CHAPTER 4

GENERALIZED FRACTIONAL KINETIC EQUATIONS

The results obtained in this chapter have been published/accepted/communicated for publication as detailed below:

A novel computable extension of fractional kinetic equations arising in astrophysics. (Communicated for publication).
4.1 INTRODUCTION

Fractional kinetic equations have gained importance during the last decade due to their occurrence in certain problem in science and engineering. Fractional calculus and special functions have also contributed a lot to mathematical physics and its various branches. Nuclear reactions are highly energetic process involving subatomic particles found within the nuclei of atoms, are in two general forms. One in which reactions involve the splitting of atomic nuclei in to smaller subatomic particles, called fisson and another one is the process which create larger atoms from the nuclei of smaller atoms, called the fusion. In the fusion the nuclei of two or more atoms collide with sufficient force, they fuse to form a single, larger nucleus. During this process part of mass of the fused nucleus is converted in to energy. This produced energy is released as heat, light and various form of radiation.

Such nuclear reactions require extremely high temperature to induce the necessary immense collisions. Because such immense amounts of heat energy are required, fusion reactions are also known as thermonuclear reactions. The stars themselves are formed and fueled by thermonuclear reactions. Thus thermonuclear fusion play an important role in the formation of stars and to keep them shining for billions of years. The assumption of thermal equilibrium and hydrostatic equilibrium indicate that there is no time dependence in the mathematical model, which involves mathematical equations describing the internal structure of the star (Kourganoff [9], Perdang [14] and Clayton [3]).
The two important nuclear reactions in stars, during their evolution, are pp chain and CNO cycle. The production and destruction of nuclei in such chemical reaction can be described by the reaction-type equations. The linear reaction-type equation \( \frac{dy}{dx} = y^q \), lead to new insights into generalized Boltzmann-Gibbs statistical mechanics can be applied to describe the modified nuclear reaction rate of stellar plasmas which is consistent with need of the modification of the nuclear reaction rates of stellar plasma and their chemical composition. If an arbitrary reaction is characterized by a time dependent quantity \( N = N(t) \) then it is possible to calculate the rate of change of \( \frac{dN}{dt} \) by mathematical equation

\[
\frac{dN}{dt} = -d + p, \tag{4.1.1}
\]

where \( d \) is the destruction rate and \( p \) is the production rate of \( N \).

Haubold and Mathai [8] have established a functional differential equation between rate of change of reaction, the destruction rate and the production rate as follows

\[
\frac{dN}{dt} = -d(N_t) + p(N_t), \tag{4.1.2}
\]

where \( N = N(t) \) the rate of reaction, \( d = d(N) \) the rate of destruction, \( p = p(N) \) the rate of production and \( N_t \) denotes the function defined by
They have studied a special case of (4.1.2), for spatial fluctuations or inhomogeneities in the quantity $N(t)$ are neglected, given by the equation

$$\frac{dN_i}{dt} = -c_i N_i(t),$$

...(4.1.3)

with the initial condition that $N_i(t = 0) = N_0$ is the number density of species $i$ at time $t = 0$, $c_i > 0$, known as standard kinetic equation.

The solution of equation (4.1.3) is given by

$$N_i(t) = N_0 e^{-c_i t}.$$  

...(4.1.4)

An alternative form of the same equation can be obtained on integration

$$N(t) - N_0 = c_0 D^{-1}_t N(t),$$

...(4.1.5)

where $D^{-1}_t$ is the standard integral operator. Haubold and Mathai [8] have given the fractional generalization of the standard kinetic equation (4.1.5) as

$$N(t) - N_0 = c^\nu D^{-\nu}_t N(t),$$

...(4.1.6)

where $D^{-\nu}_t$ is the well-known standard Riemann-Liouville fractional integral operator (Oldham and Spainer [12]; Samko et al. [18]; Miller and Ross [10]) defined by

$$D^{-\nu}_t f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-u)^{\nu-1} f(u) \, du, \quad R(\nu) > 0,$$

...(4.1.7)
the solution of the fractional kinetic equation (4.1.6) is obtained by Haubold and Mathai [12] and is given by

\[ N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(vk + 1)} (ct)^{vk}, \]

Further, Haubold and Mathai [8] have derived the fractional kinetic equation and thermonuclear function in terms of well known Mittag-Leffler functions. As an extension of the work of Haubold and Mathai [8], Saxena et al. [20] have generalized the standard kinetic equation with generalized Mittag-Leffler functions and R-function.

In recent investigation Chaurasia and Pandey [2] established a computable generalization of the fractional kinetic equations in terms of Lorenzo-Hartley function and derived the solutions for the same.

This chapter is divided into two sections ‘A’ and ‘B’. In section A, we attempt to give an introduction to the Generalized Mittag-Leffler functions and I-function.

The section B deals with the solution of generalized fractional kinetic equations associated with generalized Mittag-Leffler function and I-function. The manifold generality of the generalized Mittag-Leffler function and I-function is discussed in terms of the solution of the above fractional kinetic equations. Some special cases involving the generalized Mittag-Leffler function, \( H \)-function and Fox H-function are also considered.
4.2 DEFINITIONS

The Swedish mathematician Mittag-Leffler [11] introduced the function $E_\nu(z)$ defined as

$$E_\nu(z) = \sum_{k=0}^{N} \frac{z^k}{\Gamma(\nu k + 1)}, \ (z \in C) \quad \ldots (4.2.1)$$

A generalization of $E_\nu(z)$ was studied by Wiman [24], where he defined the function $E_{\nu,\mu}(z)$ as

$$E_{\nu,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\nu k + \mu)}, \ (z, \nu, \mu \in \mathbb{C}; R(\nu) > 0, R(\mu) > 0) \quad \ldots (4.2.2)$$

Prabhakar [13] introduced the function $E_{\nu,\mu}^\gamma(z)$ in the form

$$E_{\nu,\mu}^\gamma(z) = \sum_{k=0}^{\infty} \frac{\gamma_k z^k}{\Gamma(\nu k + \mu) k!}, \ (z, \nu, \mu, \gamma \in \mathbb{C}; R(\nu) > 0, R(\mu) > 0, R(\gamma) > 0)$$

$$\ldots (4.2.3)$$

Shukla and Prajapati [21] (see also Srivastava and Tamovski [23]) defined and investigated the function $E_{\nu,\mu}^{\gamma,\delta}(z)$ as

$$E_{\nu,\mu}^{\gamma,\delta}(z) = \sum_{k=0}^{\infty} \frac{\gamma_{\delta k} z^k}{\Gamma(\nu k + \mu) k!}, \ (z, \nu, \mu, \gamma \in \mathbb{C}; R(\nu) > 0, R(\mu) > 0, R(\gamma) > 0, \delta \in (0,1) \cup \mathbb{N})$$

$$\ldots (4.2.4)$$

Salim [16] introduced a new generalization of Mittag-Leffler function defined as
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\[ F_{\gamma,\delta,q}^{\mu,\alpha}(z) = \sum_{k=0}^{\infty} \frac{\gamma_{qk}}{\Gamma(k + \mu)} \frac{z^k}{\delta_{\alpha k}} \]  

...(4.2.5)

where

\((z, \nu, \mu, \gamma, \delta, q) \in C; \min \{R(\nu), R(\mu), R(\gamma), R(\delta)\} > 0; \alpha, q > 0 \) and \(q \leq R(\nu) + \alpha\).

### 4.3 I-FUNCTION

The generalization of the H-function introduced by Rathie [15] and is defined as Mellin-Barnes type contour integral

\[
I_{T, U, W, S}[t] = I_{T, U, W, S}[t] \left[ \frac{(a_1 \gamma_1 A_1, \ldots, a_T \gamma_T A_T)}{(b_1 \delta_1 B_1, \ldots, b_U \delta_U B_U)} \right] 
\]

\[
= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \theta(\xi) t^\xi d\xi, \quad \ldots(4.3.1)
\]

where

\[
\theta(\xi) = \frac{\prod_{j=W+1}^{W} \{\Gamma(b_j - \delta_j \xi)\} B_j \prod_{j=1}^{S} \{\Gamma(1 - a_j + \gamma_j \xi)\} A_j}{\prod_{j=W+1}^{U} \{\Gamma(1 - b_j + \delta_j \xi)\} B_j \prod_{j=S+1}^{T} \{\Gamma(a_j - \gamma_j \xi)\} A_j}, \quad \ldots(4.3.2)
\]

where \(A_j, j = 1, \ldots, T\), \(j = 1, \ldots, U\) are not in general, positive integers. Clearly for non-negative values of \(A_j\) or \(B_j\) (4.3.1) is not expressible as an H-function. Here \(t\) may be real or complex but is not equal to zero and an empty product is interpreted as unity; \(T, U, W\) and \(S\) are integers such that \(0 \leq W \leq U; 0 \leq S \leq T, \gamma_j > 0 (j = 1, \ldots, T), \delta_j > 0 (j = 1, \ldots, U); a_j (j = 1, \ldots, T)\) and \(b_j (j = 1, \ldots, U)\) are complex parameters.
The contour in (4.3.1) is presumed to be imaginary axis \( \text{Re}(\xi) = 0 \), which is suitable intended in order to avoid the singularities of the gamma functions and to keep these singularities at appropriate sides. For \( A_j \) not an integer, the pole of the gamma functions of the numerator of (4.3.2) is converted to branch points. The branch cuts can be chosen in order that the path of integration can be distorted for the contour \( \text{Re}(\xi) = 0 \), as long as there is no coincidence of poles from any \( \Gamma(b_j - \delta_j \xi) \) and \( \Gamma(l - a_j + \gamma_j \xi) \). The sufficient conditions for the absolute convergence of the contour integral (4.3.1) is given by

\[
\nabla = \sum_{j=1}^{W} |B_j \delta_j| + \sum_{j=1}^{S} |A_j \gamma_j| - \sum_{j=W+1}^{U} |B_j \delta_j| - \sum_{j=S+1}^{T} |A_j \gamma_j| > 0, \quad \ldots(4.3.2)
\]

this condition provides exponential decay of the integrand in (4.3.1) and region of absolute convergence of (4.3.1) is

\[
|\arg t| \leq \frac{\nabla \pi}{2}. \quad \ldots(4.3.3)
\]

If we take \( B_j \) (\( j = 1, \ldots, W \)) and \( A_j \) (\( j = 1, \ldots, S \)) unity in (4.3.1), I-function reduces to \( H \)-function [1]. When the exponents \( A_j = 1 \) (\( j = 1, \ldots, T \)) and \( B_j = 1 \) (\( j = 1, \ldots, U \)) in (4.3.1), the I-function reduces to the familiar Fox’s \( H \)-function defined by Fox [6].
**SECTION B**

### 4.4 EXTENSIONS OF GENERALIZED FRACTIONAL KINETIC EQUATIONS

**Theorem 1.** Let \( t, \nu, \mu, \gamma, \delta \in \mathbb{C} \), \( \min \{ R(\nu), R(\mu), R(\delta), R(\gamma) \} > 0; \alpha > 0, q > 0, c > 0, d > 0 \), then for the solution of the fractional kinetic equation

\[
N(t) - N_0 t^{\mu-1} E_{\nu,\mu,\alpha}^{\gamma,\delta,q} (-d^\nu t^\nu) = -c^\nu D_t^{-\nu} N(t), \tag{4.4.1}
\]

has a solution of the form

\[
N(t) = N_0 t^{\mu-1} \sum_{r=0}^{\infty} \left( -c^\nu t^\nu \right)^r E_{\nu,\mu+\nu,\alpha}^{\gamma,\delta,q} (-d^\nu t^\nu). \tag{4.4.2}
\]

**Proof.** We know that (Erdélyi et al. [4]) the Laplace transform of the Riemann-Liouville fractional integration is given by

\[
L\{0 D_t^{-\nu} f(t); p\} = p^{-\nu} F(p), \tag{4.4.3}
\]

where

\[
F(p) = \int_0^\infty e^{-pu} f(u) \, du, \tag{4.4.4}
\]

Now, taking Laplace transform of both sides of equation (4.4.1), we have

\[
L\{N(t); p\} - N_0 L\{t^{\mu-1} E_{\nu,\mu,\alpha}^{\gamma,\delta,q} (-d^\nu t^\nu); p\} = L\{-c^\nu D_t^{-\nu} N(t); p\}
\]

\[
N(p) - N_0 \int_0^\infty e^{-pt} t^{\mu-1} \left( \sum_{k=0}^{\infty} \frac{(\gamma)_q (-d^\nu t^\nu)^k}{\Gamma(vk+\mu)(\delta)_{\alpha k}} \right) \, dt = -c^\nu p^{-\nu} N(p), \tag{4.4.5}
\]

\[
N(p)[1 + c^\nu p^{-\nu}] = N_0 \sum_{k=0}^{\infty} \frac{(\gamma)_{q k} (-d^\nu)^k}{(\delta)_{\alpha k} p^{vk+\mu}}
\]
Taking inverse Laplace transform of both sides of equation (4.4.6), we have

\[
L^{-1}\{N(p)\} = N_0 \sum_{k=0}^{\infty} \frac{(\gamma)^{q_k}}{(\delta)_{\alpha_k}} \left\{ \sum_{r=0}^{\infty} (-1)^r (c^v p^{-v})^r \right\}. \quad \text{(4.4.7)}
\]

which is the desired result (4.4.2).

**Theorem 2.** If \( \nu > 0, \mu > 0, c > 0, d > 0, \gamma_j > 0 \) (j = 1, \ldots, T) and \( \delta_j > 0 \) (j = 1, \ldots, U), then for the solution of fractional kinetic equation

\[
N(t) = N_0 t^{\mu-1} \sum_{r=0}^{\infty} (-c^v t^v)^r E_{\nu,\nu+\mu+1}^{\nu,\nu+\mu+1} (-d^v t^v).
\]

has a solution of form

\[
N(t) = N_0 t^{\mu-1} \sum_{r=0}^{\infty} (-c^v t^v)^r I_{T+1, U+1}^{W, S+1} (-d^v t^v) \left[ \begin{array}{c}
(a_1,\gamma_1, A_1, \ldots, a_T, \gamma_T, A_T) \\
(b_1,\delta_1, B_1, \ldots, b_U, \delta_U, B_U)
\end{array} \right].
\]

**Proof.** Applying the Laplace transform both sides of equation (4.4.9), we get

\[
N(p) = N_0 \frac{1}{2\pi i} \int_{-\infty}^{\infty} \theta(\xi) (-d^v)^\xi \frac{\Gamma(v\xi + \mu)}{p^{v\xi + \mu}} d\xi = -c^v p^{-v} N(p), \quad \text{(4.4.11)}
\]

\[
N(p) = \frac{N_0}{(1+c^v p^{-v})} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \theta(\xi) (-d^v)^\xi \frac{\Gamma(v\xi + \mu)}{p^{v\xi + \mu}} d\xi
\]
\[ N(p) = \sum_{r=0}^{\infty} (-1)^r (c^v p^{-v})^r \int_{0^{\infty}}^{t^{\infty}} \theta(\xi) (-d^v)^\xi \frac{\Gamma(v\xi + \mu)}{p^{v\xi + \mu}} d\xi, \quad \text{...}(4.4.12) \]

Now, taking Inverse Laplace transform both sides of (4.4.12), we obtain the desired result (4.4.10).

\textbf{4.5 SPECIAL CASES}

If we put \( \alpha = q = 1 \) in (4.4.1), then we arrive at the following result

\textbf{Corollary 1.} If \( R(\nu) > 0 \), \( R(\mu) > 0 \), \( R(\gamma) > 0 \), \( c > 0 \) and \( d > 0 \), then for the solution of the equation

\[ N(t) - N_0 t^{\mu-1} E_{v,\mu}^{\gamma,\delta} (-d^v t^v) = -c^v D_t^{-v} N(t), \quad \text{...}(4.5.1) \]

then the following result holds

\[ N(t) = N_0 t^{\mu-1} \sum_{r=0}^{\infty} (-c^v t^v)^r E_{v,\nu+r+\mu}^{\gamma,\delta} (-d^v t^v). \quad \text{...}(4.5.2) \]

If we take \( \delta = \alpha = q = 1 \) and \( c = d \) in (4.4.1), then we arrive at the following result obtained by Saxena et al. [20].

\textbf{Corollary 2.} If \( R(\nu) > 0 \), \( R(\mu) > 0 \), \( R(\gamma) > 0 \) and \( c > 0 \) then for the solution of the equation

\[ N(t) - N_0 t^{\mu-1} E_{v,\mu}^{\gamma} (-c^v t^v) = -c^v D_t^{-v} N(t), \quad \text{...}(4.5.3) \]

then the following result holds

\[ N(t) = N_0 t^{\mu-1} E_{v,\mu}^{\gamma+1} (-c^v t^v). \quad \text{...}(4.5.4) \]

If we set \( \delta = \alpha = q = \gamma = 1 \) in equation (4.4.1), then we arrive the result obtained by the Saxena et al. [19].
Corollary 3. If \( R(\nu) > 0, R(\mu) > 0, c > 0 \) and \( d > 0 \), then for the solution of the equation

\[
N(t) - N_0 t^{\mu-1} E_{\nu,\mu} (-d^\nu t^\nu) = -c^\nu 0 D_t^{-\nu} N(t), \quad \text{...(4.5.5)}
\]

then the following result holds

\[
N(t) = N_0 \frac{t^{\mu-\nu-1}}{c^\nu - d^\nu} [E_{\nu,\mu-\nu} (-d^\nu t^\nu) - E_{\nu+\mu-\nu} (-c^\nu t^\nu)]. \quad \text{...(4.5.6)}
\]

When \( \delta = \alpha = q = \gamma = 1 \) and \( c = d \) in equation (4.4.1), then we arrive at the following result given by Saxena et al. [19].

Corollary 4. If \( R(\nu) > 0, R(\mu) > 0 \) and \( c > 0 \), then for the solution of the equation

\[
N(t) - N_0 t^{\mu-1} E_{\nu,\mu} (-c^\nu t^\nu) = -c^\nu 0 D_t^{-\nu} N(t), \quad \text{...(4.5.7)}
\]

then the following result holds

\[
N(t) = N_0 \frac{t^{\mu-1}}{\nu} [E_{\nu,\mu-1} (-c^\nu t^\nu) + (1 + \nu - \mu) E_{\nu+\mu-1} (-c^\nu t^\nu)]. \quad \text{...(4.5.8)}
\]

If we put \( A_j (j = S + 1, ..., T) \) and \( B_j (j = 1, ..., W) \), unity in equation (4.4.9), then we arrive at the result.

Corollary 5. If \( \nu > 0, \mu > 0, c > 0, d > 0, \gamma_j > 0 (j = 1, ..., T) \) and \( \delta_j > 0 (j = 1, ..., u) \),

then for the solution of the equation

\[
N(t) - N_0 t^{\mu-1} \overline{H}_{T,U}^{W,S} \left( -d^\nu t^\nu \right) \begin{pmatrix} (a_j, \gamma_j; A_j)_{1,S} & (a_j, \gamma_j)_{S+1,T} \\ (b_j, \delta_j)_{1,W} & (b_j, \delta_j; B_j)_{W+1,U} \end{pmatrix}
\]

\[
= -c^\nu 0 D_t^{-\nu} N(t), \quad \text{...(4.5.9)}
\]
then the following result holds

\[
N(t) = N_0 t^{\mu-1} \sum_{r=0}^{\infty} (-c^v t^v)^r \left[ H^{W,S+1}_{T,U+1} \right]
\]

\[
\times H^{W,S}_{T,U+1} \left[ (-d^v t^v) \left( (1-\mu,v;1,1),s_{\gamma_j j} t, S, s_{\gamma_j j} t, S+1,T \right) \right]. \quad \ldots(4.5.10)
\]

For existence conditions of $H$-function see [11].

**Corollary 6.** If $\nu > 0, \mu > 0, c > 0, d > 0, \gamma_j > 0$ and $\delta_j > 0$, then for the solution of the equation

\[
N(t) - N_0 t^{\mu-1} H^{W,S}_{T,U+1} \left[ -d^v t^v \left( (a_j j, 1), t, S, s_{a_j j} t, S+1,T \right) \right],
\]

\[
= -c^v 0 D_t^{-v} N(t), \quad \ldots(4.5.11)
\]

then the following result holds

\[
N(t) = N_0 t^{\mu-1} \sum_{r=0}^{\infty} (-c^v t^v)^r \left[ H^{W,S}_{T,U+1} \right]
\]

\[
\times H^{W,S}_{T,U+1} \left[ (-d^v t^v) \left( (1-\mu,v;1,1),s_{a_j j} t, S, s_{a_j j} t, S+1,T \right) \right]. \quad \ldots(4.5.12)
\]

Finally, for $A_j = B_j = 1$ in (4.5.11), we arrive at the following result.

**Corollary 7.** If $\nu > 0, \mu > 0, c > 0, d > 0, \delta_j > 0$ and $\gamma_j > 0$, then for the solution of the equation

\[
N(t) - N_0 t^{\mu-1} H^{W,S}_{T,U+1} \left[ -d^v t^v \left( (a_j j, 1), t, S \right) \right],
\]

\[
= -c^v 0 D_t^{-v} N(t), \quad \ldots(4.5.13)
\]

then the following result holds

\[
N(t) = N_0 t^{\mu-1} \sum_{r=0}^{\infty} (-c^v t^v)^r \left[ H^{W,S}_{T,U+1} \right]
\]

\[
\times H^{W,S}_{T,U+1} \left[ (-d^v t^v) \left( (1-\mu,v;1,1),s_{a_j j} t, S \right) \right]. \quad \ldots(4.5.14)
\]
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