1. INTRODUCTION

1.1 Introduction
The thermoelasticity is the study of the relationship between the elastic properties of a material and its temperature, or between its thermal conductivity and its thermal stresses. A challenging problem faced by engineers and applied mathematicians is to find solutions of the basic questions arising in their field of study, which in large number of cases are partial differential equations. There could be nothing more desirable than to find exact solutions of these equations. The deformation that the body undergoes the ensuing stresses under the combined influence of external loads, body forces and heat conduction is calculated by making use of the well-known principles of classical mechanics and thermodynamics. This interaction of temperature and displacement fields, that is, the situation in which deformation is due to both mechanical and thermal causes give rise to “Thermoelasticity”.

Changes in the temperature causes thermal effects on the material. Some of these thermal effects are thermal stresses, strain and deformation. Thermal deformation simply means that as thermal energy and/or temperature of material increases so does the vibration of its atom or molecules and this increases in vibration result stretching of molecular bonds, which causes the material to expand, of course if thermal energy and temperature of material decreases, the material will shrink or contract.

The thermoelastic stress analysis is being used by engineers and scientists to solve practical problems in structural and material design.

1.2 Classification of Thermoeastic Problems

Thermo Elastic Problems are classified as

i. Direct Thermoeastic Problems: Determination of Thermal Stresses, Strain, and Deflection, Deformation etc. due to the temperature and heat flux applied on the boundary surfaces.

ii. Inverse Thermolastic Problems: Determination of Temperature and heat flux due to one of the thermal effect like thermal stress, Thermal strain, Thermal deflection etc.
1.3 Material Classification

The materials are classified depending upon their transport properties as isotropic, anisotropic, and orthotropic.

a) **Isotropic material:** Solids which have transport properties that do not depend on direction are said to be isotropic. Properties:

i. The thermal conductivity remains the same in all the directions.

ii. The heat flux is directly proportional to the temperature gradient at that position.

iii. The heat flux vector is normal to the isothermal surface passing through that position.

b) **Anisotropic material:** Solids that have properties exhibit directional characteristics are said to be anisotropic.

Properties:

i. The thermal conductivity is not same in all the directions.

ii. The heat flux is not simply proportional to the temperature gradient, each of the components the temperature is weighted according to the anisotropy of the particular material.

iii. The heat flux vector is not normal to the surface and it is not parallel to the temperature gradient.

c) **Orthotropic material:** Solids that have properties completely defined in terms of three perpendicular directions. Those three principal material directions are usually defined by a user defined coordinate system or a user defined reference plane.
1.4 Heat Transfer Mechanisms

Heat can be transferred in three different modes: conduction, convection, and radiation. All modes of heat transfer require the existence of temperature difference and all the modes are from high temperature medium to a lower temperature one.

a) Conduction: Conduction is the transfer of energy from the more energetic particles of a substance to the adjacent less energetic ones as a result of interactions between the particles. Conduction can take place in solids, liquids, or gases.

According to French mathematician Jean Baptiste Joseph Fourier (1768-1830), the heat flux vector is proportional to the temperature gradient, that is,

\[ \vec{q}(\vec{r},t) = -k \nabla T(\vec{r},t) \]

where, \( \vec{q}(\vec{r},t) \) is the heat flux vector and \( \nabla T(\vec{r},t) \) is the temperature gradient vector, the minus sign is introduced in order to make the heat flow a positive quantity, the proportionality factor \( k \) is called the thermal conductivity of the material having unit \( \text{W/mK} \).

b) Convection: Convection is the mode of energy transfer between a solid surface and the adjacent liquid or gas that is in motion. The faster the fluid motion, the greater the convection heat transfers.

The rate of convection heat transfer is proportional to the temperature difference, and it is expressed by Newton’s law of cooling (1642 -1727),

\[ \vec{q} \cdot \vec{n} = h(T - T_a) \]

where, \( \vec{n} \) is the unit outward normal to the surface and \( T \) is the surface temperature,
\( T_a \) is the temperature of the fluid sufficiently away from the surface,
\( h \) is the convection heat transfer coefficient in \( \text{W/m}^2\text{K} \).
c) Radiation: Radiation is the energy emitted by matter in the form of electromagnetic waves as a result of change in electronic configuration of the atoms or molecules. All the bodies above absolute zero temperature emit thermal radiation.

The maximum rate of radiation that can be emitted from a surface at thermodynamic temperature $T$ is given by the Stefan Boltzmann law as,

$$q_{\text{max}} = \sigma T^4$$

where, $\sigma = 5.67 \times 10^{-8} \text{ W/m}^2\text{K}^4$ is the Stefan Boltzmann constant.

1.5 Boundary Conditions

The differential equation of heat conduction will have numerous solutions unless a set of boundary conditions and an initial condition (for the time-dependent problem) are prescribed. The boundary conditions that prescribe the conditions at the boundary surfaces of the region may be linear or nonlinear. The linear boundary conditions will be separated into the following three groups:

a) Boundary condition of the First kind

Temperature is prescribed along the boundary surface and for the general case it is a function of both position and time, i.e.

$$T = f_i(\vec{r}_i,t) \text{ on the boundary surface } s_i$$

Special cases include temperature at the boundary surface is a function of position only $f_i(\vec{r}_i)$ or a function of time only $f_i(t)$, or a constant.

If the temperature at the boundary surface vanishes, one have

$$T = 0 \text{ on the boundary surface } s_i$$

This special case is called the homogeneous boundary condition of the first kind.

A boundary surface which is kept at zero temperature satisfies homogeneous boundary condition of the first kind.
b) **Boundary condition of the Second kind**

The normal derivative of temperature is prescribed at the boundary surface and it may be a function of both position and time

\[ \frac{\partial T}{\partial n_i} = f_i(r_i, t) \text{ on the boundary surface } s_i \]

where \( \frac{\partial}{\partial n_i} \) denotes differentiation along the outward drawn normal at the boundary surfaces \( s_i \).

This boundary condition is equivalent to that of prescribing the magnitude of the heat flux along the boundary surface, *i.e.*

\[ \left| \frac{\partial T}{\partial n_i} \right| = \bar{q} \]

If the normal derivative of temperature at the boundary surface vanishes, one have

\[ \frac{\partial T}{\partial n_i} = 0 \text{ on the boundary } s_i \]

This special case is called the homogeneous boundary condition satisfies this condition. An insulated boundary condition satisfies this condition.

c) **Boundary condition of the Third kind**

A linear combination of the temperature and its normal derivative is prescribed at the boundary surface, *i.e.*

\[ k_i \frac{\partial T}{\partial n_i} + h_i T = f_i(r_i, t) \text{ on the boundary surface } s_i \]

The physical significance of this equation is that the boundary surface under consideration dissipates heat by convection according to Newton’s law of cooling (*i.e.* heat transfer is proportional to temperature difference) to a surrounding temperature which varies both with time and position along the boundary surface.

A special case if

\[ k_i \frac{\partial T}{\partial n_i} + h_i T = 0 \]

which is called the homogeneous boundary condition of the third kind. The physical significance is that of heat dissipation by convection from the boundary surface into a surrounding at zero temperature.
1.6 The Differential Equation of Heat Conduction

The differential equation of heat conduction for a stationary, homogeneous, isotropic solid with heat generation within the region can be obtained from energy-balance equation [Ozicik]. i.e.

Rate of energy storage in $V = \text{Rate of heat entering } V \text{ through } + \text{Rate of heat generation in } V \text{ its bouncing surfaces}$

\[
\int_V \rho c_p \frac{\partial T(r,t)}{\partial t} dv = -\int_V \nabla \cdot \bar{q} dv + \int_V g(r,t) dv
\] (1.1)

\[
\int_V \left[ \rho c_p \frac{\partial T(r,t)}{\partial t} + \nabla \cdot (\bar{q} - g(r,t)) \right] dv = 0
\] (1.2)

Equation (1.1) is derived for an arbitrary for small volume element $V$ may be chosen so small as to remove the integral. One obtains

\[
\rho c_p \frac{\partial T(r,t)}{\partial t} = -\nabla \cdot \bar{q} + g(r,t)
\] (1.3)

\[
\rho c_p \frac{\partial T(r,t)}{\partial t} = \nabla \cdot \left[ k \nabla T(r,t) \right] + g(r,t)
\] (1.4)

\[Q = -k \nabla T(r,t)\]

where

$\bar{q} = \text{heat flux}$

$k = \text{thermal conductivity}$

\[
\rho c_p \frac{\partial T(r,t)}{\partial t} = k \nabla^2 T(r,t) + g(r,t)
\] (1.5)

is called the differential equation of heat conduction for a stationary, homogeneous, isotropic solid with heat generation within the solid.
1.6.1 Consider the following cases

a) Thermal conductivity is uniform, i.e. independent of position and temperature:

\[
\frac{1}{\alpha} \frac{\partial T(\vec{r},t)}{\partial t} = \nabla^2 T(\vec{r},t) + \frac{g(\vec{r},t)}{K}
\]  

(1.6)

Where the constant is called thermal diffusivity in the medium and it is defined as

\[
\alpha = \frac{K}{\rho c_p}
\]

Uniform thermal conductivity, no heat sources then

\[
\frac{1}{\alpha} \frac{\partial T(\vec{r},t)}{\partial t} = \nabla^2 T(\vec{r},t)
\]

(1.7)

which is called the Fourier equation of heat conduction or the Diffusion equation.

b) Uniform thermal conductivity, steady state and heat generation within the solid then

\[
\nabla^2 T(\vec{r}) + \frac{g(\vec{r})}{K} = 0
\]

(1.8)

which is called Poisson’s equation.

c) Steady state, and no heat generation then

\[
\nabla^2 T(\vec{r}) = 0
\]

(1.9)

is called Laplace equation.

1.6.2 Homogeneous and Non-homogeneous boundary value problems of heat conduction

The time-dependent boundary value problems of heat conduction will be considered in two different in two different groups: Homogeneous problems and Non-homogeneous problems

a) Homogeneous problems

The time-dependent boundary value problem of heat conduction will be referred to as a homogeneous problem when both the differential equation and the boundary conditions are homogeneous.
The problem in the form

$$\nabla^2 T = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad \text{in region } R, \ t > 0 \quad (1.10)$$

$$k \frac{\partial T}{\partial n_i} + h_i T = 0 \quad \text{on boundary } s_i, t > 0 \quad (1.11)$$

$$T = F(\vec{r}) \quad \text{in region } R, \ t = 0 \quad (1.12)$$

will be referred to as the homogeneous problem because both the differential equation and boundary condition are homogeneous.

b) Non-homogeneous problems

The boundary value problem of heat conduction will be referred to as non-homogeneous if the differential equation or the boundary conditions or both are non-homogeneous.

For example, the boundary value problem of heat conduction in the form

$$\nabla^2 T + \frac{g(\vec{r}, t)}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad \text{in region } R, \ t > 0 \quad (1.13)$$

$$k_i \frac{\partial T}{\partial n_i} + h_i T = f(\vec{r}, t) \quad \text{on boundary } s_i, t > 0 \quad (1.14)$$

$$T = F(\vec{r}) \quad \text{in region } R, \ t = 0 \quad (1.15)$$

is non-homogeneous because the differential equation and boundary condition are non-homogeneous (i.e. function $g(\vec{r}, t)$ and $f_i(\vec{r}, t)$ do not include $T$ as a product).

The boundary value problem of heat conduction in the form

$$\nabla^2 T + \frac{g(\vec{r}, t)}{K} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad \text{in region } R, \ t > 0 \quad (1.16)$$
\[ k_i \frac{\partial T}{\partial n_i} + h_i T = f(r, t) \text{ on boundary } s, t > 0 \quad (1.17) \]

\[ T = F(r) \text{ in region } R, t = 0 \quad (1.18) \]

is a non-homogeneous problem because the differential equation is non-homogeneous.

### 1.7 The Integral–Transform Technique

In solving the transient heat-conduction problems it is common to use a Laplace transform to remove time variable from the partial differential equations. However, in many problems it is more convenient to apply an integral transformation that removes the space variable from the partial differential equation. The integral transform technique is especially attractive for transient and steady-state heat conduction problems in that it treats all space variables in the same manner and has no inversion difficulties as in the case of the Laplace transformation because both the integral transform and the inversion formula are defined at the onset of the problem. For a given problem, however, the type of integral transform and corresponding inversion formula depend on the range of the space variable (i.e., finite, semi-finite or infinite extend) and on the type of boundary conditions. With the aid of integral-transform tables presented by Necati Ozisik [5] the solution of linear heat-conduction problems involving more than one space variable become a relatively easy and straightforward matter as compared with the classical method of solution in which experience and ingenuity are required to find the correct form of solution. With the integral-transform technique the solutions for finite regions are in the form of infinite series, but with present day computing facilities evaluation of such series presents no serious difficulty.

### 1.8 Necessity of work

Many important stress analysis problems involve structures that are subjected to both mechanical and thermal loadings. Thermal effects within an elastic solid produce heat transfer by conduction, and this flow of thermal energy establishes a temperature field within the material. Most solids exhibit a volumetric change with temperature...
variation, and thus the presences of a temperature distribution generally induces stresses created from boundary or interval constraints. If the temperature variation is sufficiently high, these stresses can reach levels that may lead to structural failure, especially for brittle materials. Thus, for many problems involving high temperature variation, the knowledge of thermal stress analysis can be very important. From the literature studied it is found that thermomechanical problems are solved by using assumed thermal profile. Also exact solution of heat conduction equation in three dimensional problems considering conduction and convention are not available in the literature. In the present work thermal profile are determined by solving heat conduction equations in thermodynamics which gives exact solution of thermoelastic problems.

1.9 Aim and Scope:
The specific aims and objectives of the present research work are:

In the present study, thermal stress analysis of 1-D, 2-D and 3-D problems such as beams, thin and thick plates are solved as heat conduction problems with various thermodynamics boundary conditions. Integral transform techniques are used to solve these boundary value problems.

Solutions obtained are unique and can be served as benchmark solutions for the purpose of comparison of other approximate solutions based on applied thermal profiles of various degrees of thickness coordinate.

1.10 Organization:

In this introductory chapter, necessity of the thermoelasticity and their brief description is presented; the aim and scope of the study have been defined. Comprehensive review of literature on the thermal stress analysis is presented in Chapter 2. Thermal stress analysis of 1-D, 2-D and 3-D problems such as beams, thin and thick plates are solved as heat conduction problems with various thermodynamics boundary conditions. Integral transform techniques are used to solve these boundary value problems in Chapter 3. In Chapter 4 the integral
transform technique is used for the solution of heat conduction equation which gives analytical solution for temperature distribution within clamped metallic rod and plate. Further this analytical solution of temperature distribution is used to determine thermal stresses developed within metallic rod and plate. The Matlab programming is used for determination numerical values of temperature change and thermal stresses. The results for temperature and stresses have been computed numerically and illustrated graphically. In Chapter 5 overall conclusions from the present research work carried out.