, \in C$ and $x > 0, T > 0$ and $|\arg(z)| < \frac{1}{2} \pi T$ be such that $\Re(\gamma) > 0, \Re(\rho+\lambda qn+\lambda/(m+n)+\lambda re-\xi s) < 1 + \min \left[ \Re(\alpha+\alpha'-\gamma), \Re(\alpha+(\beta'-\gamma)), \Re(-\beta) \right]$, then there hold the formula

$$
\left\{ H_{0,+}^{\alpha,\beta,\gamma} z^{p-1} S_n A,B,0 \left[ z^{\gamma}; r,q,1,0, m,k, \ell \right] \left( \frac{1}{z^5} \right) \right. \\
= \lim_{m+n \to 0} \sum_{p=0}^{m+n} \sum_{e=0}^{p} \frac{(-p)_e}{p!e!} \left( A+qn+k+re \right) \left( \frac{\ell}{m+n} \right) (B)^p \\
\left. \right\} (x) \frac{1}{2\pi i} \int_L \phi(s) \left\{ I_{0,+}^{\alpha,\beta,\gamma} z^{p+\lambda qn+\lambda/(m+n)+\lambda re+\xi s} \right\} (x) ds.
$$
where

\[ \Delta = \rho + \lambda q n + \lambda \ell (m + n) + \lambda r p \]

**Proof.** By using the definitions (5.2.2), (5.3.1) and (5.4.5), and changing the order of summations and integration, which is permissible under the conditions stated with the Theorem 2, we get

\[
= \ell^{m+n} \sum_{p=0}^{m+n} \sum_{e=0}^{p} \frac{(-p)_e}{p! e!} \left( \frac{A + q n + k + r e}{\ell} \right)_{m+n} (B)^p \\
\]

\[
\times \left[ z^{-\xi} \left( (\Delta + \gamma - \alpha + \alpha' + \xi; 1)(\Delta - \alpha - \beta + \gamma + \xi; 1), (\Delta + \beta, \xi; 1), (\Delta - \alpha, \xi; 1) \right)_{l, N, 1}, (e_j, E_j)_{N+1, P} \right] \\
\]

\[
\times H^{M,N+3}_{P+3,Q+3} \left( (\xi, j; 1, M; (f_j, F_j; B_j)_{M+1, Q} \right). \\
\]

\[
\ldots (5.5.2)
\]
5.6 SPECIAL CASES

(A) If we reduce the generalized polynomial set to unity by giving the suitable values to the parameters and the $\mathcal{H}$-function to the generalized Wright hypergeometric function $\mathcal{W}_Q [7]$, for this replacing

$$
\begin{align*}
\mathcal{H}_{M,N}^{P,Q} \left[ z \mathcal{E}_{j=1}^{P} \left( \frac{(\alpha_j + \beta_j + \gamma_j)}{(\gamma_j + 1)} \right) \right] \\
\text{by } \mathcal{H}_{P,Q+1}^{1,P} \left[ -z^{\xi} \mathcal{E}_{j=1}^{P} \left( \frac{(1-a_j + \beta_j)}{(1-a_j + \gamma_j + 1)} \right) \right]
\end{align*}
$$

in equation (5.5.1), under the conditions stated for the Theorem 1, we arrive at the following interesting result

$$
\begin{align*}
\left\{ \frac{1}{a_0} a_1 a_2 \right\} z^{p-1} \mathcal{H}_{P,Q}^{1,P} \left[ -z^{\xi} \mathcal{E}_{j=1}^{P} \left( \frac{(1-a_j + \beta_j)}{(1-a_j + \gamma_j + 1)} \right) \right]
\end{align*}
$$

Further on setting $E_j = 1$ ($j = 1, \ldots, P$) and $F_j = 1$ ($j = 1, \ldots, Q$) in (5.6.1), then in similar way the $\mathcal{H}$-function reduces to the generalized hypergeometric function $\mathcal{W}_Q [7]$.

(B) If we reduce $S_{N}^{A,B,0}$ polynomial to unity by giving suitable values to the parameters, the $\mathcal{H}$-function to Mittag-Leffler function, defined in the monograph by Erdélyi et al. [5] in (5.5.1), under the conditions stated for the Theorem 1, then we obtain the following result

$$
\begin{align*}
\left\{ \frac{1}{a_0} a_1 a_2 \right\} z^{p-1} \mathcal{E}_{a}^{1}(z^{\xi}) \end{align*}
$$
\[ \times \overline{H}_{4,5}^{1.4} \left[ -x^\xi \begin{pmatrix} (0,\xi;1), (1-p,\xi;1), (1-p-\gamma+\alpha+\alpha',\beta,\xi;1), (1-p-\beta'+\alpha',1;1) \\ (1-p,\beta;1), (1-p-\gamma+\alpha+\alpha',\xi;1), (1-p-\gamma+\alpha'+\beta,1;1), (1-\sigma,\lambda;1) \end{pmatrix} \right]. \quad \ldots(5.6.2) \]

(C) Now, we reduce the \( \overline{H} \)-function to the generalized Riemann Zeta function and \( S_{n}^{A,B,0} \) polynomial to unity in (5.5.1), under the conditions stated for the Theorem 1, then we arrive at the following result after a little simplification

\[ [(I_{0,+}^{\alpha,\alpha',\beta,\gamma} z^{p-1} \phi(z^\xi, p, \eta))] (x) = x^{p-\alpha+\gamma-1} \]

\[ \times \overline{H}_{5,5}^{1.5} \left[ -x^\xi \begin{pmatrix} (0,1), (1-\eta,1,p'), (1-p,\xi;1), (1-p-\beta'+\alpha',\xi;1), (1-p-\gamma+\alpha+\alpha'+\beta,\xi;1) \\ (-\eta,1,p'), (1-p-\beta',\xi;1), (1-p-\gamma+\alpha+\alpha',\xi;1), (1-p-\gamma+\alpha'+\beta,\xi;1), (1-\sigma,\lambda;1) \end{pmatrix} \right]. \quad \ldots(5.6.3) \]

(D) Further, on reducing the \( \overline{H} \)-function to the generalized Wright-Bessel function \( \overline{J}_{g}^{\delta,\mu} (z) \) (see [7]) and \( S_{n}^{A,B,0} \) polynomial set to unity in (5.5.1), under the conditions stated for the Theorem 1, we obtain the following result

\[ [(I_{0,+}^{\alpha,\alpha',\beta,\gamma} z^{p-1} \overline{J}_{g}^{\delta,\mu} (z^\xi))] (x) = x^{p-\alpha+\gamma-1} \]

\[ \times \overline{H}_{5,5}^{1.3} \left[ x^\xi \begin{pmatrix} (1-p,\xi;1), (1-p-\gamma+\alpha+\alpha'+\beta,\xi;1), (1-p-\beta'+\alpha',\xi;1) \\ (1-p-\beta',\xi;1), (1-p-\gamma+\alpha+\alpha',\xi;1), (1-p-\gamma+\alpha'+\beta,\xi;1), (-g,\delta,\mu) \end{pmatrix} \right]. \quad \ldots(5.6.4) \]

Similarly, we can find the results for the Theorem 2.
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