CHAPTER 3

STUDY OF CERTAIN GENERALIZED ELLIPTIC-TYPE INTEGRALS PERTAINING TO EULER INTEGRALS AND GENERATING FUNCTIONS

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3.1 INTRODUCTION

Elliptic-type integrals have their importance and potential in certain problems in radiation physics and nuclear technology. A number of earlier works on the subject contains several interesting unifications and generalizations of some significant families of elliptic-type integrals. The present chapter consists of three sections ‘A’, ‘B’ and ‘C’. In this chapter, we obtain certain new theorems on generating functions. In section A, we attempt to give an introduction to the families of elliptic-type integrals have been studied and generalized by many authors.

In section B, we derived certain new theorems on generating functions associated with I-function. The results obtained in this chapter are of manifold generality and basic in nature.

In section C, we have discussed certain application of the theorems obtained in section B by using different generating functions. The importance and usefulness of the results thus established lies in the fact that a number of Euler-type integrals involving generating functions can be obtained.
SECTION A

2.2 ELLIPTIC-TYPE INTEGRALS

Elliptic integrals occur in a number of physical problems [1,3,8,10,17,18,28] in the form of single and multiple integrals. Epstein and Hubbell [7] have treated the elliptic-type integrals

\[ \Omega_j (k) = \int_0^\pi (1-k^2 \cos \theta)^{\frac{j-1}{2}} d\theta ; \quad j=0,1,2 \quad \ldots (3.2.1) \]

and \( 0 \leq k \leq 1 \)

Elliptic integrals (3.2.1) have been studied and generalized by many authors notably by Kalla [11,12] and Kalla et al. [14], Kalla and Al-Saqabi [13], Kalla et al. [15], Salman [21], Saxena et al. [25], Srivastava and Bromberg [29] and Chaurasia et al.[4,5,6] have investigated various interesting unifications of the elliptic-type integrals (3.2.1).

Some of the important generalizations of elliptic-type integrals (3.2.1) are as follows

Kalla [11,12] introduced the generalization of the form

\[ R^\mu (k, \alpha, \gamma) = \int_0^\pi \frac{\cos^{2\alpha-1}(\theta/2) \sin^{2\gamma-2\alpha-1}(\theta/2)}{(1-k^2 \cos \theta)^{\mu+\frac{1}{2}}} d\theta, \quad (3.2.2) \]

where \( 0 \leq k < 1, \text{Re}(\gamma) > \text{Re}(\alpha) > 0, \text{Re}(\mu) > -\frac{1}{2} \). Results for this generalization are also derived by Glasser and Kalla [9].
Al-Saqabi [1] defined and studied the generalization given by the integral

\[ B_\mu(k, m, v) = \int_0^\pi \frac{\cos^{2m}(\theta) \sin^{2v}(\theta)}{(1 - k^2 \cos^2 \theta)^{\mu + \frac{1}{2}}} \, d\theta, \tag{3.2.3} \]

where \( 0 \leq k < 1; m \in \mathbb{N}_0, \mu \in \mathbb{C}, \text{Re}(\mu) > -\frac{1}{2}. \)

Asymptotic expansion of (3.2.3) has recently been discussed by Matera et al. [19].

The integral

\[ \Lambda_v(\alpha, k) = \int_0^\pi \frac{\exp[\alpha \sin^2(\theta/2)]}{(1 - k^2 \cos^2 \theta)^{\nu + \frac{1}{2}}} \, d\theta, \tag{3.2.4} \]

where \( 0 \leq k < 1, \alpha, \nu \in \mathbb{R}; \) presents another generalization of (3.2.1), given by Siddiqui [28].

Srivastava and Siddiqui [30] have given an interesting unification and extension of families of elliptic-type integrals in the following form

\[ \Lambda^{(\alpha,\beta)}_{(\lambda,\mu)}(\rho ; k) = \int_0^\pi \frac{\cos^{2\alpha-1}(\theta/2) \sin^{2\beta-1}(\theta/2)}{(1 - \rho \sin^2(\theta/2))^\lambda} \, d\theta, \tag{3.2.5} \]

where \( 0 \leq k < 1, \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0, \lambda, \mu \in \mathbb{C}, |\rho| < 1. \)

Kalla and Tuan [16] generalized the result in (3.2.5) by means of the following integral and also obtained its asymptotic expansion
\[ \Lambda^{(\alpha,\beta)}_{(\lambda,\gamma;\mu)}(\rho,\delta;k) = \int_0^\pi \cos^{2\alpha-1}\left(\frac{\theta}{2}\right) \sin^{2\beta-1}\left(\frac{\theta}{2}\right) (1 - k^2 \cos\theta)^{-\frac{\mu-1}{2}} \times (1 - \rho \sin^2(\theta/2))^{-\lambda} (1 + \delta \cos^2(\theta/2))^{-\gamma} \, d\theta, \quad \ldots(3.2.6) \]

where

\[ 0 \leq k < 1, \quad \text{Re}(\alpha) > 0, \quad \text{Re}(\beta) > 0, \lambda, \mu, \gamma \in \mathbb{C} \text{ and either } |\rho|, |\delta| < 1 \text{ or } \rho \text{ (or } \delta) \in \mathbb{C} \]

where \( \lambda = m \) or \( \gamma = -m, \; m \in \mathbb{N}_0 \), respectively.

Saxena and Kalla [23] have studied families of elliptic-type integrals of the form

\[ \Omega^{(\alpha,\beta)}_{(\sigma_1,\ldots,\sigma_{n-2};\delta,\mu)}(\rho_1,\ldots,\rho_{n-2},\delta;k) \]

\[ = \int_0^\pi \cos^{2\alpha-1}\left(\frac{\theta}{2}\right) \sin^{2\beta-1}\left(\frac{\theta}{2}\right) \prod_{j=1}^{n-2} \left[ 1 - \rho_j \sin^2\left(\frac{\theta}{2}\right) \right]^{-\sigma_j} \]

\[ \times \left[ 1 + \delta \cos^2\left(\frac{\theta}{2}\right) \right]^{-\gamma} (1 - k^2 \cos\theta)^{-\frac{\mu-1}{2}} \, d\theta, \quad \ldots(3.2.7) \]

where \( 0 \leq k < 1, \; \text{Re}(\alpha) > 0, \; \text{Re}(\beta) > 0; \; \sigma_j (j = 1,\ldots,n-2), \gamma, \mu \in \mathbb{C} \);

\[ \max \left\{ \left| \rho_j \right|, \left| \frac{\delta}{1+\delta} \right|, \left| \frac{2k^2}{k^2-1} \right| \right\} < 1. \]

In a recent paper, Saxena and Pathan [26] investigated an extension of equation (3.2.7) in the form

\[ \Omega^{(\alpha,\beta)}_{(\sigma_1,\ldots,\sigma_m,\gamma;\tau_1,\ldots,\tau_n)}(\rho_1,\ldots,\rho_m,\delta;\lambda_1,\ldots,\lambda_n) \]

\[ = \int_0^\pi \cos^{2\alpha-1}\left(\frac{\theta}{2}\right) \sin^{2\beta-1}\left(\frac{\theta}{2}\right) \prod_{i=1}^{m} \left[ 1 - \rho_i \sin^2\left(\frac{\theta}{2}\right) \right]^{-\sigma_i} \]

\[ \times \left[ 1 + \delta \cos^2\left(\frac{\theta}{2}\right) \right]^{-\gamma} (1 - k^2 \cos\theta)^{-\frac{\mu-1}{2}} \, d\theta, \quad \ldots(3.2.8) \]
\[
\times \left[ 1 + \delta \cos^2 \left( \frac{\theta}{2} \right) \right]^{-\gamma} \prod_{j=1}^{n} (1 - \lambda_j^2 \cos \theta)^{-\tau_j} d\theta,
\]  
\[\ldots (3.2.8)\]

where \( \min (\text{Re}(\alpha), \text{Re}(\beta)) > 0; |\lambda_j| < 1; \sigma_j, \gamma, \tau_j \in \mathbb{C}; \)

\[
\max \left\{ |\rho_i|, |\delta_i|, \left\| \frac{2\lambda_j^2}{\lambda_j^2 - 1} \right\|, \left| \frac{\delta_i}{1+\delta_i} \right| \right\} < 1 \ (i=1,...,m; \ j=1,...,n).
\]

In recent paper [4,5,6], Chaurasia et al. have investigated a new family of unified and generalized elliptic-type integrals

\[
\overline{\Omega}^{(\alpha,\beta)}_{(\lambda_j, \tau_j)}((\rho_i), (\delta_i); k_j) = \overline{\Omega}^{(\alpha,\beta)}_{\lambda_i,...,\lambda_N, \tau_i,...,\tau_M, k_i,...,k_M} (\rho_1,...,\rho_N; \delta_1,...,\delta_N; k_1,...,k_M)
\]

\[
= \int_0^\pi \cos^{2\alpha-1} \left( \frac{\theta}{2} \right) \sin^{2\beta-1} \left( \frac{\theta}{2} \right) \prod_{i=1}^{N} \left[ 1 + \rho_i \sin^2 \left( \frac{\theta}{2} \right) + \delta_i \cos^2 \left( \frac{\theta}{2} \right) \right]^{-\lambda_i}
\]

\[
\times \prod_{j=1}^{M} (1 - k_j^2 \cos \theta)^{-\tau_j} d\theta,
\]  
\[\ldots (3.2.9)\]

where \( \min (\text{Re}(\alpha), \text{Re}(\beta)) > 0; |k_j| < 1; \lambda_i, \tau_j \in \mathbb{C}; \)

\[
\max \left\{ |\rho_i|, |\delta_i|, \left\| \frac{2k_j^2}{k_j^2 - 1} \right\|, \left| \frac{\delta_i - \rho_i}{1+\delta_i} \right| \right\} < 1 \ (i=1,...,N \text{ and } j=1,...,M),
\]

which include most of the known generalized and unified families of elliptic-type integrals (including those discussed in (3.2.1) through (3.2.8). For more details also see [12], [24] and [26].
Upon a closer examination of the above equation (3.2.9), it can be seen that the family of elliptic-type integral \( \Omega^{(\alpha, \beta)}_{(\lambda_1, \tau_1)}(\rho_1, \delta_1; k_1) \) can be put into the following form involving Euler-type integral

\[
\Omega^{(\alpha, \beta)}_{(\lambda_1, ..., \lambda_N; \tau_1, ..., \tau_M)}(\rho_1, ..., \rho_N, \delta_1, ..., \delta_N; k_1, ..., k_M) = \prod_{j=1}^{M} (1 - k_j^2)^{-\tau_j} \prod_{i=1}^{N} (1 + \delta_i)^{-\lambda_i} \int_0^1 \omega^{\beta - 1} (1 - \omega)^{\alpha - 1} \]

\[
\times \prod_{j=1}^{M} \left[ 1 - \frac{2\omega k_j^2}{k_j^2 - 1} \right]^{-\tau_j} \prod_{i=1}^{N} \left[ 1 - \frac{(\delta_i - \rho_i)\omega}{1 + \delta_i} \right]^{-\lambda_i} d\omega. \quad \text{...(3.2.10)}
\]

A two-variable generating function \( F(z, t) \) possesses a formal power series representation in \( t \) and can be written in the following form

\[
F(z, t) = \sum_{n=0}^{\infty} C_n f_n(z) t^n, \quad \text{...(3.2.11)}
\]

where each member of the set \( \{f_n(z)\}_{n=0}^{\infty} \) is independent of \( t \).

### 3.3 THE I-FUNCTION

The I-function [27], a significant generalization of Fox’s H-function is defined in the form

\[
I(z) = \Gamma_{p_1, q_1}^{m, n} \left[ z \right] = \Gamma_{p_1, q_1}^{m, n} \left[ \begin{array}{c}(A_{ji})_{1,n}; (A_{ji}^{\alpha_{ji}})_{n+1,p_i} \\
(B_{ji})_{1,m}; (B_{ji}^{\beta_{ji}})_{m+1,q_i} \end{array} \right].
\]
\[= \sum_{k=0}^{\infty} \sum_{h=1}^{m} \frac{(-1)^k \prod_{j=1}^{m} \Gamma \left( B_j - \beta_j \left( \frac{B_h + k}{\beta_h} \right) \right)}{\sum_{i=1}^{r} \beta_h^k k! \prod_{j=m+1}^{q_i} \Gamma \left( 1 - B_{ji} + \beta_{ji} \left( \frac{B_h + k}{\beta_h} \right) \right)} \]

\[\times \frac{\prod_{j=1}^{n} \Gamma \left( 1 - A_j + \alpha_j \frac{B_h + k}{\beta_h} \right) \frac{B_h + k}{\beta_h} z}{\prod_{j=n+1}^{p_i} \Gamma \left( A_{ji} - \alpha_{ji} \frac{B_h + k}{\beta_h} \right)} \]

\[\ldots (3.3.1)\]

where \( p_i < q_i \) and \(|z| < 1\).
SECTION B

3.4 THEOREMS

In this section we have studied four new theorems associated with variables generating functions and I-function. Some new families of elliptic-type integrals can also be deduced with the help of the results obtained in this section. Such generalized and new families of elliptic-type integrals play an important role in evaluation of a number of Euler-type integrals involving various generating function. The basic idea of evolving the theorems discussed in the article is inspired by the research work of Mohammed [20], Saran [22] and Srivastava and Yeh [32]. These theorems can be used to establish known and various new elliptic-type integrals.

**Theorem 1.** Consider the generating function \( F(z,t) \) and the I-function defined in (3.2.11) and (3.3.1) respectively, then

\[
\int_{0}^{1} \omega^{\alpha-1}(1-\omega)^{\gamma-\alpha-1} \prod_{i=1}^{R} I_{p_i}^{(n(t))} F(z,t) \, d\omega
\]

\[
= \Gamma(\gamma-\alpha) \sum_{n=0}^{\infty} C_n f_n(z) t^n (\gamma-\alpha) \prod_{i=1}^{R} I_{p_i}^{(n(t))} F(z,t) \, d\omega
\]

\[
= \Gamma(\gamma-\alpha) \sum_{n=0}^{\infty} C_n f_n(z) t^n (\gamma-\alpha) \prod_{i=1}^{R} I_{p_i}^{(n(t))} F(z,t) \, d\omega
\]
provided that $\text{Re} \left[ \alpha + \xi \min \left( \frac{B_j^{(\ell)}}{\beta_j^{(\ell)}} \right) \right] > 0$, $\text{Re}[\gamma - \alpha] > 0$, $\text{Re}(\eta) > 0$, $\text{Re}(\mu) > 0$

and $|\arg z^{\xi_{\ell}}| < \frac{1}{2} \pi \Delta$, $j = 1, 2, \ldots, m$, $i = 1, 2, \ldots, r$, and $\ell = 1, \ldots, R$,

where $\Delta = \sum_{j=1}^{n} \alpha_{j}^{(\ell)} - \sum_{j=n+1}^{p_i} \alpha_{j}^{(\ell)} + \sum_{j=1}^{m} \beta_{j}^{(\ell)} - \sum_{j=m+1}^{q_i} \beta_{j}^{(\ell)}$.

**Theorem 2.** Consider the generating function $F(z,t)$ and the I-function defined in (3.2.11) and (3.3.1) respectively, then

\[
\int_{0}^{1} \omega^{\alpha - 1} (1 - \omega)^{\gamma - \alpha - 1} \prod_{\ell=1}^{R} \frac{I_{m_{1}}^{(1)}(\alpha_{1}, \beta_{1})}{p_{1}^{(1)} \cdot q_{1}^{(1)} \cdot \gamma_{1}} \left( z^{(1-\omega)} \right)^{\xi_{\ell}} \left[ \left( A_{j}^{(\ell)} \right) \left( \alpha_{j}^{(\ell)} \right) _{1,n_{1}}^{(\ell)} \cdot \left( A_{ji}^{(\ell)} \right) \left( \alpha_{ji}^{(\ell)} \right) _{1,n_{1}}^{(\ell)} + 1,p_{i}^{(\ell)} \right] \left( B_{j}^{(\ell)} \right) \left( \beta_{j}^{(\ell)} \right) _{1,m_{1}}^{(\ell)} \cdot \left( B_{ji}^{(\ell)} \right) \left( \beta_{ji}^{(\ell)} \right) _{1,m_{1}}^{(\ell)} + 1,q_{i}^{(\ell)} \right] F[z,t,\omega^{n}(1 - \omega)^{\mu}] d\omega
\]

\[
= \Gamma(\alpha) \sum_{n=0}^{\infty} C_{n} f_{n}(z)^{n} \left( \alpha_{n} \right) \eta_{n} \prod_{\ell=1}^{R} \frac{I_{m_{1}}^{(1)}(\alpha_{1}, \beta_{1})}{p_{1}^{(1)} + 1,q_{1}^{(1)} + 1,r_{1}^{(1)}} \left( z^{(1-\omega)} \right)^{\xi_{\ell}} \left[ \left( A_{j}^{(\ell)} \right) \left( \alpha_{j}^{(\ell)} \right) _{1,n_{1}}^{(\ell)} \cdot \left( A_{ji}^{(\ell)} \right) \left( \alpha_{ji}^{(\ell)} \right) _{1,n_{1}}^{(\ell)} + 1,p_{i}^{(\ell)} \right] \left( B_{j}^{(\ell)} \right) \left( \beta_{j}^{(\ell)} \right) _{1,m_{1}}^{(\ell)} \cdot \left( B_{ji}^{(\ell)} \right) \left( \beta_{ji}^{(\ell)} \right) _{1,m_{1}}^{(\ell)} + 1,q_{i}^{(\ell)} \cdot (1 - \gamma - \eta n - \mu n, \xi_{\ell}) \right], \quad \ldots (3.4.2)
\]

provided that $\text{Re}(\alpha) > 0$,

\[
\text{Re} \left[ \gamma - \alpha + \xi \min \left( \frac{B_j^{(\ell)}}{\beta_j^{(\ell)}} \right) \right] > 0$, $\text{Re}(\eta) > 0$, $\text{Re}(\mu) > 0$ and

$|\arg z^{\xi_{\ell}}| < \frac{1}{2} \pi \Delta$, $j = 1, 2, \ldots, m$, $i = 1, 2, \ldots, r$, and $\ell = 1, \ldots, R$,
where \( \Delta_i = \sum_{j=1}^{n+i} \alpha_j^{(i)} - \sum_{j=n+1}^{p_i} \alpha_j^{(i)} + \sum_{j=1}^{m} \beta_j^{(i)} - \sum_{j=m+1}^{q_i} \beta_j^{(i)} \).

**Theorem 3.** Consider the generating function \( f(z,t) \) and the \( I \)-function defined in (3.2.11) and (3.3.1) respectively, then

\[
\int_0^1 \omega^{\alpha-1} (1 - \omega) \gamma^{\beta-1} \prod_{\ell=1}^{R} I_{m, n, p_i}^{(\ell)} \omega \left[ F[z, \omega^n (1 - \omega) \mu] \right] d\omega
\]

\[
= \sum_{n=0}^{\infty} C_n n_1 (z) t^n \prod_{\ell=1}^{R} I_{m, n, p_i}^{(\ell)} + 2 q_i^{(\ell)} + 1 \tau^{(\ell)}
\]

\[
\left[ z^{\xi} \left( (1-\alpha-\eta, \xi), (1+\alpha-\gamma-\mu, \xi), (A_{j}^{(\ell)}, \alpha_j^{(\ell)}), (A_{ji}^{(\ell)}, \alpha_{ji}^{(\ell)}) \right) \right]
\]

\[
\left[ z^{\xi} \left( (1-\alpha-\eta-\mu, \xi), (1+\alpha-\gamma, \xi), (A_{j}^{(\ell)}, \alpha_j^{(\ell)}), (A_{ji}^{(\ell)}, \alpha_{ji}^{(\ell)}) \right) \right], \quad \cdots (3.4.3)
\]

provided that \( \text{Re} \left[ \alpha + \xi, \min \left( \frac{B_j^{(\ell)}}{\beta_j^{(\ell)}} \right) \right] > 0, \text{Re} \left[ \gamma - \alpha + \xi, \min \left( \frac{B_j^{(\ell)}}{\beta_j^{(\ell)}} \right) \right] > 0, \)

\( \text{Re}(\eta) > 0, \text{Re}(\mu) > 0 \) and \( |\arg z^{\xi}| \leq \frac{1}{2} \pi \Delta, i = 1, 2, \ldots, r, j = 1, \ldots, m \) and \( \ell = 1, \ldots, R, \)

where \( \Delta_i = \sum_{j=1}^{n+i} \alpha_j^{(i)} - \sum_{j=n+1}^{p_i} \alpha_j^{(i)} + \sum_{j=1}^{m} \beta_j^{(i)} - \sum_{j=m+1}^{q_i} \beta_j^{(i)} \).

Now, we state another modification of the Theorem 1, which can be used to obtain various new interesting generalizations of elliptic-type integrals.
Theorem 4. Consider the generating function \( f(z, t) \) and the I-function defined in (3.2.11) and (3.3.1) respectively, then

\[
\int_0^1 \omega^{\beta-1} (1-\omega)^{\alpha-1} \prod_{\ell=1}^R \frac{I_{n_{\ell}^{(\ell)}, n_{\ell}^{(\ell)}}(t)}{p_{i_{\ell}^{(\ell)}, q_{i_{\ell}^{(\ell)}}^{(\ell)}(t)}} \nu_{\ell}
\]

\[
\left( z, \omega \right)^{\xi_{\ell}} \left[ (A_{j_1}^{(\ell)}, A_{j_2}^{(\ell)}, \ldots, A_{j_{\ell}}^{(\ell)})_{1, n_{\ell}^{(\ell)}}(t), (A_{j_1}^{(\ell)}, A_{j_2}^{(\ell)}, \ldots, A_{j_{\ell}}^{(\ell)})_{n_{\ell}^{(\ell)}+1, p_{i_{\ell}^{(\ell)}}^{(\ell)}}(t) \right] \nu_{i_{\ell}^{(\ell)}}^{(\ell)}
\]

\[
\times \prod_{a=1}^M \left[ 1 - \frac{2\omega a^2}{k^2 - 1} \right]^{-\frac{\alpha}{\alpha}} \prod_{b=1}^N \left[ 1 - \frac{(\delta_b - \rho_b)\omega}{1 + \delta_b} \right]^{-\frac{\lambda_b}{\lambda_b}} F[z, t] \omega^{\gamma \ell} (1-\omega)^{\mu \ell} \right) \right) d\omega
\]

\[
= \Gamma(\alpha) \sum_{n=0}^\infty C_n \sum_{n=0}^\infty \frac{\left( \tau_n \right)_{n_a}}{n_a!} \left( \frac{2k^2_a}{k^2 - 1} \right)^{n_a}
\]

\[
\times \prod_{b=1}^\infty \sum_{n_b=0}^\infty \left( \frac{(\delta_b - \rho_b)\omega}{1 + \delta_b} \right)^{n_b} \left( \frac{\lambda_b}{\lambda_b} \right)^{n_b} \prod_{\ell=1}^R \frac{I_{n_{\ell}^{(\ell)}, n_{\ell}^{(\ell)}+1}^{(\ell)}}{p_{i_{\ell}^{(\ell)}, q_{i_{\ell}^{(\ell)}}^{(\ell)}+1}(t)}
\]

\[
\left[ (1-\beta-n_a-n_b-\eta, \xi, \ldots, (A_{j_1}^{(\ell)}, A_{j_2}^{(\ell)}, \ldots, A_{j_{\ell}}^{(\ell)})_{1, n_{\ell}^{(\ell)}}(t), (A_{j_1}^{(\ell)}, A_{j_2}^{(\ell)}, \ldots, A_{j_{\ell}}^{(\ell)})_{n_{\ell}^{(\ell)}+1, p_{i_{\ell}^{(\ell)}}^{(\ell)}}(t) \right] \nu_{i_{\ell}^{(\ell)}}^{(\ell)}(t)
\]

\[
\left( B_{j_1}^{(\ell)}, B_{j_2}^{(\ell)}, \ldots, B_{j_{\ell}}^{(\ell)} \right)_{1, m_{\ell}^{(\ell)}}(t), (B_{j_1}^{(\ell)}, B_{j_2}^{(\ell)}, \ldots, B_{j_{\ell}}^{(\ell)})_{m_{\ell}^{(\ell)}+1, q_{i_{\ell}^{(\ell)}}^{(\ell)}}(t), (1-\alpha-\beta-n_a-n_b-\eta, \ldots, \xi, \ldots) \right)
\]

...(3.4.4)

provided that

\[
\text{Re} \left[ \beta + \xi, \text{min} \left( \frac{B_{j_1}^{(\ell)}}{B_{j_2}^{(\ell)}} \right) \right] > 0, \text{Re}(\alpha) > 0, \text{Re}(\eta) > 0, \text{Re}(\mu) > 0; \delta_b, \rho_b, \lambda_b, \tau_a \in C; |k_a| < 1
\]

and

\[
\max \left\{ \left| \rho_b \right|, \left| \delta_b \right|, \left| \lambda_b \right|, \left| \tau_a \right|, \left( \frac{2k^2_a}{k^2 - 1} \right), \left( \frac{\delta_b - \rho_b}{1 + \delta_b} \right) \right\} < 1, \ a = 1, \ldots, M, b = 1, \ldots, N
\]
and \( |\arg z_{\ell}^{\xi_{\ell}}| < \frac{1}{2}\pi \Delta_1 \), \( j = 1, \ldots, m \), \( i = 1, \ldots, r \), and \( \ell = 1, \ldots, R \),

where \( \Delta_1 = \sum_{j=1}^{n} \alpha_j^{(\ell)} - \sum_{j=n+1}^{p_1} \alpha_j^{(\ell)} + \sum_{j=1}^{m} \beta_j^{(\ell)} - \sum_{j=m+1}^{q_1} \beta_j^{(\ell)} \).

**Proof.** Expressing \( F[z, t] \) as a power series form (3.2.11) in the integral (3.4.1), changing the order of integration and summation, which is permissible due to uniform convergence of the series involved. Using the definition (3.3.1) of the I- function in the evaluation of the resulting integral, we get the result (3.4.1), which proves Theorem 1. The proof of Theorems 2, 3 and 4 are similar to that of the Theorem 1.
SECTION C

3.5 APPLICATIONS

On account of the usefulness of the theorems discussed in the section B, we consider some interesting applications, which indicates manifold generality of the results obtained in this section.

(i) Consider the generating function [31]

$$F(z,t) = (1 - zt)^{-\sigma} = \sum_{n=0}^{\infty} \left( \frac{z^n t^n}{n!} \right), \quad \ldots(3.5.1)$$

and by the use of the Theorem 1, under the stated conditions for the Theorem 1, we get the following interesting result

$$\int_0^1 \omega^{\alpha-1} (1 - \omega)^{\gamma-\alpha-1} \prod_{\ell=1}^{R} I_{p_1, a_{1 \ell}}^{(\ell)} \sum_{n=0}^{\infty} \left( \frac{z^n t^n}{n!} \right) \frac{[1 - zt \omega^{\eta} (1 - \omega)^{\mu} ]^{-\sigma}}{\omega} \, d\omega$$

$$= \Gamma(\gamma - \alpha) \sum_{n=0}^{\infty} \left( \frac{z^n t^n (\gamma - \alpha) n!}{n!} \right) \prod_{\ell=1}^{R} I_{p_1, a_{1 \ell}}^{(\ell)} + I_{p_1, a_{1 \ell}}^{(\ell)} + z^{(\ell)}$$

when we put $\omega = \cos^2 \left( \frac{\theta}{2} \right)$ and $\cos \theta = 2 \cos^2 \left( \frac{\theta}{2} \right) - 1$, the above equation (3.5.2) gives the following generalization of the elliptic-type integral
\[
\int_0^\pi \cos^{2\alpha-1}\left(\frac{\theta}{2}\right) \sin^{2\gamma-2\alpha-1}\left(\frac{\theta}{2}\right) \prod_{\ell=1}^R I_{m_{ij}^{(\ell)}}^{n_{ij}^{(\ell)}} p_{i,q_{ij}}^{(\ell)} x^{(\ell)}
\]

\[
\left[ z^z \cos^2\left(\frac{\theta}{2}\right) \right] \left[ \begin{array}{c}
i_{ij}^{(\ell)} A_j^{(\ell)} A_i^{(\ell)}_{j,1,n_{ij}^{(\ell)}} A_i^{(\ell)}_{ji,1,n_{ij}^{(\ell)}} + 1, p_{i,q_{ij}}^{(\ell)} \\
i_{ij}^{(\ell)} B_j^{(\ell)} B_i^{(\ell)}_{ji,1,m_{ij}^{(\ell)}} B_i^{(\ell)}_{ji,1,m_{ij}^{(\ell)}} + 1, q_{ij}^{(\ell)} \end{array} \right]
\times \left[ 1 - z t \cos^{2\alpha} \left(\frac{\theta}{2}\right) \sin^{2\mu} \left(\frac{\theta}{2}\right) \right]^{-\sigma} d\theta
\]

\[
= \Gamma(\gamma - \alpha) \sum_{n=0}^\infty \frac{(\sigma)_n z^n t^n (\gamma - \alpha)_n}{n!} \prod_{\ell=1}^R I_{m_{ij}^{(\ell)}}^{n_{ij}^{(\ell)}+1} p_{i,q_{ij}}^{(\ell)} x^{(\ell)}
\]

\[
\left[ z^z \sin^2\left(\frac{\theta}{2}\right) \right] \left[ \begin{array}{c}
i_{ij}^{(\ell)} A_j^{(\ell)} A_i^{(\ell)}_{j,1,n_{ij}^{(\ell)}} A_i^{(\ell)}_{ji,1,n_{ij}^{(\ell)}} + 1, p_{i,q_{ij}}^{(\ell)} \\
i_{ij}^{(\ell)} B_j^{(\ell)} B_i^{(\ell)}_{ji,1,m_{ij}^{(\ell)}} B_i^{(\ell)}_{ji,1,m_{ij}^{(\ell)}} + 1, q_{ij}^{(\ell)} \end{array} \right], \quad \ldots(3.5.3)
\]

If we setting \( \omega = \sin^2(\theta/2) \) and using \( \cos\theta = 1 - 2\sin^2(\theta/2) \) and \( \sigma \to 0 \) in (3.5.2), we get the following result

\[
\int_0^\pi \sin^{2\alpha-1}\left(\frac{\theta}{2}\right) \cos^{2\gamma-2\alpha-1}\left(\frac{\theta}{2}\right) \prod_{\ell=1}^R I_{m_{ij}^{(\ell)}}^{n_{ij}^{(\ell)}} p_{i,q_{ij}}^{(\ell)} x^{(\ell)}
\]

\[
\left[ z^z \sin^2\left(\frac{\theta}{2}\right) \right] \left[ \begin{array}{c}
i_{ij}^{(\ell)} A_j^{(\ell)} A_i^{(\ell)}_{j,1,n_{ij}^{(\ell)}} A_i^{(\ell)}_{ji,1,n_{ij}^{(\ell)}} + 1, p_{i,q_{ij}}^{(\ell)} \\
i_{ij}^{(\ell)} B_j^{(\ell)} B_i^{(\ell)}_{ji,1,m_{ij}^{(\ell)}} B_i^{(\ell)}_{ji,1,m_{ij}^{(\ell)}} + 1, q_{ij}^{(\ell)} \end{array} \right] d\theta
\]

\[
= \Gamma(\gamma - \alpha) \prod_{\ell=1}^R I_{m_{ij}^{(\ell)}}^{n_{ij}^{(\ell)}+1} p_{i,q_{ij}}^{(\ell)} x^{(\ell)}
\]

\[
\left[ \frac{1 - \alpha, z, i_{ij}^{(\ell)}, A_j^{(\ell)} A_i^{(\ell)}_{j,1,n_{ij}^{(\ell)}} A_i^{(\ell)}_{ji,1,n_{ij}^{(\ell)}} + 1, p_{i,q_{ij}}^{(\ell)} \\
B_j^{(\ell)} B_i^{(\ell)}_{ji,1,m_{ij}^{(\ell)}} B_i^{(\ell)}_{ji,1,m_{ij}^{(\ell)}} + 1, q_{ij}^{(\ell)} \end{array} \right], \quad \ldots(3.5.4)
\]
It can be seen that above elliptic integral (3.5.2), also provides generalizations to a number of new families of elliptic-type integrals, which also generalizes known families of elliptic integrals.

Also by using the generating function (3.5.1), and by the application of the Theorem 4, under the stated conditions, we have obtained the following new family of elliptic-type integrals, which also generalizes known families of elliptic-type integrals.

\[
\int_0^1 \omega^{\beta-1}(1-\omega)^{\alpha-1} \prod_{\ell=1}^R I_{n_1}^{m_1}\left(\frac{t_1}{p_1}, \frac{q_1}{n_1}\right)
\]

\[
\left(\begin{array}{c}
(z, \omega) \\
(z, \omega)
\end{array}\right)^{\xi_\ell}
\]

\[
= \Gamma(\alpha) \sum_{n=0}^{\infty} (\sigma) z^n t^n (\alpha) \prod_{n=1}^M \sum_{n_a=0} \frac{(\tau_a)^{n_a}}{n_a!} \left(\frac{2k_a^2}{k_a^2 - 1}\right)^{n_a}
\]

\[
\times \prod_{b=1}^N \sum_{n_b=0} \left\{(\delta_b - \rho_b)^n_b \right\} \left(\frac{\lambda_b}{n_b!}\right)^{n_b} \prod_{\ell=1}^R I_{n_1}^{m_1}\left(\frac{t_1}{p_1}, \frac{q_1}{n_1}\right)
\]

\[
\left(\begin{array}{c}
(1-\beta-n_a-n_b-\eta n, \xi_\ell) \\
(1-\beta-n_a-n_b-\eta n, \xi_\ell)
\end{array}\right)^{\xi_\ell}
\]

\[
\left(\begin{array}{c}
(1-\alpha-n_a-n_b-\eta n-\mu, \xi_\ell) \\
(1-\alpha-n_a-n_b-\eta n-\mu, \xi_\ell)
\end{array}\right)^{\xi_\ell}
\]

\[
\cdots (3.5.5)
\]
for $\sigma \to 0$ and $\omega = \sin^2 \left(\frac{\theta}{2}\right)$ in equation (3.5.5), we get

\[
\int_0^\pi \cos^{2\alpha-1} \left(\frac{\theta}{2}\right) \sin^{2\beta-1} \left(\frac{\theta}{2}\right) \prod_{\ell=1}^R I_{\ell}^{m(\ell), n(\ell)} P_{\ell}^{(i)} \alpha^{(i)}_{\ell} x^{(i)}_{\ell}
\]

\[
\left[ \frac{1}{z} \sin^{2} \left(\frac{\theta}{2}\right) \right]^{\xi_{\ell}} \left[ (A^{(i)}_{\ell}, \alpha_{\ell}^{(i)}_{\ell})_{1, n}^{(i)} \cdot (A^{(i)}_{\ell}, \alpha_{\ell}^{(i)}_{\ell})_{n}^{(i)} + 1 \cdot P_{\ell}^{(i)} \right]^{\eta_{\ell}}
\]

\[
\times \prod_{a=1}^M \left[ 1 - k^2 a \cos \theta \right]^{-\frac{\tau_a}{a}} \prod_{b=1}^N \left[ 1 + \rho \sin^2 \left(\frac{\theta}{2}\right) + \delta b \cos^2 \left(\frac{\theta}{2}\right) \right]^{-\lambda_b} d\theta
\]

\[
= \Gamma(\alpha) \prod_{a=1}^M \left[ 1 - k^2 a \right]^{-\frac{\tau_a}{a}} \prod_{b=1}^N \left[ 1 + \delta b \right]^{-\lambda_b} \prod_{a=1}^M \sum_{n_a=0}^\infty \frac{(\tau_a)}{n_a} \left( \frac{2k^2 a}{k^2 a - 1} \right)^{n_a}
\]

\[
\times \prod_{b=1}^N \sum_{n_b=0}^\infty \left( \frac{(\delta b - \rho q)}{1 + \delta b} \right)^{n_b} \left( \frac{(\lambda b)}{n_b} \right) \prod_{\ell=1}^R I_{\ell}^{m(\ell), n(\ell) + 1} P_{\ell}^{(i)} + 1, q_{\ell}^{(i)} + 1, x_{\ell}^{(i)}
\]

\[
\left[ \frac{1}{z} \sin^{2} \left(\frac{\theta}{2}\right) \right]^{\xi_{\ell}} \left[ (A^{(i)}_{\ell}, \alpha_{\ell}^{(i)}_{\ell})_{1, n}^{(i)} \cdot (A^{(i)}_{\ell}, \alpha_{\ell}^{(i)}_{\ell})_{n}^{(i)} + 1 \cdot P_{\ell}^{(i)} \right]^{\eta_{\ell}}
\]

\[
\left[ \frac{1}{z} \sin^{2} \left(\frac{\theta}{2}\right) \right]^{\xi_{\ell}} \left[ (A^{(i)}_{\ell}, \alpha_{\ell}^{(i)}_{\ell})_{1, n}^{(i)} \cdot (A^{(i)}_{\ell}, \alpha_{\ell}^{(i)}_{\ell})_{n}^{(i)} + 1 \cdot P_{\ell}^{(i)} \right]^{\eta_{\ell}}
\]

\[
F(z, t) = (1 - z_1 t)^{-\alpha_1}(1 - z_2 t)^{-\alpha_2} = \sum_{n=0}^\infty g_n^{\alpha_1, \alpha_2}(z_1, z_2) t^n, \quad \ldots(3.5.7)
\]

where $g_n^{\alpha_1, \alpha_2}$ is the Lagrange polynomial defined by

\[
g_n^{\alpha_1, \alpha_2}(x, y) = \sum_{n=0}^\infty \frac{(\alpha_1)_r (\alpha_2)_{n-r}}{r! (n-r)!} x^r y^{n-r} \quad \ldots(3.5.8)
\]

and by the application of the Theorem 1, under the stated conditions, we get

(ii) Consider the generating function [31]
\[
\int_0^1 \omega^{\alpha-1} (1-\omega)^{\gamma-\alpha-1} \prod_{\ell=1}^R \frac{\Gamma_{\mu_{\ell}}(t_{\ell})}{\Gamma_{\mu_{\ell}}(t_{\ell})} \Phi^{\mu_{\ell}}_{\ell}(t_{\ell}) \, \omega
\]

\[
\left[ (z, \omega) \right]^{\xi \ell} \left[ \left( A_{j_1, j_2}^{(\ell)} \right)_{n, n+1, q_i} \left( B_{j_1, j_2}^{(\ell)} \right)_{m, m+1, q_i} \right] \prod_{j=1}^2 \left[ 1 - z_j \omega^n (1-\omega)^\mu \right]^{-\sigma_j} \, d\omega
\]

\[
= \Gamma(\gamma - \alpha) \sum_{n=0}^\infty g_n^{\sigma_1, \sigma_2} (z_1, z_2) t^n (\gamma - \alpha) \prod_{\ell=1}^R \frac{\Gamma_{\mu_{\ell}}(t_{\ell})}{\Gamma_{\mu_{\ell}}(t_{\ell})} \Phi^{\mu_{\ell}}_{\ell}(t_{\ell})
\]

Also, by the application of the Theorem 4 under the stated conditions, and by the use of the generating relation (3.5.7), we get

\[
\int_0^1 \omega^{\beta-1} (1-\omega)^{\alpha-1} \prod_{\ell=1}^R \frac{\Gamma_{\mu_{\ell}}(t_{\ell})}{\Gamma_{\mu_{\ell}}(t_{\ell})} \Phi^{\mu_{\ell}}_{\ell}(t_{\ell}) \, \omega
\]

\[
\left[ (z, \omega) \right]^{\xi \ell} \left[ \left( A_{j_1, j_2}^{(\ell)} \right)_{n, n+1, q_i} \left( B_{j_1, j_2}^{(\ell)} \right)_{m, m+1, q_i} \right] \prod_{j=1}^2 \left[ 1 - z_j \omega^n (1-\omega)^\mu \right]^{-\sigma_j} \, d\omega
\]

\[
= \Gamma(\alpha) \sum_{n=0}^\infty g_n^{\sigma_1, \sigma_2} (z_1, z_2) t^n \prod_{a=1}^M \frac{\Gamma_{\mu_{\ell}}(t_{\ell})}{\Gamma_{\mu_{\ell}}(t_{\ell})} \Phi^{\mu_{\ell}}_{\ell}(t_{\ell})
\]
\[ \times \prod_{b=1}^{N} \sum_{n_b=0}^{\infty} \left\{ \left( \delta_b - \rho_b \right) \right\}^{n_b} \left( \lambda_b \right)^{n_b} \frac{R}{n_b!} \prod_{\ell=1}^{R} \frac{m_\ell^{(t)}, n_\ell^{(t)} + 1}{p_1^{(t)} + 1, q_1^{(t)} + 1} \]

\[ \times \frac{\xi^\ell}{\sum_{\ell=1}^{\infty} \left( \frac{1}{1-\beta-\eta n-\eta n_{\xi}} \right) \left( \omega \right)^\ell \left( B_j^{(t)} \beta_j^{(t)} \right)_{1, m_\ell^{(t)} + 1, q_1^{(t)} + 1} \left( B_j^{(t)} \beta_j^{(t)} \right)_{m_\ell^{(t)} + 1, q_1^{(t)} + 1} } {\left( B_j^{(t)} \beta_j^{(t)} \right)_{1, m_\ell^{(t)} + 1, q_1^{(t)} + 1} \left( B_j^{(t)} \beta_j^{(t)} \right)_{m_\ell^{(t)} + 1, q_1^{(t)} + 1}} \]

\[ (iii) \text{ Consider the well-known generating function} \]

\[ F(z, t) = e^{-zt} = \sum_{n=0}^{\infty} \frac{(-1)^n z^n t^n}{n!} , \]

\[ \ldots(3.5.11) \]

and by the use of Theorem 1, under the stated conditions, we get

\[ \int_0^1 \omega^{\alpha-1} (1-\omega)^{\gamma-\alpha} \prod_{\ell=1}^{R} \frac{1}{I_1^{(t)} + 1, q_1^{(t)} + 1} \left( B_j^{(t)} \beta_j^{(t)} \right)_{1, m_\ell^{(t)} + 1, q_1^{(t)} + 1} \left( B_j^{(t)} \beta_j^{(t)} \right)_{m_\ell^{(t)} + 1, q_1^{(t)} + 1} e^{-zt[\omega^\gamma(1-\omega)]} \, d\omega \]

\[ \Gamma(\gamma-\alpha) \sum_{n=0}^{\infty} \frac{(-z^n t^n)}{n!} \prod_{\ell=1}^{R} \frac{1}{I_1^{(t)} + 1, q_1^{(t)} + 1} \left( B_j^{(t)} \beta_j^{(t)} \right)_{1, m_\ell^{(t)} + 1, q_1^{(t)} + 1} \left( B_j^{(t)} \beta_j^{(t)} \right)_{m_\ell^{(t)} + 1, q_1^{(t)} + 1} \]

\[ \ldots(3.5.12) \]

Also by application of Theorem 4, under the stated conditions, and by the use of the generating function (3.5.11), we get the following useful integral
\[
\int_0^1 \omega^{\beta-1} (1 - \omega)^{\alpha-1} \prod_{\ell=1}^R \left[ \frac{I_{m_{\ell}}(n_{\ell})}{p_{i_{\ell}}^{(f)} q_{j_{\ell}}^{(f)} x_{k_{\ell}}} \right] \\
\left[ \frac{(A_j^{(f)}, \alpha_j^{(f)})_{1,n_{\ell}}, (A_{ji_{\ell}}^{(f)}, \alpha_{ji_{\ell}}^{(f)})_{n_{\ell}+1, p_{i_{\ell}}^{(f)}}}{(B_{ji_{\ell}}^{(f)}, \beta_{ji_{\ell}}^{(f)})_{1,m_{\ell}}, (B_{ji_{\ell}}^{(f)}, \beta_{ji_{\ell}}^{(f)})_{m_{\ell}+1, q_{j_{\ell}}^{(f)}}} \right] \\
\times \prod_{a=1}^M \left[ 1 - \frac{2\omega k_a^2}{k_a^2 - 1} \right]^{-\tau_{\ell}} \prod_{b=1}^N \left[ 1 - \frac{(\delta_b^{\alpha} - \rho_b^{\alpha})}{1 + \delta_b^{\alpha}} \right]^{-\lambda_b^{\alpha}} e^{-z\ell(\omega(1-\omega)^{\mu_1})} d\omega \\
= \Gamma(\alpha) \sum_{n=0}^\infty \frac{(-z)^n}{n!} (\alpha)^{\mu_1} \prod_{a=1}^M \sum_{n_a=0}^\infty \left( \frac{\tau_{a}}{n_a!} \left( \frac{2k_a^{2}}{k_a^2 - 1} \right)^{n_a} \right) \\
\times \prod_{b=1}^N \sum_{n_b=0}^\infty \left( \frac{\delta_b^{\alpha} - \rho_b^{\alpha}}{1 + \delta_b^{\alpha}} \right)^{n_b} \left( \frac{\lambda_b^{\alpha}}{n_b!} \right) \prod_{i=1}^R \left[ \frac{I_{m_{\ell}}(n_{\ell})}{p_{i_{\ell}}^{(f)} q_{j_{\ell}}^{(f)} x_{k_{\ell}}} \right] \\
\left[ \frac{(1-\beta-n_a-n_b-\eta n_x_{\ell}), (A_{ji_{\ell}}^{(f)}, \alpha_{ji_{\ell}}^{(f)})_{n_{\ell}+1, p_{i_{\ell}}^{(f)}}}{(B_{ji_{\ell}}^{(f)}, \beta_{ji_{\ell}}^{(f)})_{m_{\ell}+1, q_{j_{\ell}}^{(f)}}, (1-\alpha-\beta-n_a-n_b-\eta n_x_{\ell})} \right]. \right]

\ldots (3.5.13)

It can be seen that the Theorems (1 through 4) discussed in this article provide generalizations to a new and known elliptic-type integrals. Furthermore, these theorems have also their applications in formulation of various new elliptic-type integrals, and some use in representing various forms of Euler-type integrals in terms of different generalized functions.
3.6 SPECIAL CASE

If we take $r = 1$ in the results (3.4.1) through (3.4.4), reduces to the known results recently obtained by Chaurasia and Meghwal [4].
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