CHAPTER IV

INTUITIONISTIC FUZZY SOFT SET THEORY ON
SEQUENTIAL COMPACT SPACE

4.1 INTRODUCTION

We are living in a practical world where we require and need to face situations including imprecision, vagueness, imperfect, undetermined and uncertainty. Most importantly the biggest task of the data concern with the Social, Medical and Computer sciences and many other areas are always not authentic and impressive as it involves various kinds of uncertainty. But in classical approach of Mathematics, the tools all that we use to calculate, to model and to approach logically are certain. So they are failed to find solution for those complicated and significant problems in practical situations. The researchers have tremendously developed interest to deal with the complexity of this type of uncertain data for the last decay. With the various theories like fuzzy set theory, rough theory, probability theory and vague set theory which are considered reasonably the best mathematical models to approach vagueness. However these theories too have their own significant difficulties, which are explained in the paper soft set theory first results in detail (Molodtsov, 1999). The lack of using the concept of parameters is the reason for these inherent difficulties. The tools to prepare certain parameters are very weak that uncertainty of parameters has become a big task of using those theories. So Molodtsov has prepared the base for soft sets which is now made very easy to deal with uncertain, undetermined, fuzzy, roughness, vagueness and unclear information. The soft theory has been used in numerous areas such as Game theory, the Operations research, Smoothness of functions, Graph theory, Measure theory, Functional analysis, Riemann integration, Probability theory and Topology etc. A high level of research work is still going rapidly on soft sets. Next the authors Maji, Roy, (2003) have introduced few algebraic operations on soft set theory and they have also extended the crisp soft theory to fuzzy soft theory. The algebraic properties of soft sets are studied and developed by many authors. For example the authors Aktas and Cagman, (2007) have initiated the
notion of soft groups, Feng, (2008) has defined and proved some important properties on soft semi rings and Sun, (2008) has initiated and established soft modules.

Later Tanay, Bekir, and Burc Kandemir (2011) have introduced and developed the most interesting branch of pure abstract theory, topological structures in Fuzzy soft sets. Including the theoretical approach, fuzzy soft set theory has been improved and developed in application aspects too in different domains.

As a part of the vigorous study and extension of soft theory, fuzzy theory and fuzzy soft theory, in the year (Atanassov, 1986) has initiated the Intuitionistic fuzzy set. In his study, different properties which are related to the operations over sets are proved. With a proposal of distances between intuitionistic fuzzy sets, a geometrical representation of an intuitionistic fuzzy set is introduced. New definitions and properties are introduced, illustrated and compared with the approach that has been used for fuzzy sets. It is also proved that all the three parameters that are describing intuitionistic fuzzy sets should be considered into account while measuring those distances. (Maji, 2001) has introduced the concept of intuitionistic fuzzy soft set. Next Gunduz, Cigdem, and Sadi Bayramov (2011) have published, proposed and have introduced a new concept, Intuitionistic fuzzy soft modules. Later with the interest that has been increased significantly on Intuitionistic fuzzy soft sets, numerous papers were published. Many researchers have developed, investigated and improved various fundamental properties including many theorems on topology. The authors Roy, Akhil Ranjan and Maji, (2007) have given and presented a novel method to recognize the object from an inexplicit multi observer data in their paper a theoretic approach on fuzzy soft set to decision making problems. A comparison table is constructed in a view of parametric from a fuzzy soft set for decision making method. On intuitionistic fuzzy soft sets, Bora, Neog and Sut (2012) have given some results. The purpose of this study is to develop some fundamental operations and theorems and results obtainable in the literature of IFS sets. Plus some new results besides illustrating examples have been put forward in their work. In the same year, the researchers Bora, Neog and Sut (2012) have defined some new operations on IFS sets such as disjunctive sum and difference of two IFS sets and have studied their fundamental properties. Some
new notions relating to strong soft set of IFS set have been put ahead in their work. Few related properties with proofs, examples and counter examples have been established.

The authors Chetia and Das, (2013) in their paper on results of intuitionistic fuzzy soft sets and have shown and enhanced the flexibility of its application in decision making with the help of parametrization tool. They have established few results that were made by Cagman (2013) By Eulalia Szmidt (2000) a definition of the new concept intuitionistic fuzzy set is given and the latter is a generalization of the fundamental and known concept fuzzy set and relating example is described. By (Janusz Kacprzyk, 2000) a proposal on a non-probabilistic-type entropy measure related to intuitionistic fuzzy sets has come into the research. It is a consequence of a geometric presentation of intuitionistic fuzzy sets and presents a ratio of distances between them which is proposed by Szmidt and Kacprzyk, (2001). It is also established and shown that the proposed measure is to be defined to express as the ratio of intuitionistic fuzzy cardinalities of $F \cap F'$ and $F \cup F'$. In their paper, an algorithm was constructed on linear order between pairs of intervals with the help of aggregation functions. They have adapted this method to the case of Atanassov’s interval-valued intuitionistic fuzzy sets and also have applied the considered orders and these sets to a decision making problem.

Next, Çağman, Naim, and Irfan Deli (2013) have presented an article on Similarity measures of intuitionistic fuzzy soft sets and their application in decision making. There they have defined some categories of distances among two IFS sets and have proposed indistinguishable measures of two IFS sets. Then they have constructed a decision technique that can be applied to a medical diagnosis problem which is based on indistinguishable measures of IFS sets. At the end they gave two simple examples to prove the feasibility of using this technique to diagnosis the diseases which would be enhanced by assimilating clinical results and some other competing diagnosis. (Bayramov, Sadi, and Cigdem Gunduz, 2014) Few important properties of IFS topological spaces, definitions of the IFS closure and interior of an IFS set and also, IFS continuous mapping are presented and structural characteristics are given and studied. In this paper, a new similarity measure and a weighted similarity measure on intuitionistic fuzzy soft sets are proposed and some of their basic properties are discussed. A relation
\(\approx \alpha\) between two IFS sets are defined by using the proposed similarity measure and it is examined that the above defined relation is not equivalence. Moreover, the use of the proposed similarity measure is explained with an example and with the help of measure of performance and error. Moreover, medical diagnosis problems are demonstrated by means of a hypothetical case study with the help of proposed similarity measure. At the end, an application of proposed method is done on different medical data sets and has exhibited that proposed method is more accurate and enhanced than the existing methods by Muthukumar, and Sai Sundara Krishnan (2016).

In chapter IV, we give an extension to the concept of topological structures in intuitionistic fuzzy soft theory. We define IFS sequentially compact space and examine some important theorems on IFS sequentially compact space. We also introduce IFS \(\varepsilon\)-net, totally bounded IFS metric space and we will study properties of this space. Finally we define Lebesgue IFS number and uniformly continuous IFS mapping and investigate some important theorems. We present the basic definitions and theorems of soft, fuzzy, intuitionistic fuzzy and intuitionistic fuzzy soft set theories that are useful for further discussions.

Throughout this chapter IV, X means an initial universe, \(\rho(X)\) is the set of sub sets of \(X\), \(E\) is the set of all specifications or parameters describing objects of \(X\). These parameters may be attributes, characteristics or properties of some objects in \(X\).

### 4.2 INTUITIONISTIC FUZZY SOFT SET THEORY

In this section, we give the basic notation for intuitionistic fuzzy soft sets and fundamental operations on IFS sets.

#### 4.2.1. IMPORTANT NOTATION.

1. \(\mathrm{SX}_A\) – A Soft set over \(X\) with parameters \(A \subseteq E\), and there is a function \(s: A \rightarrow \rho(X)\).
2. \(F_X\) – A fuzzy set over \(X\), \(F_X = \{\mu_X(x)/x \in X\}\) where \(\mu_X : X \rightarrow [0,1]\), \(\mu_X\) called membership function of \(X\), and \(\mu_X(x)\) is the degree or value of membership of \(x \in X\).
3. **IFX** - An Intuitionistic Fuzzy set over the universe X, \(\text{IF}_X = \{(x, \mu(x), \gamma(x)): x \in X\}\), where \(\mu: X \rightarrow [0, 1]\), \(\gamma: X \rightarrow [0, 1]\), with \(0 \leq \mu(x) + \gamma(x) \leq 1, \forall x \in X\). The values \(\mu(x)\) and \(\gamma(x)\) represent the value of membership and non-membership of X to [0, 1] respectively.

4. IF(\(\tilde{X}\)) – The set of all intuitionistic fuzzy sub sets of X.

5. IF\(\hat{B}\) - \(\{(x, 0, 1): x \in X\}\) - intuitionistic fuzzy null set.

6. IF\(\hat{I}\) - \(\{(x, 1, 0): x \in X\}\) - intuitionistic fuzzy absolute set.

7. IF\(B\) = \(\{(x, \mu_B(x), \gamma_B(x)): x \in X\}\) be intuitionistic fuzzy sets of X. where \(\mu_A: X \rightarrow [0, 1]\), \(\gamma_A: X \rightarrow [0, 1]\), \(\mu_B: X \rightarrow [0, 1]\), \(\gamma_B: X \rightarrow [0, 1]\).

8. IFS\(X\)\(A\) - A IFS Soft set over X with parameters \(A \subseteq E\), and there is a function \(s: A \rightarrow \text{IF}(\tilde{X})\).

9. IFS\(X\)\(B\) - A IFS Soft set over X with parameters \(B \subseteq E\), and there is a function \(q: B \rightarrow \text{IF}(\tilde{X})\).

10. IFS\(\{\tilde{X}\}_E\) - The set of all Intuitionistic Fuzzy Soft sets IFS\(X\)\(A\) over X with parameters from E.

11. IFS\(X\)\(A\) \(\subseteq\) IFS\(X\)\(B\) if \(A \subseteq B\), \(s: A \rightarrow \text{IF}(\tilde{X}), q: B \rightarrow \text{IF}(\tilde{X})\) and \(s(e) \subseteq q(e) \forall e \in A\).

12. (IFS\(X\)\(A\))\(\prime\) - \(s'(e)\) for all \(e \in A\) where \(s': A \rightarrow \text{IF}(\tilde{X})\).

To be clear, when \(s(e) = \{(x, \mu_A(x), \gamma_A(x)): x \in X\}\), \(s'(e) = \{(x, \gamma_A(x), \mu_A(x)): x \in X\}\).

13. IFS\(\tilde{X}\)\(X\) - absolute IFS set over X.

14. IFS\(\tilde{X}\)\(\varphi\)\(A\).

**Proposition 4.1.** Let IF\(A\) = \(\{(x, \mu_A(x), \gamma_A(x)): x \in X\}\) and

Then the following properties hold.

(1) IF\(A\) \(\subseteq\) IF\(B\) if and only if \(\mu_A(x) \leq \mu_B(x)\) and \(\gamma_A(x) \geq \gamma_B(x)\) for all \(x \in X\);

(2) IF\(A\) \(\cong\) IF\(B\) if and only if \(\mu_A(x) = \mu_B(x)\) and \(\gamma_A(x) = \gamma_B(x)\) for all \(x \in X\);
(3) $\text{IF}_A \cap \text{IF}_B = \{(x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\gamma_A(x), \gamma_B(x)\}): \text{for all } x \in X\};$

(4) $\text{IF}_A \cup \text{IF}_B = \{(x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\gamma_A(x), \gamma_B(x)\}): \text{for all } x \in X\};$

(5) $\text{IF}_A^c = \{(x, \gamma_A(x), \mu_A(x))): \text{for all } x \in X\};$

(6) $\text{IF}_A \cdot \text{IF}_B = \{(x, \mu_A(x)\mu_B(x), y_A(x) + y_B(x) - y_A(x) y_B(x)): \text{for all } x \in X\};$

(7) $\text{IF}_A + \text{IF}_B = \{(x, \mu_A(x) + \mu_B(x), y_A(x) - \mu_A(x) y_B(x)): \text{for all } x \in X\};$

(8) For any arbitrary collection of IF sets $\text{IF}_{A_i} = \{(x, \mu_{A_i}(x), \gamma_{A_i}(x)): \text{for all } x \in X\}$, their intersection is defined as $\text{IF}_{A_i} = \{(x, \min\{\mu_{A_i}(x)\}, \max\{\gamma_{A_i}(x)\}): \text{for all } x \in X\};$

(9) For any arbitrary collection of IF sets $\text{IF}_{A_i} = \{(x, \mu_{A_i}(x), \gamma_{A_i}(x)): \text{for all } x \in X\}$, their union is defined as $\text{IF}_{A_i} = \{(x, \max\{\mu_{A_i}(x)\}, \min\{\gamma_{A_i}(x)\}): \text{for all } x \in X\};$

(10) For any IF sets $\text{IF}_A, \text{IF}_B, \text{IF}_C, \text{IF}_D$ we have $\text{IF}_A \subseteq \text{IF}_B$ and $\text{IF}_C \subseteq \text{IF}_D \Rightarrow$

$\text{IF}_A \cup \text{IF}_C \subseteq \text{IF}_B \cup \text{IF}_D$ and $\text{IF}_A \cap \text{IF}_C \subseteq \text{IF}_B \cap \text{IF}_D$.

(11) For any IF sets $\text{IF}_A, \text{IF}_B, \text{IF}_C$, such that $\text{IF}_A \subseteq \text{IF}_B$ and $\text{IF}_B \subseteq \text{IF}_C$ then $\text{IF}_A \subseteq \text{IF}_C$.

(12) For any IF sets $\text{IF}_A, \text{IF}_B, \text{IF}_C$, such that $\text{IF}_A \subseteq \text{IF}_B$ and $\text{IF}_A \subseteq \text{IF}_C$, then $\text{IF}_A \subseteq \text{IF}_B \cap \text{IF}_C$ also $\text{IF}_A \subseteq \text{IF}_B \cup \text{IF}_C$.

(13) $(\text{IF}_A \cap \text{IF}_B)^\prime = (\text{IF}_A)^\prime \cup (\text{IF}_B)^\prime$.

(14) $(\text{IF}_A \cup \text{IF}_B)^\prime = (\text{IF}_A)^\prime \cap (\text{IF}_B)^\prime$.

(15) $(\text{IF}_A^c)^\prime = \text{IF}_A$.

(16) For any IF sets $\text{IF}_A, \text{IF}_B$, such that $\text{IF}_A \subseteq \text{IF}_B \Rightarrow \text{IF}_B^c \subseteq \text{IF}_A^c$.

(17) For IF null set $\text{IF}_0$ and IF full set $\text{IF}_1$ we have $\text{IF}_1^c = \text{IF}_0$, $\text{IF}_0^c = \text{IF}_1$. 

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Example 4.1. Let $IFS_X^A$ describe the character of the employees in an organization with respect to the given parameters, for finding the best employee in the performance in a year. Let the set of employees under consideration be $X = \{p_1, p_2, p_3, p_4\}$. Let $E = \{e_1, e_2, e_3, e_4, e_5\}$. Suppose Mr. X has the parameter set $A = \{e_1, e_3, e_5\}$. Then

$IFS_X^A = \{IFS(e_1) = \{(p_1; 0.8, 0.2); (p_2, 0.7, 0.1), (p_3, 0.9, 0.1), (p_4, 0.7, 0.2)\};$

$IFS(e_3) = \{(p_1, 0.6, 0.2), (p_2, 0.7, 0.1), (p_3, 0.5, 0.3), (p_4, 0.3, 0.6)\};$

$IFS(e_5) = \{(p_1, 0.6, 0.2), (p_2, 0.7, 0.1), (p_3, 0.5, 0.3), (p_4, 0.3, 0.6)\}$

4.3. IFS TOPOLOGY

As an extension of the intuitionistic fuzzy soft theory which is the generalization of soft sets and fuzzy soft sets, introduced by Maji the researchers Yin, Li and Jun, (2012) have further studied and discussed the operations, algebraic structure and properties of intuitionistic fuzzy soft sets. They have derived lattice structures of intuitionistic fuzzy soft sets. The notions of various intuitionistic fuzzy soft equalities and relationships are developed and their fundamental properties have been investigated by Qin and Hong. They have also given the notion of a mapping on intuitionistic fuzzy soft families is introduced and different properties of the images and inverse images have been presented. Later, Osmanoglu, Ismail Deniz and Tokat, (2013) have studied Intuitionistic Fuzzy Soft Topology and have given the notions for subspace, separation axioms, connectedness, compactness and proved few base theorems on intuitionistic fuzzy soft topological spaces. In this section we give fundamental definitions in IFS topology, IFS subspace, IFS open cover, IFS finite subcover.

Definition 4.3.1. Let $IFS_{\tau} \subseteq IFS[\overline{X}_E]$, then $IFS_{\tau}$ is said to be a IFS topology on $X$ if the following conditions hold.

(i) $IFS_{\tau} \subseteq IFS_X$, $IFS_{\tau}$ belong to $IFS_{\tau}$.

(ii) The union of any number of IFS sets in $IFS_{\tau}$ belongs to $IFS_{\tau}$.

(iii) The intersection of any two IFS sets in $IFS_{\tau}$ belongs to $IFS_{\tau}$.
Here IFS\(\tau\) is called an Intuitionistic fuzzy soft topology over \(X\) and the ordered pair

\((\text{IFS}[\bar{X}_E], \text{IFS}\tau)\) is called a IFS topological space over \(X\).

The members of IFS\(\tau\) are said to be IFS open sets in \(X\).

An IFS set over \(X\) is said to be an IFS closed set in \(X\), if its complement belongs to IFS\(\tau\).

**Definition 4.3.2.** Let IFS\(Y_A\) be a non-empty Intuitionistic fuzzy soft subset of IFS\(X_E\) over \(Y\) where \(Y \subseteq X\), \(A \subseteq E\), \(q(e) \subseteq s(e) \forall e \in A\). Let IFS\(\tau^*\) be a collection of all intersections with IFQ\(Y_A\) of soft open sets in IFS\(\tau\). Then IFS\(Y_A\) equipped with its relative topology IFS\(\tau^*\) is called an intuitionistic fuzzy soft subspace of (IFS\([\bar{X}_E]\), IFS\(\tau\)) and is denoted by (IFS\([\bar{Y}_B]\), IFS\(\tau^*\)).

i.e. IFS\(Y_C \in \text{IFS}\tau^* \Rightarrow \text{IFS}Y_C = \text{IFS}Y_B \cap \text{IFS}X_E\)

where IFS\(X_E \in \text{IFS}\tau\), an IFS soft open set over \(X\).

Here clearly A, C \(\subseteq E\) and for all e \(\in C\), t(e) = q(e) \(\cap s(e)\).

**Definition 4.3.3.** Let (IFS\([\bar{X}_E]\), IFS\(\tau\)) be an intuitionistic fuzzy soft topological space. A class \(\{\text{IFS}(X, G_i)\}_i\) of fuzzy soft subsets of IFS\(X_A\) is said to be a fuzzy soft open cover of IFS\(X_A\) if each IFS point in IFS\(X_A\) belongs to at least one IFS\(X, G_i\).

i.e. IFS\(X_A = \cup IFS(X,G_i)\).

**Definition 4.3.4.** Let (IFS\([\bar{X}_E]\), IFS\(\tau\)) be IFS topological space and IFS\(X_A \in \text{IFS}\tau\). An intuitionistic fuzzy soft set IFS\(X_A\) is said to be intuitionistic fuzzy soft compact set if each IFS open cover of IFS\(X_A\) has a finite sub cover. Also IFS topological space (IFS\([\bar{X}_E]\), IFS\(\tau\)) is called IFS compact space if each IFS open cover of IFS\([\bar{X}_E]\) has a finite sub cover.
4.4. INTUITIONISTIC FUZZY SOFT REAL NUMBERS

Here we introduce IFS real number and its properties, IFS point and related properties and definitions are discussed.

**Definition 4.4.1.** An IFSX set \( A = \{ (x, \mu A (x), \gamma A (x)) : x \in X \} \) over the universe \( X \) is said to bounded if there is a real number \( r \) where \( 0 < r < 1 \) such that \( \mu A (x) < 1 - r \) and \( \gamma A (x) < r \).

**Definition 4.4.2.** Let \( R \) be the set of all real numbers and \( IFR \) be the collection of all non-empty bounded IF subsets of \( R \) and \( E \) taken as a set of parameters. Then a mapping \( f: E \to IFR \) is called an IFS real set. It is denoted by \([f, E]_R\). If specifically \([f, E]_R\) is a singleton IFS set, then identifying \([f, E]_R\) with the corresponding IFS element, it will be called an IFS real number and denoted by \( \tilde{r}, \tilde{s}, \tilde{t} \) etc.

**Note:** (i) \( \tilde{0} = \{(0,0,1) : 0,1 \in R\} \)

(ii) \( \tilde{1} = \{(1,1,0) : 0,1 \in R\} \)

(iii) \( \tilde{r} = \{(r, \mu_A (r), \theta_A (r)) : r \in R\} \)

(iv) \( \tilde{s} = \{(s, \mu_A (s), \theta_A (s)) : s \in R\} \) where \( A \subseteq E \) etc.

**Definition 4.4.3.** For two IFS real numbers

(i). \( \tilde{r} \geq \tilde{s} \) if \( \tilde{r} (e) \geq \tilde{s} (e) \) \( \forall e \in E \).

(ii). \( \tilde{r} \leq \tilde{s} \) if \( \tilde{r} (e) \leq \tilde{s} (e) \) \( \forall e \in E \).

(iii). \( \tilde{r} \succ \tilde{s} \) if \( \tilde{r} (e) > \tilde{s} (e) \) \( \forall e \in E \).

(iv). \( \tilde{r} \prec \tilde{s} \) if \( \tilde{r} (e) < \tilde{s} (e) \) \( \forall e \in E \).

**Definition 4.4.4.** An IFS set \( IFS_X A \) is said to be a IFS point, denoted by \( e_P x \), if there is one \( e \in E \), such that \( f(e) = \{x\} \) for some \( x \in X \) and \( f(e') = \varphi, \forall e' \in E - \{e\} \).

**Definition 4.4.5.** Two IFS points \( e_P x, e'_P y \) corresponding to the IFS sets \( IFS_X A \) and \( IFS_X B \) respectively, i.e. \( s(e) = \{x\} \); \( q(e') = \{y\} \), are said to be equal if \( e = e' \) and \( s(e) = q(e') \). i.e. \( x = y \). Thus \( e_P x \neq e'_P y \) iff \( x \neq y \) or \( e \neq e' \).
Definition 4.4.6. An IFS point $e_F x$ in an IFS set $IFS_X A$ is said to be in another IFS set $IFS_X B$, denoted by $e_F x \in IFS_X B$ if for the element $e \in A$; $s(e) \leq q(e)$.

Definition 4.4.7. Let $(IFS[\tilde{X}_E], IFS\tau)$ be an IFS topological space over $X$. Let $e_F x$ and $e_F y$ be two unequal IFS points in $IFS[\tilde{X}_E]$. Then $(IFS[\tilde{X}_E], IFS\tau)$ is said to be a

IFS $T_0$ Space if $IFS_X A$ and $IFS_X B$ are two IFS sets such that $e_F x \in IFS_X A$ but $e_F y \notin IFS_X A$ or $e_F y \in IFS_X B$ but $e_F x \notin IFS_X B$.

Definition 4.4.8. Let $(IFS[\tilde{X}_E], IFS\tau)$ be an IFS topological space over $X$. Let $e_F x$ and $e_F y$ be two unequal IFS points in $IFS[\tilde{X}_E]$. Then $(IFS[\tilde{X}_E], IFS\tau)$ is said to be a

IFS $T_1$ Space if $IFS_X A$ and $IFS_X B$ are two IFS sets such that $e_F x \in IFS_X A$ but $e_F y \notin IFS_X A$ and $e_F y \in IFS_X B$ but $e_F x \notin IFS_X B$.

Definition 4.4.9. Let $(IFS[\tilde{X}_E], IFS\tau)$ be an IFS topological space over $X$. Let $e_F x$ and $e_F y$ be two unequal IFS points in $IFS[\tilde{X}_E]$. Then $(IFS[\tilde{X}_E], IFS\tau)$ is said to be a

IFS Hausdorff space or $T_2$ Space if $IFS_X A$ and $IFS_X B$ are two IFS sets such that $e_F x \in IFS_X A$ and $e_F y \in IFS_X B$ and $IFS_X A \cap IFS_X B = \emptyset$.

Definition 4.4.10. Suppose $(IFS[\tilde{X}_E], IFS\tau)$ is a IFS topological space over $X$, $IFS_X B$ be a IFS closed set in $X$ and $e_F x \in IFS[\tilde{X}_E]$, such that $e_F x \notin IFS_X B$. If there exist IFS open sets $IFS_X A_1$ and $IFS_X A_2$ such that $e_F x \in IFS_X A_1$, $IFS_X B \subseteq IFS_X A_2$ and $IFS_X A_1 \cap IFS_X A_2 = IFS[\tilde{X}_E]$, then $(IFS[\tilde{X}_E], IFS\tau)$ is said to be a IFS regular space.

Definition 4.4.11. Let $(IFS[\tilde{X}_E], IFS\tau)$ be a IFS topological space over $X$, $IFS_X A$ and $IFS_X B$ be IFS closed sets over $X$ such that $IFS_X A \cap IFS_X B = IFS[\tilde{X}_E]$. If there exist

IFS open sets $IFS_X A_1$ and $IFS_X A_2$ such that $IFS_X A \subseteq IFS_X A_1$, $IFS_X B \subseteq IFS_X A_2$ and $IFS_X A_1 \cap IFS_X A_2 = IFS[\tilde{X}_E]$, then $(IFS[\tilde{X}_E], IFS\tau)$ is said to be a IFS normal space.

Definition 4.4.12. Let $(IFS[\tilde{X}_E], IFS\tau)$ be an IFS topological space over $X$. 86
Then IFS interior of $\text{IFS}_{A}$ denoted by $\text{IFS}_{A}^0$ is defined as the union of all IFS open sets contained in $\text{IFS}_{A}$.

**Definition 4.4.13.** Let $(\text{IFS}[X], \text{IFS}_\tau)$ be an IFS topological space over $X$. Then IFS closure of $\text{IFS}_{A}$, denoted by $\overline{\text{IFS}_{A}}$, is defined as the intersection of all IFS closed supersets of $\text{IFS}_{A}$.

**Proposition 4.2** For any IFS set, the IFS closure, $\overline{\text{IFS}_{A}}$, and IFS interior $\text{IFS}_{A}^0$ we have the following properties.

$$(\overline{\text{IFS}_{A}})^\prime = ((\text{IFS}_{A})^\prime)^0.$$  

$\overline{\text{IFS}_{A}} = ((\text{IFS}_{A})^0)^\prime.$

$\text{IFS}_{A}^0 \subseteq \text{IFS}_{A}.$

$\text{IFS}_{A} \subseteq \overline{\text{IFS}_{A}}.$

If $\text{IFS}_{A} \subseteq \text{IFQ}_{B}$ then $\text{IFS}_{A}^0 \subseteq \text{IFS}_{B}^0.$

If $\text{IFS}_{A} \subseteq \text{IFQ}_{B}$ then $\overline{\text{IFS}_{A}} \subseteq \overline{\text{IFS}_{B}}.$

$((\text{IFS}_{A})^0)^0 = \text{IFS}_{A}.$

$\overline{\text{IFS}_{A}} = \text{IFS}_{A}.$

The largest IFS open subset of $\text{IFS}_{A}$ is its interior, $\text{IFS}_{A}^0$.

The smallest IFS closed superset of $\text{IFS}_{A}$ is its closure $\overline{\text{IFS}_{A}}$.

**Definition 4.4.14** Let $(\text{IFS}[X], \text{IFS}_\tau)$ be an IFS topological space and $\text{IFS}_{A}$ be an IFS set in $(\text{IFS}[X], \text{IFS}_\tau)$. An IFS set $\text{IFS}_{B}$ in $(\text{IFS}[X], \text{IFS}_\tau)$ is said to be an IFS neighbourhood of $\text{IFS}_{A}$ if there exists an IFS open set $\text{IFS}_{C} \in \text{IFS}_\tau$ such that $\text{IFS}_{A} \subseteq \text{IFS}_{C} \subseteq \text{IFS}_{B}$.
**Definition 4.4.15** Let \((\text{IFS}[\bar{X}_E], \text{IFS}\tau)\) and \((\text{IFS}[\bar{X}_E], \text{IFS}\tau^*)\) be two IFS topological spaces, \(\tilde{f}: (\text{IFS}[\bar{X}_E], \text{IFS}\tau) \rightarrow (\text{IFS}[\bar{X}_E], \text{IFS}\tau^*)\) be a mapping. For each IFS neighborhood \(\text{IFS}X_E\) of \(\tilde{f}(\text{IFS}[\bar{X}_c])\), for some \(e \in E\) if there exists an IFS neighborhood \(\text{IFS}X_A\) of \(\text{IFS}[\bar{X}_c]\), such that \(\tilde{f}(\text{IFS}X_A) \subseteq \text{IFS}X_E\), then \(\tilde{f}\) is said to be IFS continuous mapping at \(\text{IFS}X_e\).

Let \(\text{IFS}\bar{X}_X\) be the IFS absolute set i.e., \(s(e) = 1\) \(\forall e \in E\), \(\text{IFS}A = 1\) and \(\text{IFS}(\text{IFS}X_E)\) be the collection of all IFS points of \(\text{IFS}[\bar{X}_E]\) and \(\text{IFR}(E)\) denote the set of all non negative IFS real numbers.

**Definition 4.4.16** A mapping \(\bar{d}: \text{IFS}(\text{IFS}X_E) \times \text{IFS}(\text{IFS}X_E) \rightarrow \text{IFR}(E)\), is said to be IFS metric on the IFS set \(\text{IFS}[\bar{X}_E]\) if \(\bar{d}\) satisfies the following conditions.

(i). \(\bar{d}(e_Fx, e_Fy) \geq 0\),

(ii). \(\bar{d}(e_Fx, e_Fy) = 0\) if and only if \(e_Fx = e_Fy\).

(iii). \(\bar{d}(e_Fx, e_Fy) = \bar{d}(e_Fy, e_Fx)\)

(iv). \(\bar{d}(e_Fx, e_Fy) \leq \bar{d}(e_Fx, e_Fz) + \bar{d}(e_Fz, e_Fy)\) for all \(e_Fx, e_Fy, e_Fz \in \text{IFS}(\text{IFS}X_E)\).

The IFS set \(\text{IFS}(\text{IFS}X_E)\) with the IFS metric \(\bar{d}\) on \(\text{IFS}(\text{IFS}X_E)\) is called a IFS metric space and denoted by \(\text{IFS}(\text{IFS}X_E, \bar{d}, E)\).

**Definition 4.4.17** Let \(\text{IFS}(\text{IFS}X_E)\) be a IFS metric space and \(\bar{\epsilon}\) be a non negative IFS real number.

Then \(\text{IFS}(\text{IFS}X_E, \bar{\epsilon}) = \{ e_F' \bar{y} \in \text{IFS}(\text{IFS}X_E): \bar{d}(e_Fx, e_F' \bar{y}) \leq \bar{\epsilon} \} \subseteq \text{IFS}(\text{IFS}X_E)\) is called the IFS open ball with center \(e_Fx\) and radius \(\bar{\epsilon}\) and

\(\text{IFS}(e_Fx, \bar{\epsilon}) = \{ (e_Fx \in \text{IFS}(\text{IFS}X_E): \bar{d}(e_Fx, e_F' \bar{y}) \leq \bar{\epsilon} \} \subseteq \text{IFS}(\text{IFS}X_E)\) is called the IFS closed ball with center \(e_Fx\) and radius \(\bar{\epsilon}\).
Definition 4.4.18 Let \( \{e_Fx_{\alpha,n}\}_n \) be a sequence of IFS points in a IFS metric space \( \text{IFSP}(\tilde{X}_E, \tilde{d}, E) \). The sequence \( \{e_Fx_{\alpha,n}\}_n \) is said to be convergent in \( \text{IFSP}(\tilde{X}_E, \tilde{d}, E) \) if there is a IFS point \( e_Fy_\omega \in \text{IFSP}(\tilde{X}_E) \) such that

\[
\tilde{d}(e_Fx_{\alpha,n}, e_Fy_\omega) \to 0 \text{ as } n \to \infty.
\]

This means for every \( \tilde{\epsilon} \gs 0 \), chosen arbitrarily, there exists a natural number \( N = N(\tilde{\epsilon}) \), such that \( 0 \ls \tilde{d}(e_Fx_{\alpha, n}, e_Fy_\omega) < \tilde{\epsilon} \), whenever \( n > N \).

Theorem 4.1 Limit of a sequence in a IFS metric space, if exists is unique. //

4.5. IFS SEQUENTIALLY COMPACT SPACE

Here we give the notion for IFS Cauchy sequence, IFS sequentially compact space and related theorems are proved.

Definition 4.5.1. A sequence \( \{e_Fx_{\alpha,n}\}_n \) of IFS points in \( \text{IFSP}(\tilde{X}_E, \tilde{d}, E) \) is considered as a cauchy sequence in \( \text{IFSP}(\tilde{X}_E) \) if corresponding to every \( \tilde{\epsilon} \gs 0 \), there exists \( n \in \mathbb{N} \) such that \( \tilde{d}(e_Fx_{\alpha,i}, e_Fx_{\alpha,j}) \ls \tilde{\epsilon} \), \( \forall i, j \geq m \), i.e., \( \tilde{d}(e_Fx_{\alpha,i}, e_Fx_{\alpha,j}) \to 0 \) as \( i,j \to \infty \).

Definition 4.5.2. Let \( \text{IFSP}(\tilde{X}_E, \tilde{d}, E) \) be a IFS metric space. \( \text{IFSP}(\tilde{X}_E, \tilde{d}, E) \) is called IFS sequentially compact space if every IFS sequence has a IFS subsequence that converges in \( \text{IFSP}(\tilde{X}_E) \).

i.e., Suppose \( \{e_Fx_{\alpha,n}\}_n \) is a IFS sequence in \( \text{IFSP}(\tilde{X}_E) \) then there exists a subsequence \( \{e_Fx_{\alpha,n,k}\}_n \) from \( \{e_Fx_{\alpha,n}\}_n \) such that \( \lim_{n \to \infty} e_Fx_{\alpha,n,k} = e_Fx \).

Theorem 4.2 A IFS metric space \( \text{IFSP}(\tilde{X}_E, \tilde{d}, E) \) is IFS sequential compact sapce if and only if every infinite IFS subset of \( \text{IFSP}(\tilde{X}_E) \), has a limit point.

Proof. Let \( \text{IFSP}(\tilde{X}_E, \tilde{d}, E) \) be a IFS metric space.

Let \( \text{IFSP}(\tilde{X}_E) \) be a IFS sequentially compact space.

Now we show that every infinite IFS subset of \( \text{IFSP}(\tilde{X}_E) \), has a limit point.
Let \( \text{IFSP}(\tilde{X}_\alpha) \) be an infinite subset of \( \text{IFSP}(\tilde{X}_E) \).

Since \( \text{IFSP}(\tilde{X}_\alpha) \) is infinite, a sequence \( \{e_{P}x_{\alpha,n}\}_n \) of distinct points can be extracted from \( \text{IFSP}(\tilde{X}_\alpha) \).

By definition of IFS sequentially compact space has a convergent subsequence.

i.e., There exists \( \{e_{P}x_{\alpha,n,k}\}_n \) such that \( \lim_{n \to \infty} e_{P}x_{\alpha,n,k} = e_{P}x \).

And clearly \( e_{P}x \) is a limit point of \( \{e_{P}x_{\alpha,n}\}_n \).

Therefore is a limit point of the set \( \text{IFSP}(\tilde{X}_\alpha) \) of IFS points of \( \{e_{P}x_{\alpha,n}\}_n \).

Conversely suppose that every infinite IF soft subset of \( \text{IFSP}(\tilde{X}_E) \), has a limit point.

Now we show that \( (\text{IFSP}(\tilde{X}_E), \tilde{d}, E) \) is IFS sequential compact space.

Consider an arbitrary IFS sequence \( \{e_{P}x_{\alpha,n}\}_n \) in \( \text{IFSP}(\tilde{X}_E) \).

If \( \{e_{P}x_{\alpha,n}\}_n \) has a point which is repeated infinite times, then possesses a constant convergent subsequence .

If \( \{e_{P}x_{\alpha,n}\}_n \) has no IFS point which is repeated then the set \( \text{IFSP}(\tilde{X}_\alpha) \) of infinite IFS points of this sequence is infinite.

By hypothesis \( \text{IFSP}(\tilde{X}_\alpha) \) contains a limit point in \( \text{IFSP}(\tilde{X}_E) \), say \( e_{P}x \).

And easily we can find a subsequence of \( \{e_{P}x_{\alpha,n}\}_n \) which converges to \( e_{P}x \).

Hence \( \text{IFSP}(\tilde{X}_E, \tilde{d}, E) \) is IFS sequentially compact space. \( // \)

**Theorem 4.3.** Let \( (\text{IFSP}(\tilde{X}_E), \tilde{d}, E) \) be IFS compact space. Then every infinite subset of \( \text{IFSP}(\tilde{X}_E) \) has a limit point.

**Proof.** Suppose \( (\text{IFSP}(\tilde{X}_E), \tilde{d}, E) \) is a IFS compact space.

Let \( \text{IFSP}(\tilde{X}_\alpha) \) be an infinite subset of \( \text{IFSP}(\tilde{X}_E) \).
In a contrary way suppose that \( \text{IFSP}(\tilde{X}_\lambda) \) has no limit point in \( \lambda \).

Let \( e_\rho x \in \text{IFSP}(\tilde{X}_E) \)

Then \( e_\rho x \) is not a limit point of \( \text{IFSP}(\tilde{X}_\lambda) \).

So there a IFS neighborhood \( \text{IFSN}(e_\rho x, \tilde{\varepsilon}) \) such that \( \text{IFSN}(e_\rho x, \tilde{\varepsilon}) \not\subseteq \text{IFSP}(\tilde{X}_\lambda) \)

This can be done for each IFS point \( e_\rho x \) in \( \text{IFSP}(\tilde{X}_E) \).

Consider the class \{ \( \text{IFSN}(e_\rho x, \tilde{\varepsilon})/ e_\rho x \in \text{IFSP}(\tilde{X}_E) \) \}.

Trivially \( \text{IFSP}(\tilde{X}_E) = \bigcup \text{IFSN}(e_\rho x, \tilde{\varepsilon}) \).

Since each is IFS open set, \{ \( \text{IFSN}(e_\rho x, \tilde{\varepsilon})/ e_\rho x \in \text{IFSP}(\tilde{X}_E) \) \} forms an IFS open cover for \( \text{IFSP}(\tilde{X}_E) \) and this open cover contains a finite sub cover (Since \( \text{IFSP}(\tilde{X}_E) \) is a IFS compact space).

\[
\therefore \text{IFSP}(\tilde{X}_E) = \bigcup \text{IFSN}(e_\rho x 1, \tilde{\varepsilon}) \bigcup \text{IFSN}(e_\rho x 2, \tilde{\varepsilon}) \bigcup \text{IFSN}(e_\rho x 3, \tilde{\varepsilon})... \bigcup \text{IFSN}(e_\rho x k, \tilde{\varepsilon}),
\]

finite union.

But \( \text{IFSP}(\tilde{X}_\lambda) \not\subseteq \text{IFSP}(\tilde{X}_E) \).

Which is not possible as \( \text{IFSP}(\tilde{X}_\lambda) \) is infinite.

This is a contradiction.

Hence the theorem is proved. \( // \)

**Definition 4.5.3.** Suppose \( \text{IFSP}(X_E, \tilde{d}, E) \) is a IFS metric space and

\( \text{IFSP}(\tilde{X}_\lambda) \subset \subset \text{IFSP}(\tilde{X}_E) \).

Then diameter of \( \text{IFSP}(\tilde{X}_\lambda) \) is denoted by \( \text{Diam}(\text{IFSP}(\tilde{X}_\lambda)) \) or \( D(\text{IFSP}(\tilde{X}_\lambda)) \)

and is defined as \( \text{Diam}(\text{IFSP}(\tilde{X}_\lambda)) = \text{Sup} \{ \tilde{d}(e_\rho x, e_\rho y) : e_\rho x, e_\rho y \in \text{IFSP}(\tilde{X}_\lambda) \} \).
**Definition 4.5.4.** Suppose \( \text{IFSP}(X_{\text{E}}), \tilde{d}, \text{E}) \) is a IFS topological space and

\[
\Omega = \{ \text{IFSP}(\tilde{B}_n) : n \in \Delta \}
\]

be a IFS open cover for \( \text{IFSP}(X_{\text{E}}) \).

Then a IFS real number \( \tilde{r} > 0 \) is called a Leabegue’s IFS number (\( \text{LIFSN} \) in short) if every subset \( \text{IFSP}(\tilde{X}_\lambda) \) of \( \text{IFSP}(X_{\text{E}}) \) with diameter less than \( \tilde{r} \) and \( \text{IFSP}(\tilde{X}_\lambda) \) is contained in atleast one member of the IFS open cover \( \Omega \).

**Definition 4.5.5.** A subset \( \text{IFSP}(\tilde{X}_\lambda) \) of \( \text{IFSP}(X_{\text{E}}) \) is said to be bounded if its diameter is finite.

In particular a IFS metric space \( \text{IFSP}(\tilde{X}_{\text{E}}), \tilde{d}, \text{E}) \) is bounded if \( \text{Diam}(\text{IFSP}(X_{\text{E}})) < \infty \).

**Lemma 4.1.** In a IFS sequentially compact space every open cover has a IFS Lebesgue’s number.

**Proof.** Let \( \text{IFSP}(\tilde{X}_{\text{E}}), \tilde{d}, \text{E}) \) be a IFS sequentially compact space.

Let \( \Omega = \{ \text{IFSP}(\tilde{B}_n) : n \in \Delta \} \) be a IFS open cover for \( \text{IFSP}(X_{\text{E}}) \).

We show that this open cover has a \( \text{IFSL} \) number.

In a contrary way suppose that \( \Omega \) has no \( \text{IFSL} \) number.

Then \( \exists \ \text{IFSP}(\tilde{C}_n) \) from \( \text{IFSP}(\tilde{X}_{\text{E}}) \) such that \( \text{Diam}(\text{IFSP}(\tilde{C}_n)) < \frac{1}{n} \) and \( \text{IFSP}(\tilde{C}_n) \not\subseteq \text{IFSP}(\tilde{B}_n) \) for any \( n \) \hspace{2cm} (4.1)

Now choose IFS point \( e_{\tilde{F}}x_n \) from \( \text{IFSP}(\tilde{C}_n) \) and construct the sequence \( \{ e_{\tilde{F}}x_n \} \).

Since \( \text{IFSP}(\tilde{X}_{\text{E}}) \) is IFS sequential compact, the above sequence has a convergent sub sequence, say \( \{ e_{\tilde{F}}x_{n,k} \} \).

\[ \Rightarrow \{ e_{\tilde{F}}x_{n,k} \} \rightarrow e_{\tilde{F}}x \text{ as } n \rightarrow \infty. \]
\( \therefore e_Fx \) belongs to atleast one member of \( \Omega \), say \( \text{IFSP}(\overline{B_{n_0}}) \). Since \( \text{IFSP}(B_{n_0}) \) is IS soft open set there is a IFS open ball \( \text{IFSN}(e_Fx, \varepsilon) \) with center at \( e_Fx \) such that \( \text{IFSN}(e_Fx, \varepsilon) \subseteq \text{IFSP}(B_{n_0}) \) \hspace{1cm} (4.2)

Now we consider \( \text{IFSN}(e_Fx, \frac{\varepsilon}{2}) \), a concentric circle of \( \text{IFSN}(e_Fx, \varepsilon) \).

Clearly \( \text{IFSN}(e_Fx, \frac{\varepsilon}{2}) \subseteq \text{IFSN}(e_Fx, \varepsilon) \) \hspace{1cm} (4.3)

\( \therefore \{e_Fx_{n,k}\} \) converges to \( e_Fx \), \( \exists n_0 \in \mathbb{N} \) such that \( nk \geq n_0 \), \( e_Fx_{n,k} \subseteq \text{IFSN}(e_Fx, \frac{\varepsilon}{2}) \).

Now choose an integer \( k_0 \) such that \( \frac{1}{k_0} < \frac{\varepsilon}{2} \).

\[ \Rightarrow \text{Diam}(\text{IFSP}(\overline{C_{k_0}})) \geq \frac{1}{k_0} < \frac{\varepsilon}{2} \]

\[ \Rightarrow (\text{IFSP}(\overline{C_{k_0}}) \subseteq \text{IFSN}(e_Fx, \frac{\varepsilon}{2}) \] \hspace{1cm} (4.4)

\[ \Rightarrow (\text{IFSP}(\overline{C_{k_0}}) \subseteq \text{IFSP}(\overline{B_{n_0}})) \text{ from equations (4.2), (4.3) and (4.3).} \]

Which contradicts equation (4.1).

Hence the Lemma is proved. //

4.6. TOTALLY BOUNDED IFS SET

In this section we introduce IFS \( \varepsilon \)- net, totally bounded IFS metric space and give the equivalent conditions between IFS compact space, IFS sequentially compact space and totally bounded IFS metric space.

**Definition 4.6.1.** Suppose \((\text{IFSP}(\overline{X_E}, \overline{d}, E))\) is a IFS metric space, then a subset \((IFSP)(\overline{A})\) of \(\text{IFSP}(\overline{X_E})\) is said to be IFS \(\varepsilon\)- net for \(\text{IFSP}(\overline{X_E})\) if \(\text{IFSP}(\overline{X_A})\) is finite and \(\text{IFSP}(\overline{X_E}) \subseteq \bigcup_{e_Fa \in IFSP(\overline{A})} \text{IFSN}(e_Fx, \varepsilon) \).

i.e., \(\text{IFSP}(\overline{X_A}) = \{ e_Fa1, e_Fa2, \ldots, e_{Fan} \} \)
\[ \Rightarrow \text{IFSP}(\vec{X}_E) = \text{IFS}(e_p a_1, \vec{E}) \cup \text{IFS}(e_p a_2, \vec{E}) \cup \ldots \cup \text{IFS}(e_p a_n, \vec{E}). \]

**Definition 4.6.2.** A IFS metric space IFSP(\( \vec{X}_E \), \( \vec{d} \), \( E \)) is said to be totally bounded IFS metric space if it has IFS \( \vec{E} \)- net.

**Theorem 4.4** If IFSP(\( \vec{X}_E \)) is totally bounded IFS metric space, then it is bounded.

**Proof.** Assume that IFSP(\( \vec{X}_E \)) is totally bounded IFS metric space.

Then it possesses a \( \vec{I} \)-net.

Let IFSP(\( \vec{X}_E \)) = \{ e_p a_1, e_p a_2, \ldots, e_p a_n \}, finite set.

\[ \Rightarrow \text{IFSP}(\vec{X}_E) = \text{IFS}(e_p a_1, \vec{E}) \cup \text{IFS}(e_p a_2, \vec{E}) \cup \ldots \cup \text{IFS}(e_p a_n, \vec{E}). \]

For any \( e_p x, e_p y \in \text{IFSP}(\vec{X}_E), e_p x \in \text{IFS}(e_p a_i, \vec{I}), e_p y \in \text{IFS}(e_p a_j, \vec{I}) \) for some \( i, j \in \{1, 2, \ldots, n\} \).

\[ d(e_p x, e_p a_i) \preceq \vec{I} \text{ and } d(e_p y, e_p a_j) \preceq \vec{I}. \]

Now \( d(e_p x, e_p y) \preceq d(e_p x, e_p a_i) + d(e_p a_i, e_p a_j) + d(e_p y, e_p a_j) \)

\[ \preceq \vec{2} + d(e_p a_i, e_p a_j) \]

\[ \preceq \vec{2} + \text{Diam(IFS}(\vec{X}_E)) \preceq \infty (\text{Since IFSP}(\vec{X}_A) \text{ is finite}). \]

This is true for all \( e_p x, e_p y \in \text{IFSP}(\vec{X}_E). \)

Hence IFSP(\( \vec{X}_E \)) is bounded. //

**Theorem 4.5** A IFS sequentially compact space is totally bounded IFS metric space.

**Proof.** Assume that IFSP(\( \vec{X}_E \)) is IFS sequentially compact space.

We show that IFSP(\( \vec{X}_E \)) is totally bounded IFS metric space.
Let \( e_{f}a1 \in \text{IFSP}(\tilde{X}_{E}) \) and consider the IFS neighborhood of \( e_{f}a1 \),

say \( \text{IFSN}(e_{f}a1, \xi) \) for \( \xi > 0 \).

If \( \text{IFSP}(\tilde{X}_{E}) \subseteq \text{IFSN}(e_{f}a1, \xi) \), then \( \text{IFSP}(\tilde{X}_{E}) = \{ e_{f}a1 \} \) forms a IFS \( \xi \)-net for \( \text{IFSP}(\tilde{X}_{E}) \).

If \( \text{IFSP}(\tilde{X}_{E}) \not\subseteq \text{IFSN}(e_{f}a1, \xi) \) then we can find \( e_{f}a2 \in \text{IFSP}(\tilde{X}_{E}) - \text{IFSN}(e_{f}a1, \xi) \).

And consider \( \text{IFSN}(e_{f}a1, \xi) \bar{\cup} \text{IFSN}(e_{f}a2, \xi) \).

If \( \text{IFSP}(\tilde{X}_{E}) \not\subseteq \text{IFSN}(e_{f}a1, \xi) \bar{\cup} \text{IFSN}(e_{f}a2, \xi) \), then we can find \( e_{f}a3 \in \text{IFSP}(\tilde{X}_{E}) - \{ \text{IFSN}(e_{f}a1, \xi) \bar{\cup} \text{IFSN}(e_{f}a2, \xi) \} \).

If the process continued indefinitely we get a sequence \( \{ e_{f}a_{n,k} \} \) in \( \text{IFSP}(\tilde{X}_{E}) \) such that \( d(e_{f}a_{i}, e_{f}a_{j}) \geq \xi \).

Which shows that the sequence \( \{ e_{f}a_{n,k} \} \) has no convergent subsequence.

This is a contradiction to the hypothesis.

So this process must terminate after a finite stage.

& we must have \( \text{IFSP}(\tilde{X}_{E}) \subseteq \text{IFSN}(e_{f}a1, \xi) \bar{\cup} \text{IFSN}(e_{f}a2, \xi) \bar{\cup} \ldots \text{IFSN}(e_{f}an, \xi) \) for some \( n \).

Hence \( \text{IFSP}(\tilde{X}_{E}) = \{ e_{f}a1, e_{f}a2, \ldots e_{f}an \} \) forms a IFS \( \xi \)-net.

Hence \( \text{IFSP}(\tilde{X}_{E}) \) is totally bounded IFS metric space. //

**Theorem 4.6** A IFS sequentially compact space is IFS compact.

**Proof.** Assume that \( \text{IFSP}(\tilde{X}_{E}) \) is IFS sequentially compact space.
We show that IFSP($\bar{X}_E$) is IFS compact space.

Let \{IFSP($A_\alpha$)}$_{\alpha \in \Delta}$ be a IFS open cover for IFSP($\bar{X}_E$).

Then by Lemma 4.1, this IFS open cover has a $LIFSN$ number, Say $\bar{\alpha}$.

\begin{align*}
\therefore & \text{For any IFSP}($\bar{X}_\Lambda$) $\subseteq$ IFSP($\bar{X}_E$) with \(\text{Diam(IFSP}($\bar{X}_\Lambda$)) \leq \bar{\alpha}\).

\text{IFSP}($\bar{X}_\Lambda$) & \text{lies in exactly one member of } \{(IFSP($A_\alpha$))$_{\alpha \in \Delta}$.
\end{align*}

(4.5)

Let $\bar{\varepsilon} = \frac{\bar{\alpha}}{2} \geq \bar{\delta}$.

Since IFSP($\bar{X}_E$) is IFS sequentially compact, it is totally bounded IFS metric space.

\begin{align*}
\therefore & \text{it has a IFS } \bar{\varepsilon}\text{- net.}

& \text{IFSP}($\bar{X}_\Lambda$) = \{ e_p a1, e_p a2, \ldots, e_p a^n \} be a IFS $\bar{\varepsilon}$- net.

\therefore & \text{IFSP}($\bar{X}_E$) $\subseteq$ IFSN($e_p a1, \bar{\varepsilon}$) $\sim$ IFSN($e_p a2, \bar{\varepsilon}$) $\ldots$ $\sim$ IFSN($e_p a^n, \bar{\varepsilon}$) \quad (4.6)

\Rightarrow & \text{Diam(IFSN}($e_p xai, \bar{\varepsilon}$)) $\leq 2\bar{\varepsilon} \leq \bar{\alpha}$ for $i = 1, 2, \ldots, n$.

\Rightarrow & \text{By (4.5) there is exactly one } a, i \in \Delta \text{ such that IFSN}($e_p a i, \bar{\varepsilon}$) $\subseteq$ IFSN($A_{a,i}$) for $i = 1, 2, \ldots, n$ \quad (4.7)

\text{From (4.6) and (4.7) IFSP}($\bar{X}_E$) $\subseteq$ IFSN($A_{a,1}$) $\sim$ IFSN($A_{a,2}$) $\ldots$ $\sim$ IFSN($A_{a,n}$).

\Rightarrow \{(IFSP($A_{a,i}$), (IFSP($A_{a,2}$), \ldots, (IFSP($A_{a,n}$)) forms a IFS open cover for IFSP($\bar{X}_E$).

Hence IFSP($\bar{X}_E$) is IFS compact space. //

**Definition 4.6.3.** Let (IFSP($\bar{X}_E$), $\bar{d}_1$, E) and (IFSP($\bar{Y}$), $\bar{d}_2$, E) be two IFS metric spaces. A mapping $\bar{f}$ : (IFSP($\bar{X}_E$), $\bar{d}_1$, E) $\rightarrow$ (IFSP($\bar{Y}$), $\bar{d}_2$, E) is said to be IFS uniformly continuous on IFSP($\bar{X}_E$) if for each $\bar{\varepsilon} \geq 0 \exists \bar{\delta} \geq 0$ such that $\bar{d}_1(e_p x, e_p y) \leq \bar{\delta}$ \Rightarrow $\bar{d}_2(\bar{f}(e_p x), \bar{f}(e_p y)) \leq \bar{\varepsilon}$. This is true for every $e_p x$, $e_p y \in$ IFSP($\bar{X}_E$).
**Theorem 4.7** Suppose \((\text{IFS}(\tilde{X}_E), \tilde{d}, E)\) is a IFS compact space and \(\tilde{f} : (\text{IFS}(\tilde{X}_E), \tilde{d} \tilde{1}, E) \rightarrow (\text{IFS}(\tilde{Y}), \tilde{d} \tilde{2}, E)\) is IFS continuous where \((\text{IFS}(\tilde{Y}), \tilde{d} \tilde{2}, E)\) is any arbitrary IFS topological space then \(\tilde{f}\) is IFS uniformly continuous.

**Proof.** Given \((\text{IFS}(\tilde{X}_E), \tilde{d}, E)\) is a IFS compact space and \(\tilde{f} : (\text{IFS}(\tilde{X}_E), \tilde{d} \tilde{1}, E) \rightarrow (\text{IFS}(\tilde{Y}), \tilde{d} \tilde{2}, E)\) is IFS continuous.

Now we prove that \(\tilde{f}\) is IFS uniformly continuous.

For given \(\varepsilon > 0\) and for any \(e_p x \in \text{IFS}(\tilde{X}_E)\).

Let \(\text{IFS}(\tilde{V}x) = \text{IFSN}(\tilde{f}(e_p x), \tilde{2})\)

Since \(\tilde{f}\) is IFS continuous the inverse image \(\tilde{f}^{-1}(\text{IFS}(\tilde{V}x))\) of \(\text{IFS}(\tilde{V}x)\) is in \(\text{IFS}(\tilde{X}_E)\).

Let \(\text{IFS}(\tilde{G}x) = \tilde{f}^{-1}(\text{IFS}(\tilde{V}x))\).

\(\Rightarrow\) For any \(e_p x \in \text{IFS}(\tilde{X}_E)\) we have a IFS open set \(\text{IFS}(\tilde{G}x)\) in \(\text{IFS}(\tilde{X}_E)\).

So \(\{\text{IFS}(\tilde{G}x)\}_{e_p x \in \text{IFS}(\tilde{X})}\) forms a IFS open cover for \(\text{IFS}(\tilde{X}_E)\).

And \(\text{IFS}(\tilde{X}_E)\) is IFS compact

\(\Rightarrow\) \(\text{IFS}(\tilde{X}_E)\) is IFS sequentially compact space by Theorems 4.2 and 4.3.

\(\Rightarrow\) The IFS open cover has a IFSL number by Lemma 4.1.

Let it be \(\delta\).

\(\therefore\) For any \(\text{IFS}(\tilde{X}_A) \subseteq \text{IFS}(\tilde{X}_E)\) with \(\text{Diam}(\text{IFS}(\tilde{X}_A)) < \delta\).

\(\text{IFS}(\tilde{X}_A)\) lies exactly in one \(\text{IFS}(\tilde{G}x)\) such that \(e_p x \not\in \text{IFS}(\tilde{X}_E)\).

Let \(e_p x, e_p y \in \text{IFS}(\tilde{X}_E)\) with \(\tilde{d} 1(e_p x, e_p y) < \delta\).

Then the two elements set \(\text{IFS}(\tilde{X}_A) = \{ e_p x, e_p y \}\) with \(\text{Diam}(\text{IFS}(\tilde{X}_A)) < \delta\).
⇒ IFSP($\bar{X}_A$) lies exactly in one IFSP($G\bar{x}0$).
⇒ IFSP($\bar{X}_A$) ⊆ IFSP($G\bar{x}0$).
⇒ $x$IFSP$_A, e_F, y \in$ IFSP($G\bar{x}0$).
⇒ $e_Fx, e_Fy \in \tilde{f}^{-1}(IFSP(V\bar{x}0)).$
⇒ $\tilde{f}(e_Fx), \tilde{f}(e_Fy) \in$ IFSP($V\bar{x}0$).
⇒ $\tilde{f}(e_Fx), \tilde{f}(e_Fy) \in$ IFSN($\tilde{f}(e_Fx0), \frac{\varepsilon}{2}$).
⇒ $\tilde{d}^2(\tilde{f}(e_Fx), \tilde{f}(e_Fx0)) \lt \frac{\varepsilon}{2}$
and $\tilde{d}^2(\tilde{f}(e_Fy), \tilde{f}(e_Fx0)) \lt \frac{\varepsilon}{2}$.

Now $\tilde{d}^2(\tilde{f}(e_Fx), \tilde{f}(e_Fy))$
\[\lt \tilde{d}^2((\tilde{f}(e_Fx), \tilde{f}(e_Fx0)) + \tilde{d}^2(\tilde{f}(e_Fx0), \tilde{f}(e_Fy)) \lt \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\]

Hence the theorem. //

4.7 CONCLUSION

We have given the notion for Intuitionistic fuzzy soft (IFS) Real number, IFS point, corresponding basic definitions such as IFS metric and IFS Metric space. Then we have defined a convergent sequence of IFS metric spaces, Cauchy sequence in IFS metric spaces, IFS compact metric space and also IFS sequentially compact metric space. By using these concepts, we have proved some important theorems. Later the diameter of an IFS metric space and its Lebesgue’s number are discussed. The boundedness of a IFS is verified by using it. Finally with the help of IFS $\varepsilon$- net, the conditions for totally bounded of a IFS metric space are established and also we have proved that every IFS sequentially compact space is IFS totally bounded IFS metric space and also IFS compact space. At the end we have introduced the uniformly continuous IFS mapping and proved that an IFS continuous function on an IFS compact space results as an IFS uniformly continuous mapping.