CHAPTER 1

ON THE SOLUTION OF INTEGRAL EQUATIONS OF FREDHOLM TYPE WITH ALEPH-FUNCTION

The results established in this chapter have been published as detailed below:

1.1 INTRODUCTION

In the last several years a large number of Fredholm type integral equations involving various polynomials or special functions as their kernel have been studied by several authors notably Buschman [2], Higgins [8], Love ([12] and [13]), Prabhakar and Kashyap [15], Srivastava and Buschman [19], Srivastava and Raina [21], Chaurasia and Patni [4], Chaurasia and Kumar [3] and others. The main object of this chapter is to obtain the solution of the following Fredholm type integral equation

\[ \int_0^\infty y^{-\alpha} K^{m,n,p}_{q_1,q_2} \left[ \left( \frac{x}{y} \right)^p \sum \left( \frac{b^*_{ji}B_{ji}}{r_i} \right)^{\tau_i} \right] \left( \frac{V_{ji}}{V_{ji}} \right)^{\tau_j} f(y) \, dy = g(x) \quad (0 < x < \infty) \]

...(1.1.1)

This chapter is divided into two sections ‘A’ and ‘B’. In section A, firstly, we attempt to give an introduction to the Aleph-function, Riemann-Liouville fractional integral operator and Weyl fractional integral operator. Later on, as we know the Aleph-function is the most generalized special function, many special cases with potentially useful functions, such as Mittag-Leffler functions, Fox H-function, I-function and Meijer G-function etc.

Secondly, the section B turns to establish a solution of the integral equation of Fredholm type whose kernel involving the product of the Aleph-
function by using Riemann-Liouville and Weyl fractional integral operators. Further, the Fredholm integral equation involving the product of Aleph-function in the kernel is also solved by Mellin transform technique. Thus, the results presented in this chapter would at once yield a very large number of results involving a large variety of special functions occurring in the fields of mathematical physics and mechanics.
SECTION A

1.2 ALEPH-FUNCTION

The Aleph-function introduced by Südland et al. [22], however the notation and complete definition is presented here in the following manner in terms of Mellin-Barnes type integrals [see also 22]

\[ \mathcal{R}[x] = \text{Re} \left[ \frac{m', n'}{p_1 \cdot q_i \cdot r_i \cdot x'} \Gamma \left( \frac{b_j \cdot B_j}{1 \cdot n', r_i \cdot x'} \right) \Gamma \left( \frac{b_j \cdot B_j}{1 \cdot m', r_i \cdot x'} \right) \right] \]

\[ = \frac{1}{2\pi i} \int_{L} \Omega^{m', n', \ldots, \ldots, \ldots, \ldots} (s) x^{-s} ds, \quad (1.2.1) \]

for all \( x \neq 0 \), where \( \omega = \sqrt{-1} \) and

\[ \Omega^{m', n', \ldots, \ldots, \ldots, \ldots} (s) = \prod_{j=1}^{n'} \frac{\Gamma \left( v_j \cdot V_j \cdot s \right) \prod_{j=1}^{n'} \Gamma \left( 1 - b_j \cdot B_j \cdot s \right)}{\prod_{j=n'+1}^{\infty} \Gamma \left( 1 - v_j \cdot V_j \cdot s \right) \prod_{j=m'+1}^{\infty} \Gamma \left( 1 - b_j \cdot B_j \cdot s \right)} \cdot \quad (1.2.2) \]

The integration path \( L = L_{\infty \omega \gamma} \), \( \gamma \in \mathbb{R} \) extends from \( \gamma - \omega \infty \) to \( \gamma + \omega \infty \), and is such that the poles of the gamma functions \( \Gamma \left( 1 - b_j \cdot B_j \cdot s \right), \) \( j=1, n \) do not coincide with the poles of the gamma functions \( \Gamma \left( v_j \cdot V_j \cdot s \right), j=1, m \). The parameter \( p_i, q_i \) are non-negative integers satisfying \( 0 \leq n' \leq p_i, 0 \leq m' \leq q_i, \tau_i > 0 \) for \( i=1, r' \). The parameter \( V_j, B_j, v_j, B_j \) are positive numbers and \( v_j, b_j, v_j, b_j \) are complex. All poles of the integrand (1.2.2) are assumed to be simple, and
the empty product is interpreted as unity. The existence conditions for the
defining integral (1.2.1) are given below

\[ \phi_\ell > 0, |\text{arg}(x)| < \frac{\pi}{2} \phi_\ell, \ell = 0, r'; \]  
\[ \phi_\ell \geq 0, |\text{arg}(x)| < \frac{\pi}{2} \phi_\ell \text{ and } R \{ \xi_\ell \} + 1 < 0, \]  

…(1.2.3)

…(1.2.4)

where

\[ \phi_\ell = \sum_{j=1}^{n^\prime} B_j - \tau_\ell, \sum_{j=m^n+1}^{q_j} V_j - \tau_\ell, \sum_{j=m^n+1}^{1} \sum_{j=m^n+1}^{1} V_j \]  

…(1.2.5)

\[ \xi_\ell = \sum_{j=1}^{n^\prime} v_j - \sum_{j=1}^{n^\prime} b_j + r\ell, \sum_{j=m^n+1}^{q_j} v_j - \sum_{j=m^n+1}^{1} b_j + \frac{1}{2}(p_{j, \ell} - q_{j, \ell}), \ell = 0, r'. \]  

…(1.2.6)

For detail account of the Aleph (\(\aleph\))-function see Südland et al. ([22],[23]).

The series representation of the Aleph-function [1, 23] is as follows

\[ \aleph[y] = \aleph^{m^*, n^*}_{p^*_1, q_j^*_1, \tau^*_1, y} \left\{ \prod_{j=1}^{n^*} (b_j^*, B_j^*)_{m^*+1, n^*+1, p^*_j, \tau^*_j} \prod_{j=1}^{m^*} (v_j^*, V_j^*)_{1, m^*+1, q_j^*_j, \tau^*_j} \right\} \]

\[ = \sum_{h=1}^{m^*} \sum_{k=0}^{\infty} (-1)^k \frac{\phi^*(\eta_{h, k}) y^{-\eta_{h, k}}}{v_{h, k}^{*} k!}, \]  

…(1.2.7)

where

\[ \phi^*(\eta_{h, k}) = \prod_{j=1}^{m^*} \Gamma(v_j^* + V_j^* \eta_{h, k}) \prod_{j=1}^{n^*} \Gamma(1 - b_j^* - B_j^* \eta_{h, k}) \]

\[ \sum_{\ell=1}^{r^*} \tau^*_\ell \prod_{j=n^*+1}^{m^*} \Gamma(b_j^* + B_j^* \eta_{h, k}) \prod_{j=m^*+1}^{q_j^*_j} \Gamma(1 - v_j^* - V_j^* \eta_{h, k}) \]  

…(1.2.8)
and

\[ \eta_{h,k} = \frac{\nu_h^* + k}{\nu_h^*}, \quad p_i^* < q_i^* |y| < 1. \]

**Remark 1.** If the sum in the denominator of (1.2.1) can be simplified in terms of a polynomials in \( s \), the factors of the polynomial can be expressed by a fraction of Euler’s Gamma function leading to an H-function [6].

**Remark 2.** It is noticed that there is no historical name given to (1.2.1), compared to [21]. The Mellin transform of the function is the coefficient of \( x^{-s} \) in the integrand of (1.2.1). There are no references containing tables of \( \mathcal{K} \)-function in the literature.

**Remark 3.** Setting \( \tau'_{i} = 1, \forall i = 1,...,r' \) in (1.2.1) the definition of I-function [16] is obtained

\[
I[x] = \mathcal{K}^{m,n'}_{p_1q_i,1,r'}[x] = \mathcal{K}^{m,n'}_{p_1q_i,1,r'} \left[ x^{(b_i' B_i')_{p_i',n_i',1}(l(b_i' B_i')_{p_i',n_i',1})_{n_i+1,p_i'}} \right] \right] \\
\left[ x^{(v_i' V_i')_{p_i',n_i',1}(l(v_i' V_i')_{p_i',n_i',1})_{m_i+1,q_i',r'}} \right] \\
= \frac{1}{2\pi i} \int_{\Omega} \Omega^{m,n'}_{p_1q_i,1,r'}(s) x^{-s} ds, \quad ...(1.2.9)
\]

where \( \Omega^{m,n'}_{p_1q_i,1,r'}(s) \) is defined in (1.2.2). The existence conditions for the integral in (1.2.9) are the same as given in (1.2.3) through (1.2.6) with \( \tau'_{i} = 1, i = 1,...,r' \).
Remark 4. For $r' = 1$, the equation (1.2.9) reduces to the familiar H-function of Fox [6]

$$H_{p',q'}^{m,n'}[x] = \mathcal{H}_{p',q'}^{m,n'}[x] = \mathcal{H}_{p',q'}^{m,n'}[x] = \mathcal{H}_{p',q'}^{m,n'}[x]$$

$$= \frac{1}{2\pi i} \int_{L} \Omega^{m,n'}_{p',q'}(s) x^{-s} \ ds, \quad \cdots(1.2.10)$$

where the kernel $\Omega^{m,n'}_{p',q'}(s)$ is given in (1.2.2).

Remark 5. Putting $B_{i}^{1}, ..., B_{p'}^{1} = 1 = V_{1}^{1}, ..., V_{q'}^{1}$ in (1.2.10), it reduces to Meijer’s G-function [6]

$$G^{m,n'}_{p',q'}[x] = \mathcal{G}^{m,n'}_{p',q'}[x] = \mathcal{G}^{m,n'}_{p',q'}[x]$$

$$= \frac{1}{2\pi i} \int_{L} \Omega^{m,n'}_{p',q'}(s) x^{-s} \ ds, \quad \cdots(1.2.11)$$

where the kernel $\Omega^{m,n'}_{p',q'}(s)$ is given in (1.2.2).

Remark 6. Letting $p'_{i} = 0, m'_{i} = q'_{i} = 1, v' = 0, n' = 0, V' = 1$ and writing $sx$ for $x$ in (1.2.10), we have the following relation

$$H_{0,1}^{1,0}[sx|(0,1)] = \mathcal{H}_{0,1,1,1}^{1,0}[sx] = \mathcal{H}_{0,1,1,1}^{1,0}[sx]$$

$$= e^{-sx}$$

$$= e^{-x} \quad \text{(for $s = 1$).} \quad \cdots(1.2.12)$$
Remark 7. If \( m' = 0 \) and corresponding the suitable variation of the contour, the integrand is analytic on and with the contour and so on, from (1.2.11)

\[
G_{p_1,q_1}^{0,n'} \begin{bmatrix} x \\ (v) \end{bmatrix} = K_{p_1,q_1}^{0,n'} \begin{bmatrix} b \\ (v) \end{bmatrix} = 0
\]

It is clear that \( K \)–function is a many valued function of \( x \) with a branch point at the origin. In what follows, the Aleph-function will be represented by the contracted notation \( K_{p_1,q_1}^{m,n'} \begin{bmatrix} x \\ (v) \end{bmatrix} \) or \( K(x) \).

Let \( \mathcal{F} \) denote the space of all functions \( f \) which are defined on \( R^+\) and satisfy

(i) \( f \in \mathcal{F}(R^+) \),

(ii) \( \lim_{x \to \infty} [x^y f^\gamma (x)] = 0 \) for all non-negative integers \( \gamma \) and \( r \),

(iii) \( f(x) = 0(1) \) as \( x \to 0 \).

For correspondence to the space of good functions defined on the whole real line \( (-\infty, \infty) \) see Lighthill [11].

The Riemann-Liouville fractional integral (of order \( \mu \)) is defined by

\[
D^{-\mu} \{f(x)\} = \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} f(t) \, dt, (\text{Re}(\mu) > 0, f \in \mathcal{F}),
\]

...(1.2.11)
where $D^\mu\{f(x)\} = \phi(x)$ is understood to mean that $\phi$ is a locally integrable solution of $f(x) = D^{-\mu}\{\phi(x)\}$, implying that $D^\mu$ is the inverse of the fractional operator $D^{-\mu}$.

The Weyl fractional integral operator of order $h$ is defined by

$$W^{-h}\{f(x)\} = \frac{1}{\Gamma(h)} \int_{x}^{\infty} (\xi - x)^{h-1} f(\xi) d\xi, \quad (\text{Re}(h) > 0, f \in L). \quad \text{ ...(1.2.12)}$$
SECTION B

1.3 SOLUTION OF THE INTEGRAL EQUATION INVOLVING THE PRODUCT OF THE ALEPH-FUNCTIONS

In this section, we begin our study by establishing the following Lemma:

Lemma 1.

Let

(i) \( p_i^*(i,...,r'), q_i^*=(i=1,...,r'), m', n' \) be non-negative integers satisfying

\[ 0 \leq n' \leq p_i^*, 0 \leq m' \leq q_i^*, \tau_i^* > 0 \text{ for } i=1,...,r', \]

(ii) \( \Re(\alpha) > \Re(\beta), |\arg(z)| < \frac{\pi}{2} \phi_{\ell}, \) where

\[ \phi_{\ell} = \sum_{j=1}^{n'} B_j^* + \sum_{j=1}^{m'} V_j^* - \tau_{\ell}^* \left( \sum_{j=n'+1}^{p_i^*} B_j^* + \sum_{j=m'+1}^{q_i^*} V_j^* \right); (\ell = 1,...,r') \]

(iii) \( p_i^* < q_i^* \) and \(|u| < 1\).

Then

\[
W^{\beta-\alpha} \begin{bmatrix}
\frac{1}{y^{\alpha}} \mathbb{K}_{m',n'}^{n^*,n^*} p_i^*,q_i^*,\tau_i^*,x^*
\end{bmatrix}
+ \begin{bmatrix}
\frac{x}{y} \mathbb{P}_{1,n^*}^{1,n^*} (b_j^*,B_j^*)_{1,n^*}, n_{n^*+1,p_i^*}^{1,n^*}
\end{bmatrix}
\times \begin{bmatrix}
\frac{1}{z^{\beta}} \mathbb{K}_{m',n'}^{n^*,n^*} p_i^*,q_i^*,\tau_i^*,x^*
\end{bmatrix}
\times \begin{bmatrix}
\frac{1}{z} \mathbb{Q}_{1,m^*}^{1,m^*} (b_j^*,B_j^*)_{1,m^*}, m_{m^*+1,q_i^*}^{1,m^*}
\end{bmatrix}
\]
\[ y^{-\beta} \sum_{h=1}^{m^*} \sum_{k=0}^{\infty} (-1)^k \phi^* (\eta_{h,k}) u^{-\eta_{h,k}} V_h^* k! \]

\[ \left( -\frac{x}{y} \right)^{-p \eta_{h,k}} R^{m',n'}_{p_1', q_1', \tau_1', x'} \left[ z \left( -\frac{x}{y} \right)^q \right] \]

\[ (1-\beta+\mu h_k, q) (b_j, B_j)_{l,n} [\tau_j (b_j, B_j)]_{n+1, p_j' x'} \]

\[ (v_j, V_j)_{l,m} [\tau_j (v_j, V_j)]_{m' + 1, q_j' x'} (1-\alpha+\mu h_k, q) \]  

\[ \cdots (1.3.1) \]

**Proof of the Lemma**

By using the definition of the Weyl fractional integral in (1.3.1), we have

\[ W^{\beta-\alpha} \left[ y^{-\alpha} \sum_{p_1, q_1, \tau_1 x}^* \left[ u \left( \frac{x}{y} \right)^p \right] R^{m',n'}_{p_1', q_1', \tau_1', x'} \left[ z \left( \frac{x}{y} \right)^q \right] \right] \]

\[ = \frac{1}{\Gamma(\alpha-\beta)} \int_y^{\infty} (t-y)^{-\alpha-1} t^{-\alpha} R^{m',n'}_{p_1', q_1', \tau_1', x'} \left[ u \left( \frac{x}{t} \right)^p \right] R^{m',n'}_{p_1', q_1', \tau_1', x'} \left[ z \left( \frac{x}{t} \right)^q \right] dt \]

\[ \cdots (1.3.2) \]

Now expressing one of the Aleph-function in series form and other in term of the contour integral (1.2.1), interchanging the order of integrations and summations which is justified under the conditions, we have

\[ \frac{1}{\Gamma(\alpha-\beta)} \sum_{h=1}^{m^*} \sum_{k=0}^{\infty} (-1)^k \phi^* (\eta_{h,k}) u^{-\eta_{h,k}} V_h^* k! \]

\[ \times \frac{1}{2\pi i} \int_L \prod_{j=1}^{m'} \Gamma(v_j + V_j s) \prod_{j=1}^{n'} \Gamma(1-b_j - B_j s) \sum_{i=1}^{r'} \tau_i \prod_{j=n+1}^{q_i} \Gamma(b_j + B_j s) \prod_{j=m+1}^{q_j} \Gamma(1-v_j - V_j s) \]

\[ \times \left[ \int_0^\infty (t-y)^{\alpha-\beta-1} t^{-\alpha+\mu h_k+qs} dt \right] ds, \quad \cdots (1.3.3) \]
Now evaluate the t-integral (1.3.3) and reinterpreting the resulting Mellin-Barnes contour integral in terms of the Aleph-function, we easily arrive at the desired result (1.3.1).

**Theorem 1.** Let the sufficient conditions (i), (ii) and (iii) of Lemma 1 be satisfied then

\[
\int_0^\infty y^{-\beta} \sum_{h=1}^{m^*} \sum_{k=0}^{\infty} \frac{(-1)^k \phi^* (\eta_{h,k}) u^{-\eta_{h,k}}}{\chi_{h,k}^* \eta_{h,k}!} \left( \frac{x}{y} \right)^{-\eta_{h,k}} \\
\times R_{m^*+1, n^*+1, p_{i+1, q_{i+1, x''\tau^\prime}}} \left[ \left( \frac{x}{y} \right)^q \left( b_{j_i}^* B_{j_i}^* \right)_{1,n^*} \left[ \xi_i^* B_{j_i}^* \right]_{n^*+1,p_{j_i}^* x''} \right] f(y) dy \\
= \int_0^\infty y^{-\alpha} R_{m^*, n^*} \left[ \left( \frac{x}{y} \right)^p \left( b_{j_i}^* B_{j_i}^* \right)_{1,n^*} \left[ \xi_i^* \right]_{n^*+1,p_{j_i}^* x''} \right] \left( v_{j_i}^* V_{j_i}^* \right)_{1,m^*} \left[ \xi_i^* \right]_{m^*+1,a_{j_i}^* x''} \right] D^{\beta-\alpha} f(y) dy. \quad \ldots(1.3.4)
\]

**Proof.** Let \( \Delta \) denote the left hand side of the assertion (1.3.4), then using Lemma 1 and applying (1.2.12), we get

\[
\Delta = \int_0^\infty \frac{f(y)}{\Gamma(p_{i+1, q_{i+1, x''\tau^\prime}})} \left( \int_0^\infty (1-y)^{\alpha-1} t^{-\alpha} R_{m^*, n^*} \left[ \left( \frac{x}{t} \right)^p \right] \\
\times R_{m^*, n^*} \left[ \left( \frac{x}{t} \right)^q \right] \right] dt \right) dy,
\]

\[
\int_0^\infty \left( \frac{x}{y} \right)^{\eta_{h,k}} \phi^* (\eta_{h,k}) u^{-\eta_{h,k}} dy.
\]
Now change the order of integration is assumed to be permissible just as in the proof of Lemma 1.

\[ \Delta = \int_0^\infty t^{-\alpha} K_{m^*,n^*}^{p_1,q_1,\tau_{x^*}} \left[ u \left( \frac{x}{t} \right)^p \right] K_{m',n'}^{p_1',q_1',\tau_{x'}} \left[ z \left( \frac{x}{t} \right)^q \right] \times \left\{ \int_0^t (t-y)^{\alpha-\beta-1} f(y) \, dy \right\} \, dt, \quad \ldots(1.3.5) \]

Finally, using (1.2.11), we arrive at the desired result (1.3.4).

### 1.4 APPLICATION OF THE MELLIN TRANSFORM

Fredholm integral equation (1.1.1) involving the product of Aleph-function in the kernel can also be solved by using the application of Mellin transform.

**Theorem 2.** If \( f \in \mathcal{D}^{\alpha-\beta}\{f(x)\} \) exists, \( q > 0, x > 0, |\arg(z)| < \frac{1}{2} \pi, \phi > 0, \)

\( \Re(\alpha) > \Re(\beta) > 0, p_i^* < q_i^* \) and \( |u| < 1 \), then the solution of the integral equation

\[ \int_0^\infty y^{-\alpha} K_{m^*,n^*}^{p_1,q_1,\tau_{x^*}} \left[ u \left( \frac{x}{y} \right)^p \left( \frac{b_j B_j}{y} \right) \frac{\tau_i^{(b_j B_j)}}{\tau_j^{(v_j V_j)}} \frac{n_1 + 1, p_i}{m_1 + 1, q_i} \right] \times K_{m',n'}^{p_1',q_1',\tau_{x'}} \left[ z \left( \frac{x}{y} \right)^q \left( \frac{b_j' B_j'}{y} \right) \frac{\tau_i^{(b_j' B_j')}}{\tau_j^{(v_j' V_j')}} \frac{n_1 + 1, p_i'}{m_1 + 1, q_i'} \right] f(y) \, dy = g(x) \quad (0 < x < \infty) \]

\[ \ldots(1.4.1) \]

is given by
\[ f(x) = \frac{q}{2\pi i} x^{\alpha - \frac{1}{2}} \int_{c-i\infty}^{c+i\infty} x^{-s} \left\{ \sum_{h=1}^{m^*} \sum_{k=0}^{\infty} (-1)^k \phi^*(\eta_{h,k}) U_h^* k! \right\} \]

\[ \theta \left( \frac{-s + p\eta_{h,k}}{q} \right) \left[ \frac{s-p\eta_{h,k}}{q} \right] \phi(s) ds , \quad \ldots(1.4.2) \]

provided that

\[ \max \{ \text{Re}[(b_{i}^{'},1)/B_{i}']\} < -\text{Re} \left( \frac{s-p\eta_{h,k}}{q} \right) < \min \left\{ \text{Re} \left( \frac{V_{j}}{V_{j}} \right) \right\}, \]

\[ (j = 1, \ldots, m', i = 1, \ldots, n'). \quad \ldots(1.4.3) \]

**Proof.** First, we replace \( f \) by \( D^{\alpha-\beta}\{f\} \) in (1.3.4) and applying (1.4.1), we have

\[ g(x) = \int_{0}^{\infty} y^{-\frac{1}{2}} \left( \sum_{h=1}^{m'} \sum_{k=0}^{\infty} (-1)^k \phi^*(\eta_{h,k}) U_h^* k! \right) \left( x^{-s} \right) \left( \frac{x}{y} \right)^{-p\eta_{h,k}} \]

\[ \times R_{p+1,q+1}^{m',n+1} \left[ q \left( x \right) \left( y \right) \left( \frac{x}{y} \right)^{q} \right] \left( v_{j}^{'},v_{j} \right)^{1,m'} \left( v_{j}^{'},v_{j} \right)^{1,m'} \left( 1+q,\eta_{h,k}+q \right) \left( 1+\alpha+q,\eta_{h,k}+q \right) \left( 1+\alpha+q,\eta_{h,k}+q \right) \]

\[ D^{\alpha-\beta}\{f(y)\} dy \]

\[ \ldots(1.4.4) \]

Multiplying both side of (1.4.4) by \( x^{s-1} \) and integrating with respect to \( x \) from 0 to \( \infty \), we have

\[ \phi(s) = \int_{0}^{\infty} x^{s-1} g(x) dx \]
\[ \int_{0}^{\infty} y^{-\beta} D^{\alpha-\beta} \{ f(y) \} \sum_{h=1}^{m^*} \sum_{k=0}^{\infty} \frac{(-1)^{k} \phi^*(\eta_{h,k}) u^{-\eta_{h,k}}}{V_h^* k! y^{-p_{h,k}}} \]

\[ \times \left( \int_{0}^{\infty} x^{s-p_{h,x}-1} K^{m',n'+1}_{p_{1}+1,q_{1}+1,1,x'} \right) \]

\[ \times \left[ \left( \frac{x}{y} \right)^{q} \right] \left( (1-\beta+p_{h,k},q)(B_j^*B_j^*)_{1,n}(b_j^*b_j^*)_{n+1,p_{1}x'} \right) \]

\[ \times \left[ \left( \frac{y}{x} \right)^{q} \right] \left( (y_j^*y_j^*)_{1,m}(y_j^*y_j^*)_{m+1,q_{1}x'} \right) \]

... (1.4.5)

where we have assumed the uniform convergence of the integrals involved, with a view to justifying the inversion of the order of integration.

Now, evaluate the inner integral in (1.4.5) by a simple change of variable in the familiar results (cf. for example [7] and [18]), equation (1.4.5) reduces to

\[ = \frac{1}{q} \sum_{h=1}^{m^*} \sum_{k=0}^{\infty} \frac{(-1)^{k} \phi^*(\eta_{h,k}) u^{-\eta_{h,k}}}{V_h^* k! z} \left( \frac{s-p_{h,k}}{q} \right) \theta \left( \frac{-s+p_{h,k}}{q} \right) \]

\[ \times \frac{\Gamma(\beta-s)}{\Gamma(\alpha-s)} \int_{0}^{\infty} y^{-\beta} D^{\alpha-\beta} \{ f(y) \} dy. \]

...(1.4.6)

Inverting (1.4.6), by applying the Mellin-inversion theorem [24], we get

\[ D^{\alpha-\beta} \{ f(y) \} = \frac{q}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^{\beta-s-1} \left\{ \sum_{h=1}^{m^*} \sum_{k=0}^{\infty} \frac{(-1)^{k} \phi^*(\eta_{h,k}) u^{-\eta_{h,k}}}{V_h^* K!} \frac{\Gamma(\beta-s)}{\Gamma(\alpha-s)} \right\} \]

\[ \times \left[ \frac{s-p_{h,k}}{q} \right] \theta \left( \frac{-s+p_{h,k}}{q} \right) \phi(s) ds. \]

...(1.4.7)
Chapter 1

Operating both side by $D^{\beta-\alpha}$ (1.4.7), gives us

$$f(y) = \frac{q}{2\pi i} D^{\beta-\alpha} \left[ \int_{c-i\infty}^{c+i\infty} y^{\beta-s-1} \left( \sum_{h=1}^{m^*} \sum_{k=0}^{\infty} \frac{(-1)^k \phi^*(\eta_{h,k})}{\nu_h^k k!} u^{-\eta_{h,k}} \theta \left( \frac{-s+p\eta_{h,k}}{q} \right) \right) \right]^{\eta+\theta} \phi(s) ds,$$

which finally gives the desired result

$$f(x) = \frac{q}{2\pi i} x^{\alpha-1} \int_{c-i\infty}^{c+i\infty} x^{-s} \left( \sum_{h=1}^{m^*} \sum_{k=0}^{\infty} \frac{(-1)^k \phi^*(\eta_{h,k})}{\nu_h^k k!} u^{-\eta_{h,k}} \theta \left( \frac{-s+p\eta_{h,k}}{q} \right) \right) \times z \left( \frac{s-p\eta_{h,k}}{q} \right)^{-1} \phi(s) ds,$$

this is the solution of the integral equation (1.4.1).

1.5 APPLICATION

For $\tau_{1}' = \tau_{2}' = ... = \tau_{r}' = 1$ and $r'=1$ in (1.2.1), then the Aleph-function reduces to familiar the H-function introduced by Fox [7]

$$H^{m',n'}_{p',q'}[x] = \mathbf{H}^{m',n'}_{p',q'}[x] = \mathbf{H}^{m',n'}_{p',q'}[x] \left( \frac{b_{p'} B_{p'}}{v_{q'} V_{q'}} \right).$$

If now, we set $m'=n'=p'=q'=1$ and use the identity [6, section 18.1]

$$E_{\lambda,\sigma}(x) = H^{1,1}_{1,2}[x] \left( \begin{array}{c} (0,1) \\ (0,1), (1-\sigma, \lambda) \end{array} \right),$$
where $E_{\lambda,\sigma}(x)$ stands for the Mittag-Leffler function, defined in the monograph by Erdélyi et al. [5] as

$$E_{\lambda,\sigma}(x) = \sum_{\rho=0}^{\infty} \frac{x^\rho}{\Gamma(\lambda \rho + \sigma)}, \text{ Re}(\lambda) > 0, \text{ Re}(\sigma) > 0, \quad \ldots(1.5.2)$$

the H-function reduces to the Mittag-Leffler and consequently the following results are obtained.

**Lemma 2.** Let

(i) $\lambda, \sigma \in \mathbb{C}, \text{ Re}(\lambda) > 0, \text{ Re}(\sigma) > 0$;

(ii) $\text{ Re}(\alpha) > \text{ Re}(\beta), \text{ Re}\left[\beta + q \left(\frac{V_j}{V_i}\right)\right] > 0$, $(j=1,\ldots,m'); q > 0$;

(iii) $p_i^* < q_i^*$ and $|u| < 1$.

Then

$$W^{\beta-\alpha} \left[ y^{-\alpha} \Gamma_{p_i,q_i}^{m_i,n_i} \left[ u \left(\frac{x}{y}\right)^p \right] \cdot E_{\lambda,\sigma} \left[ z \left(\frac{x}{y}\right)^q \right] \right]$$

$$= y^{-\beta} \sum_{h=1}^{m^*} \sum_{k=1}^{\infty} \frac{(-1)^k \phi^{*} ((\eta_{h,k})^u)}{V_{h}^{*} k !} \left(\frac{x}{y}\right)^{-p \eta_{h,k}}$$

$$\times H^{1,2}_{2,3} \left[ -z \left(\frac{x}{y}\right)^q \left(0,1,1-\beta+p \eta_{h,k} q\right) \left(0,1,1-\sigma,\lambda,1-\alpha+p \eta_{h,k} q\right) \right], \quad \ldots(1.5.3)$$

**Proof.** The Lemma 2 can be easily established by using the same technique as used in Lemma 1.
Theorem 3. Let the sufficient conditions (i),(ii) and (iii) of Lemma 2 be satisfied then

\[
\int_0^\infty y^{-\beta} \sum_{h=1}^m \sum_{k=0}^\infty \frac{(-1)^k \phi^*(\eta_{h,k})}{\psi_{h,k}^{*} k!} u^{-\eta_{h,k}} \left(\frac{x}{y}\right)^{-\eta_{h,k}} \times H_{2,3}^{1,2}\left[-z\left(\frac{x}{y}\right)^q \left(\frac{(0,1,1-\beta+p \eta_{h,k},q)}{(0,1,1-\alpha,\lambda,1-\alpha+p \eta_{h,k},q)}\right)\right]
\]

\[
= \int_0^\infty y^{-\alpha} \sum_{p_i,q_i,\tau_i,\sigma_i} \left[\left(\frac{x}{y}\right)^p \left(\frac{(b'_j,B'_j)_{n'1} \tau_i (b'_i,B'_i)_{n'1+p_i',\tau_i'}}{\left(v'_j,\lambda'_j\right)_{m'1} \tau_i (v'_i,\lambda'_i)_{m'1+p_i',\tau_i'}}\right)\right] \times E_{\lambda,\sigma}\left[z\left(\frac{x}{y}\right)^q\right] D^{\beta-\alpha}\{f(y)\}dy.
\]

...(1.5.4)

**Proof.** In our proof of Theorem 3, we shall use Lemma 2 and equation (1.2.11), then we apply same technique as we used in Theorem 1.

### 1.6 SPECIAL CASES

(i) If we set \( \tau_1' = \tau_2' = ... = \tau_r' = 1 \), \( r' = 1 \) and \( p = 0 \), then the result in (1.3.1), (1.3.4) and (1.4.1) reduces to the known results with a slight modification obtained by Chaurasia and Patni [4].

(ii) When \( \tau_1' = \tau_2' = \tau_3' = ... = \tau_r' = 1 \), \( \tau_1^* = \tau_2^* = ... = \tau_r^* = 1 \), the Aleph-functions reduces to the I-functions [17], then the results in (1.3.1), (1.3.4) and (1.4.1) reduces to the known results recently derived by Chaurasia and Kumar [3].
(iii) On specializing the parameters in (1.3.1), (1.3.4) and (1.4.1), we arrive at the results obtained by Srivastava and Raina [21].

A number of special cases can be deduced by making suitable changes in parameters with the help of our equations (1.3.1), (1.3.4) and (1.4.1).
REFERENCES

Braaksma, B.L.J.


Buschman, R.G.


Chaurasia, V.B.L. and Kumar, D.


Chaurasia, V.B.L. and Patni, Rinku


Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F.G.


Fox, C.


Higgins, T.P.

Hilfer (Ed.), R.


Kilbas, A.A. and Saigo, M.


Lighthill, M.J.


Love, E.R.


Mathai, A.M. and Saxena, R.K.


Saxena, V.P.


Srivastava, H.M.

Srivastava, H.M. and Buschman, R.G.

Srivastava, H.M. and Panda, R.

Srivastava, H.M. and Raina, R.K.

Südland, N., Baumann, B. and Nonnenmacher, T.F.

Titchmarsh, E.C.