Chapter 3

χ* = χ Of Some Special Graphs

3.1 Triangle Free Planar Family With χ* = χ

Definition 3.1: A wheel $W_k = (v, u_1, ..., u_k)$ is a graph consisting of a k-cycle $C_k = (u_1, u_2, ..., u_k)$ with a center vertex v adjacent to all vertices of $C_k$. An edge incident to the center is called spoke; an edge not incident to the center is called rim.

We say that a graph G is triangle-free if G contains no $K_3$ as its subgraph. If a graph contains a triangle then its star chromatic number and its chromatic number are at least 3. The following class of graphs was one of the first examples of triangle-free graphs with star chromatic number 3.

Definition 3.2: Let $W'_n$ denote the triangle free planar graph obtained from $W_n = (v, w_1, w_2, ..., w_n)$ by subdividing each spoke $vw_i$ by $w'_i$ ($i = 1, 2, ..., n$). From the construction, $W'_n$ has $(2n + 1)$ vertices and $3n$ edges. The graph $W'_3$, with each face being a $C_5$, is shown in Figure 3.1.
If \( n \) is even, then it is easily shown that \( W_2^* \) and \( C_5 \) are homomorphically equivalent. It follows that \( \chi^*(W_2^*) = \chi^*(C_5) = 5/2 \), for an even integer \( n \geq 4 \) which is smaller than the chromatic number 3. In the following, we will show that \( \chi^*(W_{2n+1}^*) = \chi(W_{2n+1}^*) = 3 \) for \( n \geq 2 \). We begin by proving a series of lemmas.

**Lemma 3.3:** Let \( r = \lfloor (4n - 2) / 3 \rfloor \), where \( n \geq 2 \). \( W_{2n+1}^* \) is not \((3r + 5, r + 2)\)-colorable.

**Proof:** Suppose, by the contrary, \( W_{2n+1}^* \) is \((3r + 5, r + 2)\)-colorable.
We let a mapping \( \phi: V(W'_{2n+1}) \rightarrow [3r + 5] \) be a \((3r + 5, r + 2)\)-coloring of \(W'_{2n+1}\). Without loss of generality, we assume that \( \phi(v) = 0 \). Then, it is immediate to see that

\[
\phi(w'_i) \in \{r + 2, r + 3, \ldots, 2r + 3\}, \text{ for } i \in [2n+1].
\]

For any vertex \( w_i \) \((i \in [2n + 1])\), we note that \( \phi(w_i) \neq 0 \). Otherwise, contracting the two vertices \( v \) and \( w_i \) (into a new vertex \( v \diamond w_i \)) induces a triangle \((v \diamond w_i, w_{i+1}, w'_{i+1})\) whose star chromatic number is strictly less than 3. This is a contradiction. Moreover, \( \phi(w_i) \not\in \{r + 2, r + 3, \ldots, 2r + 3\} \).

Otherwise, adding a new edge \( vw_i \) between two vertices \( v \) and \( w_i \) induces a triangle \((w_i, w'_i, v)\) whose star chromatic number is strictly less than 3, a contradiction again. Therefore, for \( i \in [2n + 1] \), we have that

\[
\phi(w_i) \in \{1, 2, \ldots, r+1, 2r+4, 2r+5, \ldots, 3r+4\}. \quad \text{We let } g_1 = \{1, 2, \ldots, r+1\} \text{ and } g_2 = \{2r+4, 2r+5, \ldots, 3r+4\}. \quad \text{Since } |g_1| = |g_2| = r + 1 \text{ and } w_i \text{ is adjacent to } w_{i+1},
\]

then \( w_i \) and \( w_{i+1} \) cannot be in \( g_1 \) and \( g_2 \) simultaneously. Without loss of generality, we assume that \( w_0 \in g_1 \), then \( w_1 \in g_2 \). By the same token, we can easily show that \( w_{2i} \in g_1 \), for every \( i \in [n + 1] \) and \( w_{2i+1} \in g_2 \) for each \( i \in [n] \).

Because \( w_0 \) and \( w_{2n} \) are adjacent, \( w_0 \in g_1 \) forces \( w_{2n} \in g_2 \). This contradicts \( w_{2n} \in g_1 \). Therefore \( W'_{2n+1} \) is not \((3r + 5, r + 2)\)-colorable.
**Lemma 3.4:** For any integer \( n \geq 2 \), let \( r = \left\lceil \frac{(4n - 2)}{3} \right\rceil \), then there exists no integer pair \( k \) and \( d \) satisfying the inequalities:

\[
\frac{(3r + 5)}{r + 2} < \frac{k}{d} < 3, \quad k \leq 4n + 3
\]

**Proof:** Suppose, by contrary, that there exists an integer pair \( k \) and \( d \) satisfying the inequalities. Then we have

\[
d < \frac{r+2}{3r+5}k \leq \frac{r+2}{3r+5}(3r+7) = r + 2 + \frac{2r + 4}{3r+5}
\]

or \( d \leq r+2 \). Assume that \( d = (r+2) - a \), where \( a \) is an integer such that \( 0 \leq a \leq r+1 \). From \( k/d < 3 \), we have that \( k < 3d = 3r + 6 - 3a \). From

\[
\frac{(3r+5)}{(r+2)} < \frac{k}{d}
\]

we have that

\[
k > \frac{3r + 5}{r+2} (r+2-a) = 3r + 6 -3a + \frac{a}{r+2} - 1
\]

Therefore, we have the following inequalities:

\[
a - 1 < k - 3(r+2) + 3a < 0.
\]

Because \( 0 \leq a \leq r + 1 \), then \( -1 \leq \frac{a}{r+2} - 1 < 0 \). Thus,

\[-1 < k - 3(r+2) + 3a < 0.
\]

This implies that \( k \) is not an integer, contradicting the assumption.
Theorem 3.5 (G. G. Gao [G]): For any integer $n \geq 2$, the triangle-free planar graph $W_{2n+1}^*$ has star chromatic number 3.

**Proof:** For any integer $n \geq 2$, let $r = \lfloor (4n - 2) / 3 \rfloor$. By Lemma 3.3, we know that $W_{2n+1}^*$ is not $(3r + 5, r + 2)$-colorable. By Lemma 3.4, there exists no integer pair $k$ and $d$ such that $(3r + 5) / (r + 2) < k / d < 3$ and $k \leq 4n + 3$. Thus, $\chi^*(W_{2n+1}^*) \geq 3$. A 3-coloring of $W_{2n+1}^*$ can be easily given (in fact, it is proved that a planar graph containing no more than three triangles is always 3-colorable). Hence, $\chi^*(W_{2n+1}^*) = 3$. 
3.2 \( \chi_c = \chi \) For Mycielskian Of Some Graphs

We begin the section by defining Mycielskian of a graph \( G \) denoted by \( \mu(G) \). For \( t \geq 2 \) let \( \mu^t(G) \) denote \( \mu(\mu^{t-1}(G)) \).

**Definition 3.6**: Mycielskian of \( G \), \( \mu(G) \) of a graph \( G = (V, E) \) has vertex set \( V \cup V' \cup \{u\} \), where \( V' = \{x' : x \in V\} \), and edge set is the union of the edges \( E \cup \{xy' : xy \in E\} \cup \{x'u : x' \in V'\} \). The vertex \( x' \) is called the twin of the vertex \( x \) (and \( x \) is also called the twin of \( x' \)); and the vertex \( u \) is called the root of \( \mu(G) \).

The Mycielskian of \( G \), \( \mu(G) \) has the same clique number as \( G \) and has chromatic number \( \chi(G) + 1 \).

We consider the question of equality of circular chromatic number and chromatic number of Mycielskian of a graph \( G \). Although a characterization of graphs \( G \) for which \( \chi_c(\mu(G)) = \chi(\mu(G)) \) is not known, several classes of graphs for which the equality holds or fails have been found. We present here two important results in this direction. The first one shows that for graphs \( G^d_k \) (\( k > 2d \)), \( \chi_c(\mu(G^d_k)) = \chi(\mu(G^d_k)) \). The next result says that if a graph has at least two universal vertices, then their Mycielskians have same
\( \chi_c \) and \( \chi \). It is also known that \( \chi_c(\mu(K_n)) = \chi(\mu(K_n)) = n + t \) for very large \( n \), \( n \geq 2^t + 2 \). There is a conjecture that it must be true for \( n \geq t + 2 \). 

Theorem 3.7 (Huang and Chang [HC]): \( \chi_c(\mu(G^d_k)) = \chi(\mu(G^d_k)) = \left\lfloor \frac{k}{d} \right\rfloor + 1 \) for any positive integer \( k > 2d \).

Note that for \( k = 2d \), we have \( G^d_k \cong dK_2 \) and \( \mu(G^d_k) \) is the graph obtained from \( d \) copies of \( C_5 \) by identifying one vertex in each copy. Therefore \( \chi_c(\mu(G^d_k)) = 2.5 < 3 = \chi(\mu(G^d_k)) \).

Also since \( \chi_c(\mu(G^d_k)) = k/d \) and \( \chi_c(\mu(G^d_k)) = \chi(\mu(G^d_k)) \) for any positive integers \( k > 2d \) we also have following consequence.

Corollary 3.8: There exists a graph \( G \) such that \( \chi_c(G) \) is as close to \( \chi(G) - 1 \) as we want, but \( \chi_c(\mu(G)) = \chi(\mu(G)) \).

The following special case of Theorem 3.7 was proved in [CHZ].

Lemma 3.9: If \( \chi(G) = 3 \), then \( \chi_c(\mu(G)) = \chi(\mu(G)) = 4 \).
For an n-coloring \( c : V(G) \rightarrow \{0, 1, ..., n-1\} \) of \( G = (V, E) \), we denote by \( D_c(G) \) the directed graph with vertex set \( V \) such that there exists an arc from \( x \) to \( y \) if and only if \( xy \in E \) and \( c(x) + 1 \equiv c(y) \mod n \). It was shown in [Gu] that an n-chromatic graph \( G \) satisfies \( \chi_c(G) < n \) if and only if \( G \) has an n-coloring \( c \) for which \( D_c(G) \) is acyclic. The following improvement of this result was proved in [CHZ].

**Lemma 3.10:** \( \chi_c(\mu(G)) < \chi(\mu(G)) = n \), then there exists an n-coloring \( c \) of \( \mu(G) \) such that \( D_c(\mu(G)) \) is acyclic, \( c(u) = 1 \), and \( c(x') \notin \{0, 1\} \) for all \( x' \in V' \).

Moreover, for any such coloring \( c \), there is an edge \( xy \in E(G) \) such that \( \{c(x), c(y)\} = \{0, 1\} \) and \( c(x') = c(y') \).

**Proof of Theorem 3.7:** Write \( k = dr + i \), where \( d \geq 2 \) and \( 1 \leq i \leq d \). Note that \( \mu(G_{d^r+1}^{d}) \) is a subgraph of \( \mu(G_{d^r+i}^{d}) \). If the theorem holds for the special case \( i = 1 \) then

\[
r + 2 \leq \chi_c(\mu(G_{d^r+i}^{d})) \leq \chi_c(\mu(G^{d}_{d^r+i})) \leq \chi(\mu(G^{d}_{d^r+i})) = r + 2
\]
and so the general case follows. Hence, it remains to prove the theorem for the special case when \( k = dr + 1 \).

For clarity of notation, we consider \( G_{dr+1}^d \) as the graph with vertex set \( V = \{x_0, x_1, \ldots, x_{dr}\} \) and edge set
\[
E = \{x_ix_j : d \leq |i − j| \leq (dr + 1) - d\};
\]
and the Mycielskian \( \mu(G_{dr+1}^d) \) as the graph with vertex set \( V \cup V' \cup \{v\} \), where \( v' = \{x_i' : x_i \in V\} \) and edge set \( E \cup \{x_ix_i' : x_i, x_i' \in E\} \cup \{x_j'u : x_j' \in V'\} \). Indices of the vertices \( x_i \) and \( x_i' \) are taken modulo \( dr + 1 \), if arithmetic operations are performed on them. Let \( V_{ij} = \{x_i, x_{i+1}, \ldots, x_j\} \) and \( V_{ij}' = \{x_i', x_{i+1}', \ldots, x_j'\} \).

Note that \( V_{ij}'(i \leq j) \) and \( V_{ij'}(i > j) \) are of size \( j - i + 1 \) and \( dr + 1 + j − i + 1 \).

It is known that \( \chi_c(G_{dr+1}^d) = r + 1/d = \chi(G_{dr+1}^d) = r + 1 \) and \( \chi(\mu(G_{dr+1}^d)) = r + 2. \)

We now show that \( \chi_c(\mu(G_{dr+1}^d)) = \chi(\mu(G_{dr+1}^d)) = r + 2 \) for any positive integer \( r \geq 2 \).

Note that \( G_{2d+1}^d \) is in fact the odd cycle \( C_{2d+1} \). According to Lemma 3.9, the theorem holds for \( r = 2 \). So, we may assume that \( r \geq 3 \). Let \( G = (V, E) \) be the graph \( G_{dr+1}^d \).

Suppose that the theorem does not hold, i.e., \( \chi_c(\mu(G)) < \chi(\mu(G)) = r + 2 \). Then, by Lemma 3.10, there exists an \( (r + 2) \)-coloring \( c \) such that \( D_c(\mu(G)) \)
is acyclic, \( c(u) = 1 \), and \( c(x') \not\in \{0,1\} \) for all \( x' \in V' \). Note that if \( x_i \) is a vertex of \( V \) such that \( c(x_i) \not\in \{0,1\} \) and \( c(x_i) \neq c(x') \), then we can replace the color of \( x_i \) with \( c(x_i) \) and still preserve \( D_c(\mu(G)) \) being acyclic. Hence, we may assume that \( c(x_i) = c(x'_i) \) for each \( x_i \in V \) with \( c(x_i) \not\in \{0,1\} \).

Moreover by Lemma 3.10, there exists an edge \( x_a x_b \in E \) such that \( \{c(x_a), c(x_b)\} = \{0,1\} \) and \( c(x'_a) = c(x'_b) = t \not\in \{0,1\} \). Clearly, \( |V_{a,b}| \geq d + 1 \), since \( x_a x_b \) is an edge. Without loss of generality, we may assume that \( x_a x_b \) is chosen to satisfy the property that \( |V_{a,b}| \) is minimum and, under this condition, \( |c(V_{a,b})| \) is also minimum. Finally, we may also assume that \( c(x_a) = 0 \) and \( 0 < a < d \leq b \leq (dr+1)/2 \).

Let \( A_i = \{x_i + d_j : 0 \leq j \leq r\} \) for \( 0 \leq i \leq dr \). It is clear that any two vertices of \( A_i \), except the pair \((x_i, x_{i-1})\), are adjacent and hence, have different colors. Note that \( x_{i-1} = x_{i+dr} \).

**Claim 1:** If \( x_i, x_j \in A_i \setminus \{x_i, x_{i-1}\} \) are vertices such that \( 2 \leq |c(x_i) - c(x_{i-1})| \leq r \), and \( 2 \leq |c(x'_j) - c(x'_j')| \leq r \), then \( A_i \cup \{x'_j\} \) induces a directed cycle in \( D_c(\mu(G)) \).

**Proof of Claim 1:** Since \( x_j \in A_i \setminus \{x_i, x_{i-1}\} \), any two vertices of \( A_i \cup \{x'_j\} \) are adjacent, except the two pairs \((x_i, x_{i-1})\) and \((x_j, x'_j)\). Also, \( c(x_i) \neq c(x_{i-1}) \) and
c(x_i) \neq c(x_j) \) imply \( |c(A_i \cup \{x_j\})| = r + 2\). Since any two vertices of \( A_i \cup \{x_j\} \) with consecutive colors are adjacent, \( A_i \cup \{x_j\} \) induces a directed cycle in \( D_c(\mu(G)) \).

For any color \( k \in c(V) \), let \( V_{l(k),e(k)} \) be the \( V_{i,j} \) of smallest size including \( c^{-1}(k) \) as a subset. Clearly, \( c(x_{l(k)}) = c(x_{e(k)}) = k \), but \( c(x_{l(k)-1}) \neq k \) and \( c(x_{e(k)+1}) \neq k \). Also, \( V_{l(k),e(k)} \) has a size at most \( d \), since any two vertices in it are nonadjacent. Also, \( V = \bigcup_{k \in c(V)} V_{l(k),e(k)} \) implies \( r + 1 \leq |c(v)| \leq r + 2 \).

**Claim 2:** \( |c(V)| = r + 2 \).

**Proof of Claim 2:** Suppose to the contrary that \( |c(V)| = r + 1 \), say \( p \notin c(V) \) for some \( p \) with \( 2 \leq p \leq r + 1 \). Let

\[
S = \{x_i \in V : 2 \leq |c(x_i) - c(x_{i-1})| \leq r \text{ or } 2 \leq |c(x_i) - c(x_{i+1})| \leq r \}.
\]

Suppose that \( p - 1 \notin c(S) \). Then, \( c(x_{l(p-1)}) = c(x_{e(p-1)}) = p - 1 \) imply \( x_{l(p-1)} \notin S \) and \( x_{e(p-1)} \notin S \). Therefore, \( c(x_{l(p-1)-1}) \) and \( c(x_{e(p-1)+1}) \) are in \( \{p - 2, p - 1\} \). However, \( c(x_{l(p-1)-1}) \neq p - 1 \) and \( c(x_{e(p-1)+1}) \neq p - 1 \) by the definition of \( V_{l(p-1),e(p-1)} \). Hence, \( c(x_{l(p-1)-1}) = c(x_{e(p-1)+1}) = p - 2 \), and so \( V_{l(p-1),e(p-1)} \) is a subset of \( V_{l(p-2),e(p-2)} \). Then, \( V \) is the union of \( r \) sets \( V_{l(k),e(k)} \).
each of size at most \( d \), for \( k \in \{0, 1, \ldots, r + 1\} \setminus \{p - 1, p\} \), a contradiction to the fact that \(|V| = dr + 1\). This proves that \( p - 1 \in c(S) \). Similarly, \( p + 1 \in c(S) \).

We then choose a vertex \( x_i \in S \) such that \( c(x_i) = 1 \), when \( p = 2 \) and \( c(x_i) = p + 1 \) otherwise. For the case of \( 2 \leq |c(x_i) - c(x_{i-1})| \leq r \), \( c(A_i) = c(V) \) and \( \{c(x_i), c(x_{i-1})\} \neq \{0, 1\} \). Then, there exists a vertex \( x_j \in A_i \setminus \{x_i, x_{i-1}\} \) of color 0 or 1. Since \( x_j \) is adjacent to all vertices of \( A_i \setminus \{x_j\} \) the color of \( x_j \) must be \( c(x_j) \) or \( p \). However, a vertex in \( V' \) cannot colored by 0 or 1, hence, \( c(x_j') = p \). Recall that \( 2 \leq p \leq r + 1 \) and \( c(x_j) \in \{0, 1\} \). If \(|c(x_i) - c(x_j)| \in \{0, 1, r + 1\}\), then either \( c(x_j) = 0 \), \( p = r + 1 \), or \( c(x_j) = 1 \), \( p = 2 \). Both cases lead to \( c(x_j) = c(x_i) \), a contradiction to the fact that \( x_i \) and \( x_j \) are adjacent. Hence, \( 2 \leq |c(x_i) - c(x_j')| \leq r \). According to Claim 1, \( A_i \cup \{x_j'\} \) induces a directed cycle in \( D_c(\mu(G)) \) a contradiction. Similar arguments also lead to a contradiction for the case \( 2 \leq |c(x_i) - c(x_{i+1})| \leq r \). So, \(|c(V)| = r + 2 \).

According to Claim 2, \( t \in c(V) \). Since \( c(x_j') = c(x_j) = t \), we have \( V_{l(t),c(t)} \subseteq V_{d+1, d-1} \cap V_{b+d+1, b+d-1} \), and so \( d \leq b \leq 2d - 2 \) and \( V_{l(t),c(t)} \subseteq V_{b+d+1, b+d-1} \subseteq V_{0,b} \). Therefore, \( \{0, 1, t\} \subseteq c(V_{0,b}) \).

Claim 3: For any vertex \( x_j \in V_{0,b} \) colored by \( t \), \( \{c(x_{j-1}), c(x_{j+1})\} \subseteq \{0, 1, t\} \).

Proof of Claim 3: Suppose to the contrary that there exist vertices \( x_i, x_{i+1} \) in
FIGURE 3.2: Vertex colors near $V_{0,b}$ (for Claim 3)

$V_{0,b}$ such that $\{c(x_i), c(x_{i+1})\} = \{t, q\}$ for some $q \notin \{0,1,t\}$. Then $b \leq i + d$; and $c(x'_{i-d}) = l$ and $c(x'_{i+1+d}) = m$ are not in $\{t, q\}$. If $r \geq 4$, then $l \neq m$. 

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Otherwise, since every vertex of \( V \) is adjacent to at least one of \( x'_{i-d}, x'_{i+1+d} \), we would have \( l \not\in c(V) \), contrary to Claim 2. Hence, none of the vertices of \( V_{i+1+2d, i-2d} \) can be colored by 0, 1, t, q, l, m. This implies that the \( dr + 1 - 4d \) vertices of \( V_{i+1+2d, i-2d} \) must be colored by only \( r - 4 \) colors, a contradiction (see Fig. 3.2). If \( r = 3 \), then \( |V_{i+1+d, i-d}| = dr + 1 - 2d = d+1 \) and \( l = m \) as there are only five colors to be used. Therefore, \( c(x_{i+1+d}) = 1, c(x_{i-d}) = 0 \) and \( c(V_{i+1+d, i-d}) = \{0, 1, l\} \). It follows therefore that \( x_{i+1+d}x_{i-d} \) is an edge with \( \{c(x_{i+1+d}), c(x_{i-d})\} = \{0, 1\} \) and \( c(x'_{i+1+d}) = c(x'_{i-d}) \), but \( |V_{i+1+d, i-d}| = d + 1 \leq |V_{0,b}| \) and \( |c(V_{i+1+d, i-d})| < |c(V_{0,b})| \), a contradiction to the choice of \( x_0x_b \). Hence, Claim 3 holds.

**Claim 4:** \( c(V_{0,b}) = \{0, 1, t\} \).
Proof of Claim 4: To the contrary, assume that there exists a color \( p \in c(V_{0,b}) \setminus \{0, 1, t\} \). Choose any \( x_j \in V_{0,b} \) such that \( c(x_j) = t \). By Claim 3, \( c(x_{j-1}) \neq p \) and \( c(x_{j+1}) \neq p \). If there exists some vertex in \( V_{1,j-2} \) colored by \( p \), then choose the largest integer \( i \leq j - 1 \) such that \( c(x_i) \in \{0, p\} \) and \( c(x'_i) = p \).

We now consider the following three cases according to the color of \( x_{i+1} \).

**Case 1:** \( c(x_{i+1}) = 0 \). By the choice of \( i \), we have \( c(x'_{i+1}) \neq p \). Suppose that \( c(x_{i+1+d}) \neq 1 \).

Then \( c(x_{i+1+d}) = q \) for some \( q \notin \{0, 1, p, t\} \), since \( x_{i+1+d} \) is adjacent to \( x_0, x'_0 \) and \( x'_i \). If \( r \geq 4 \), then \( c(x'_{i+1+2d}) = m \) for some \( m \notin \{0, 1, p, t, q\} \). Hence, \( dr + 1 - 4d \) vertices of \( V_{i+1+3d, i+d} \) are colored by \( r - 4 \) colors, a contradiction (see Fig. 3.3). If \( r = 3 \), then \( x_{i+1+2d} = x_{i-d} \). Hence \( c(x_{i-d}) = 1 \) and \( c(x'_{i-d}) = p \). Therefore, \( c(x_i) = 0 \) and \( b \geq i+1+d \). Since \( t, q \notin c(V_{i-d,i}) \), we
have that \( x_{i+d}x_i \) is an edge with \( \{c(x_{i-d}), c(x_i)\} = \{0, 1\} \) and \( c(x'_{i+d}) = c(x'_i) \), but \( |V_{i+d}| = d+1 \leq |V_{0,b}| \) and \( |c(V_{i+d})| < |c(V_{0,b})| \), a contradiction to the choice of \( x_0x_b \). Hence \( c(x_{i+1+d}) = 1 \).

We now consider the set \( c(V'_{i+1, i+1+d}) \). If \( q \in c(V'_{i+1, i+1+d}) \) for some \( q \notin \{0, 1, p, t\} \), then the \( dr+1-3d \) vertices of graph \( V_{i+1+2d, i-d} \) are colored by \( r+2-|\{0, 1, p, t, q\}| = r-3 \) colors, a contradiction. Hence, \( c(x'_{i+1}) = c(x'_{i+1+d}) = t \) by claim 2; and moreover, \( c(V_{i+1, i+1+d}) \subseteq \{0, 1, t, p\} \).

If \( p \in c(V_{i+1, i+1+d}) \), then \( p \in c(V_{j+2, i+1+d}) \), since \( i \) is maximal and since \( c(x_{j+1}) \neq p \). Because \( p \in V_{1,j-2} \), we have \( c(x'_{j-d}), c(x'_{j+d}) \notin \{0, 1, p, t\} \), lets call \( c(x'_{j-d}) = l \) and \( c(x'_{j+d}) = m \).

If \( r \geq 4 \), then \( l \neq m \) by Claim 2. The \( dr+1-(4d-1) \) vertices of \( V_{j+2d, j-2d} \) must be colored by \( r-4 \) colors, a contradiction (see Fig. 3.4). If \( r = 3 \), then \( l = m \) by Claim 2 and \( |V_{j+d,j-d}| = (dr+1)-(2d-1) = d+2 \). Since all vertices of \( V'_{j+d, j-d} \) are also colored by \( l = m \), we have \( c(x_{j+d}) = c(x_{j,d-1}) = 1, c(x_{j-d}) = c(x_{j-d-1}) = 0 \) and \( c(V_{j+d, j-d}) = \{0, 1, l\} \). Hence \( x_{j+d}\tilde{x}_{j-d-1} \) is an edge with \( \{c(x_{j+d}), c(x_{j,d-1})\} = \{0, 1\} \) and \( c(x'_{j+d}) = c(x'_{j,d-1}) \), but \( |V_{j+d,j-1}| = d+1 \leq |V_{0,b}| \) and \( |c(V_{i+1, i+1+d})| < |c(V_{0,b})| \), a contradiction to the choice of \( x_0x_b \). Therefore, \( c(V_{i+1, i+1+d}) = \{0, 1, t\} \), which also implies that \( x_{i+1}x_{i+1+d} \) is an edge with \( \{c(x_{i+1}), c(x_{i+1+d})\} = \{0, 1\} \) and \( c(x'_{i+1}) = c(x'_{i+1+d}) \).
but \( |V_{i+1, i+1+d}| = d + 1 \) \( d+1 \leq |V_{0,b}| \) and \( |c(V_{i+1, i+1+d})| < |c(V_{0,b})| \), again a contradiction to the choice of \( x_0 x_b \).

**Case 2:** \( c(x_{i+1}) = 1 \). If \( c(x_i) = p \), then \( c(x'_{i-d}) \notin \{p, t\} \) say \( c(x'_{i-d}) = l \).

Therefore, the \( dr+1-3d \) vertices of \( V_{i+1+d, i-2d} \) are colored by \( r+2-\{0,1,p,t,l\} = r - 3 \) colors, a contradiction.

If \( c(x_i) = 0 \), then since \( c(x'_{i}) = p \), the \( dr + 1 - 2d \) vertices of \( V_{i+1+d, i-d} \) are colored by \( r - 2 \) colors, also a contradiction.

![Diagram](image)

**Figure 3.4:** Vertex colors near \( V_{0,b} \)(for the last paragraph of Case 1)
Case 3: \( c(x_{i+1}) \notin \{0, 1, p\} \).

If \( c(x_{i+1}) = t \), then \( c(x_i) = 0 \) and \( c(x'_i) = p \). Since some vertex of \( V_{1,j-2} \) is colored by \( p \), we have \( c(x'_{i+1+d}) \notin \{p, t\} \), say \( c(x'_{i+1+d}) = m \). Hence, \( d + 1 - 3d \) vertices of \( V_{i+1+2d,i-d} \) are colored by \( r - 3 \) colors, a contradiction.

If \( c(x_{i+1}) = q \) for some \( q \notin \{0,1,p,t\} \), then \( c(x'_{j+d}) \notin \{0,1,p,q,t\} \), say \( c(x'_{j+d}) = m \). Then we have \( r \geq 4 \). If \( c(x_i) = 0 \), then, since \( c(x'_i) = p \), the vertices of \( V_{j+2d,i-d} \) are colored by \( r - 4 \) colors, a contradiction to the fact that
\[
|V_{j+2d,i-d}| = dr + 1 - 3d - (j - i - 1) > dr + 1 - 4d.
\]
If \( c(x_i) = p \), then \( c(x'_{i+d}) \) cannot be colored by 0, 1, p, q, t say \( c(x'_{i-d}) = l \). Note that \( 0 > j - d > i - d, \)
\( b < j + d \) and \( j < d \). If \( r \geq 5 \) then \( |V_{j+2d,i-2d}| > dr + 1 - 5d \) and \( l \neq m \), i.e., at least \( dr + 1 - 5d \) vertices of \( V \) are colored by \( r - 5 \) colors, a contradiction (see Fig. 3.5). If \( r = 4 \), then \( l = m \) and \( |V_{j+d,i-d}| > dr + 1 - 3d = d + 1 \). It is clear that \( c(x_{i-d}) = 0 \). Since \( x_{i-2d} \in V_{j+d,i-d} \) we have \( c(x_{i-2d}) = 1 \) and \( c(V_{j+d,i-d}) = \{0, 1, l\} \).

Therefore \( x_{i-2d}x_{i-d} \) is an edge with \( \{c(x_{i-2d}), c(x_{i-d})\} = \{0, 1\} \) and
\( c(x'_{i-2d}) = c(x'_{i-d}) \), but \( |V_{i-2d,i-d}| = d + 1 \leq |V_{0,b}| \) and \( |c(V_{i-2d,i-d})| < |c(V_{0,b})| \), a contradiction to the choice of \( x_0x_b \).
By the three cases above, we conclude that no vertex of $V_{1, j-2}$ can be colored by $p$. A similar argument shows that no vertex of $V_{j+2, b-1}$ can be colored by $p$. Hence, $c(V_{0,b}) = \{0, 1, t\}$. This completes the proof of Claim 4.

**FIGURE 3.5:** Vertex colors near $V_{0,b}$ (for Case 3)
Having proved the claims, we are now ready to prove the theorem. Suppose $3 \leq t \leq r$. Since $t \not\in c(V \setminus V_{0, b})$, there exists an integer $i$ such that $c(x_i), c(x_{i-1}) \not\in \{0, 1, t\}$ and $2 \leq |c(x_i) - c(x_{i-1})| \leq r$. Since $c(A_i) = r + 1$, there must be some vertex $x_j \in A_i \setminus \{x_i, x_{i-1}\}$ such that $c(x_j) \in \{0, 1\}$. Assume that $c(x_j) = 0$ (the case of $c(x_j) = 1$ is similar). Since the color of $x_j$ say $q$, cannot belong to $c(A_i)$, i.e., $q$ is the only color not in $c(A_i)$ and $q \not\in \{0, 1\}$. Hence, $1 \in c(A_i)$, some vertex $x'_j \in A_i \setminus \{x_i, x_{i-1}, x_j\}$ is colored by 1 and $c(x'_j) = q$ is clear. It follows that either $2 \leq |c(x_j) - c(x'_j)| \leq r$ or $2 \leq |c(x_j) - c(x'_j)| \leq r$. By Claim 1, there is a directed cycle in $D_c(\mu(G))$, a contradiction. So, $t \in \{2, r + 1\}$.

Assume that $t = 2$ (the case of $t = r + 1$ is similar). Let $i$ be the smallest integer such that $c(x_i) = 2$; and let $j$ be the largest integer such that $c(x_j) = 1$ and $i \leq j \leq i + d$. Such a $j$ exists. In fact, one may take $j = b$ if there is no larger value.

For the case of $j = i + d$, $c(x_{i+d}) = 1$ implies $c(x_{i-1}) \neq 1$. By the definition of $x_i$, $c(x_{i-1}) \neq 2$. Hence, by Claim 4, $c(x_{i-1}) = 0$. Also, $c(x_i) = 2$ implies $c(x'_{i+d}) \neq 2$. By Claim 1, $A_i \cup \{x'_{i+d}\}$ induces a directed cycle of $D_c(\mu(G))$, a contradiction.
For the case of $j < i + d$, $c(x_{j+1}) \neq 1$ by choice of $x_j$; and $c(x_{j+1}) \notin \{0, 2\}$, since $j + 1 > b$ and $c(x_0) = 0$ and $c(x'_0) = 2$. According to $0 \leq b - d \leq j - d < i$, we have $c(x_{j-d}) \neq 2$ by the choice of $x_i$, and $c(x_{j-d}) \neq 1$ as $c(x_j) = 1$. Hence, by Claim 4, $c(x_{j-d}) = 0$. We conclude that $c(x'_{j-d}) = 2$. By Claim 1, $A_{j+1} \cup \{x'_{j-d}\}$ induces a directed cycle of $D_c(\mu(G))$, a contradiction.

Therefore, $\chi_c(\mu(G'_{dr+1})) = \chi(\mu(G'_{dr+1})) = r + 2$. This completes the proof of the theorem.

We now turn our attention to graphs that have more than one universal vertex. Before we prove their Mycielskians have circular chromatic number equal to chromatic number, we need the following lemma.

**Lemma 3.11:** Suppose $G = (V, E)$ is a graph with $\chi_c(G) = k/d$ and that $c : V \rightarrow \{0, 1, \ldots, k - 1\}$ is a $(k, d)$-coloring of $G$. Then for each $i \in \{0, 1, \ldots, k - 1\}$, there exists a vertex $x$ with $c(x) = i$ which is adjacent to a vertex $y$ with $c(y) = i + d$. Here the addition is modulo $k$.

**Proof:** Assume to the contrary that there exists an index $i$ such that no vertex of color $i$ is adjacent to a vertex of color $i + d$. Let $e = i(i + d)$. Then the
coloring c is a homomorphism of G to $G'_k - e$. However, it was proved in [Z1] that $\chi_c(G'_k - e) < k/d$, which implies that $\chi_c(G) \leq \chi_c(G'_k - e) < k/d$, contrary to our assumption.

**Theorem 3.12** (Hajiabolhassan and Zhu [HZ2]): Let $G = (V, E)$ be a graph on $n \geq 3$ vertices. If $G$ has 2 vertices of degree $n - 1$, then $\chi_c(\mu(G)) = \chi(\mu(G))$.

**Proof:** Let $a, b$ be two vertices of $G$ of degree $n - 1$. Let $a', b'$ be the twins of $a, b$ respectively in $\mu(G)$, and let $u$ be the root of $\mu(G)$. Assume to the contrary that $\chi_c(\mu(G)) = k/d$ for some $d \geq 2$ (and $\gcd(k, d) = 1$). Let $f$ be a $(k, d)$-coloring of $\mu(G)$. Assume $f(a) = i$. Then none of $f^1(i - d + 1), \ldots, f^1(i + d - 1)$ is empty and each contains only vertices that are not adjacent to $a$. However, the only vertices of $\mu(G)$ not adjacent to $a$ are $a'$ and $u$. So we must have $d = 2$ and, say, $f^1(i - 1) = \{a'\}$, $f^1(i) = \{a\}$ and $f^1(i + 1) = \{u\}$.

By symmetry, $f^1(i + 2) = \{b\}$ and $f^1(i + 3) = \{b'\}$. Since $n \geq 3$, $G$ has vertices other than $a$ and $b$. Now for each vertex $x \in V - \{a, b\}$, because $f(x) \notin \{f(u) - 1, f(u), f(u) + 1\}$, we can assume that $f(x') = f(x)$ (in the case $f(x) = f(x')$, we can recolour $x'$ with the color of $x$ to obtain another $(k, d)$-coloring of $\mu(G)$). Since every vertex of $V - \{b\}$ is adjacent to $b'$, and since
f(x) = f(x') for every $x \in V - \{a, b\}$, we conclude that $f^{-1}(i + 4) = \phi$. But this is contrary to Lemma 3.11.

This theorem is sharp in the sense that there are graphs $G$ with one universal vertex for which $\chi_d(\mu(G)) \neq \chi(\mu(G))$. The odd wheel $W_{2n+1}$ serves as a counter example.
3.3. $\chi_e = \chi$ FOR SOME Kneser GRAPHS

**Definition 3.13:** For $m \geq 2n \geq 2$, the *Kneser graph* $KG(m, n)$ has the vertex set of all $n$-subsets of the set $[m] = \{1, 2, \ldots, m\}$. Two vertices are defined to be adjacent in $KG(m, n)$ if they have empty intersection as subsets.

**Definition 3.14:** The vertex set of the *reduced Kneser graph* $KG_2(m, 2)$ consists of all pairs $\{a, b\}$ such that $a, b \in \{1, 2, \ldots, m\}$ and $2 \leq |a - b| \leq m - 2$. Two vertices are defined to be adjacent if they are disjoint.

It was conjectured by Kneser [K] in 1955 and proved by Lovász [L] in 1978 that $\chi(KG(m, n)) = m - 2n + 2$. Johnson et al. [JHS] in 1997 have proved that $\chi_e(KG(m, n)) = \chi(KG(m, n))$ if $m \leq 2n + 2$ or $n = 2$. They conjectured that the circular chromatic number and the chromatic number were equal for every Kneser graph $KG(m, n)$. Hajiabolhassan and Zhu [HZ3] in 2003 have proved that the conjecture is true for $m \geq 2n^2(n-1)$.

Lih et. al. [LL] in 2002 reproved the result of Johnson et. al. that $\chi_e(KG(m, 2)) = m - 2$ and showed that the same is true for the corresponding reduced Kneser graphs, i.e., $\chi_e(KG_2(m, 2)) = m - 2$. We
present the proof of these results here. The analogous conjecture for reduced
Kneser graph \( KG_2(m, n) \) has also been proved by Hajiabolhassan and Zhu
[HZ3] in 2003 for very large \( m \). A good upper bound \( t(n) \) for \( m \) is yet to be
found.

The following lemma is a \((k, d)\)-partition version of Lemma 3.11. We
state it without rewriting the proof.

**Lemma 3.15:** Let \( (V_0, V_1, ..., V_{k-1}) \) be a \((k, d)\)-partition of \( G \) and \( \chi_c(G) = k/d \),
where \( \gcd(k, d) = 1 \). Then for every \( i, 0 \leq i \leq k-1 \), there are vertices \( x \) in \( V_i \)
and \( y \) in \( V_{i+d} \) such that \( x \) and \( y \) are adjacent.

**Theorem 3.16** (Lih and Liu 02[LL]): If \( m \geq 4 \), then \( \chi_c(KG(m, 2)) = m - 2 \).

**Proof:** Suppose to the contrary that \( \chi_c(KG(m, 2)) = k/d < m - 2 \), where
\( \gcd(k, d) = 1 \) and \( d \geq 2 \). Let \( (V_0, V_1, ..., V_{k-1}) \) be a \((k, d)\)-partition of \( KG(m, 2) \)
with nonempty color classes \( V_i \)'s.

**Case 1:** For some \( i, |V_i| \geq 2 \).

Without loss of generality, let \( \{1, 2\} \) and \( \{1, 3\} \) belong to \( V_i \). By Lemma
3.15 there are vertices \( x \) in \( V_{i-1} \) and \( y \) in \( V_{i+d-1} \) such that \( x \) and \( y \) are
adjacent. Since both \( x \) and \( y \) are adjacent to neither \( \{1, 2\} \) nor \( \{1, 3\} \), the
only vertices of KG(m,2) that could be chosen as x and y are the vertices 
\{1, 4\} \{1, 5\}, ..., \{1, m\} and \{2, 3\}. However, \{2, 3\} is adjacent to every 
vertex of the independent set \{\{1, 4\}, \{1, 5\}, ..., \{1, m\}\}. Therefore, one of 
x and y must be \{2, 3\}. If \(d > 2\), then \{2, 3\} is adjacent to all vertices of \(V_{i+1}\).
This adjacency contradicts the defining properties of a (k, d)-coloring. It 
follows that \(d = 2\) and at least one of \(V_{i-1}\) and \(V_{i+1}\) is a singleton.

Next, we claim that \(|V_j| \leq 2\) for any j. Let \{a, b\} belong to \(V_{j-1}\) and 
\{c, d\} belong to \(V_{j+1}\) such that \{a, b\} and \{c, d\} are adjacent. Thus a, b, c 
and d are distinct numbers. The vertices of KG(m,2) that are adjacent to 
neither \{a, b\} nor \{c, d\} belong to \{\{a, c\},\{a, d\},\{b, c\}, \{b, d\}\}, which 
consists of two independent edges of KG(m,2). Thus, the independent set \(V_j\) 
contains at most two vertices.

We conclude that \(2k > |V(KG(m,2))| = m(m - 1)/2\). Now, the fact 
\(\chi(G) - 1 < \chi_c(G) \leq \chi(G)\) implies that

\[
\chi_c(KG(m,2)) = k/2 = m - 2 - \frac{1}{2}.
\]

It follows that \(2(2m - 5) > m(m - 1) / 2\). However, no integer \(m \geq 4\) 
satisfies this inequality.

**Case 2:** For all \(i, |V_i| = 1\).

Suppose that \(d \geq 4\). We may suppose that \(V_0\) consists of the unique vertex 
\{1, 2\} and the unique vertex of \(V_1\) also contains the number 1. Moving along
increasing indices, we reach some $j$ such that $V_j = \{\{1,a\}\}$, $V_{j+1} = \{\{1, b\}\}$, and $V_{j+2} = \{\{a, b\}\}$. This would force $V_{j+3}$ to be empty. Hence $d \leq 3$.

In this case, $k = |V(KG(m,2))|$. If $d = 2$, then $2m - 5 = k = m(m - 1)/2$, i.e., $m^2 - 5m + 10 = 0$. No integer $m \geq 4$ satisfies this identity. If $d = 3$, then we have two possibilities: (i) $k/3 = m - 2 - 1/3$ or (ii) $k/3 = m - 2 - 2/3$. We can derive the identities $m^2 - 7m + 14 = 0$ for (i) and $m^2 - 7m + 16 = 0$ for (ii). Both identities have no solutions for integers $m \geq 4$.

**Theorem 3.17 (Lih and Liu 02[LL]):** If the integer $m \geq 4$ and $m \neq 5$, then $\chi_c(KG_2(m,2)) = m - 2$.

**Proof:** The graph $KG_2(4,2)$ consists of one edge and so $\chi_c(KG_2(4,2)) = 2$.

Assume that $m \geq 6$. Suppose to the contrary $\chi_c(KG_2(m,2)) = k/d < m - 2$,

where $gcd(k,d) = 1$ and $d \geq 2$. Let $(V_0, V_1, \ldots, V_{k-1})$ be a $(k, d)$-partition of $KG_2(m,2)$ with nonempty color classes.

**Case 1:** For some $i$, $|V_i| \geq 2$.

We first make the following observation. If

$$V_i = \{\{x, y\}, \{x, z\}\}, \text{ then } \{y, z\} \text{ is a } 2\text{-stable set} \quad (3.1)$$

For otherwise, all the vertices that are adjacent to neither $\{x, y\}$ nor $\{x, z\}$ would contain the number $x$, hence form an independent set. By Lemma 3.15, there are vertices $u$ in $V_{i-1}$ and $w$ in $V_{i+d-1}$ such that $u$ and $w$ are
adjacent. Since u and w are adjacent to neither \{x, y\} nor \{x, z\}, we have obtained a contradiction.

The same argument for Case 1 of Theorem 3.16 can be used to show that \(d = 2\) and at least one of \(V_{i-1}\) and \(V_{i+1}\) is a singleton. Further more, the argument also shows that \(|V_j| \leq 2\) for all \(j\).

Now let \(p\) denote the number of color classes of size 2. Since no three consecutive color classes are of size 2, we have \(p \leq 2k/3\). It follows that

\[
|V(KG_2(m,2))| = \binom{m}{2} - m = m(m - 3)/2 = 2p + (k - p) \leq k + (2k/3).
\]

Substituting \(k = 2m - 5\) into this inequality, we obtain \(3m^2 - 29m + 50 \leq 0\) which can be satisfied only by \(m = 6\) and 7.

Assume that \(m = 6\) and there is \((7, 2)\)-partition \((V_0, V_1, ..., V_5)\) of \(KG_2(6, 2)\). Because \(|V(KG_2(6,2))| = 9\), at least one color class is of size 2. Also note that, for each \(x \in \{6\}\), there are exactly three vertices in \(KG_2(6, 2)\) that contain \(x\): \(\{x, x+2\}, \{x, x+3\}, \{x, x+4\}\) (additions modulo 6). Hence, we may assume that \(V_0 = \{\{1, 3\}, \{1, 5\}\}\) by (3.1). Then \(V_6\) and \(V_1\) are singletons and \(V_6 \cup V_1 = \{\{3, 5\}, \{1, 4\}\}\). Without loss of generality, assume that \(V_6 = \{\{3, 5\}\}\) and \(V_1 = \{\{1, 4\}\}\). By (3.1), all the four vertices of \(p; \{2, 4\}, \{3, 6\}, \{2, 5\}, \{4, 6\}\) belong to at least three distinct color classes among \(V_j, 2 \leq j \leq 5\). Since two consecutive vertices on \(P\) cannot
occur in the same or consecutive color classes, \( P \) starts from \( V_3 \), then successively moves to \( V_5, V_2 \), and terminates in \( V_4 \). Then \( \{3, 6\} \) or \( \{2, 5\} \) is in color class \( V_2 \), which is a contradiction as \( V_1 = \{\{1, 4\}\} \).

Next, assume that \( m = 7 \) and there is a \((9, 2)\)-partition \((V_0, V_1, ..., V_8)\) of \( KG_2(7, 2) \). Because \( |V(KG_2(7, 2))| = 14 \), there are exactly five color classes of size 2 and 4 color classes size 1. Hence, there exists at least one pair of consecutive color classes \( V_i \) and \( V_{i+1} \) of size 2. The intersection of all vertices in \( V_i \) and \( V_{i+1} \) is a certain integer \( p, 1 \leq p \leq 7 \). By (3.1), we may suppose that \( V_1 = \{\{1, 3\}, \{1, 5\}\} \) and \( V_2 = \{\{1, 4\}, \{1, 6\}\} \). Then, they force \( V_0 = \{\{3, 5\}\} \) and \( V_3 = \{\{4, 6\}\} \).

If both \( V_4 \) and \( V_8 \) are singletons, then we would have three consecutive color classes of size 2, which is not allowed. Thus, one of \( V_4 \) or \( V_8 \) is of size 2.

Next we claim that, besides \( V_1 \) and \( V_2 \), it is impossible to have another pair of consecutive color classes of size 2. If \( |V_4| = |V_5| = 2 \), then the only possibility is \( V_4 = \{\{2, 4\}, \{2, 6\}\} \) and \( V_5 = \{\{2, 5\}, \{2, 7\}\} \). They, in turn force \( V_6 = \{5, 7\} \). Moreover, \( \{4, 7\} \) can only belong to \( V_7 \). Since \( \{3, 6\} \) is adjacent to \( \{4, 7\} \), we see nowhere can \( \{3, 6\} \) be placed properly. By similar reasons, it is impossible to have \( |V_8| = |V_7| = 2 \).
If $|V_6| = |V_7| = 2$, then $V_6$ and $V_7$ are either the pairs $\{2, 4\}$, $\{2, 6\}$ and $\{2, 5\}$, $\{2, 7\}$ or the pairs $\{2, 7\}$, $\{4, 7\}$ and $\{3, 7\}$, $\{5, 7\}$. In any case, $\{3, 5\}$ or $\{4, 6\}$ would be forced to occupy an adjacent color class. However, this is not allowed since they have already appeared in $V_0$ and $V_2$. A similar argument can be used to show that it is impossible to have $|V_5| = |V_6| = 2$.

Therefore, the only case remaining to be considered is when $|V_4| = |V_6| = |V_8| = 2$ and $|V_5| = |V_7| = 1$. In this case, the vertices of $V_4$ must belong to $\{2, 4\}$, $\{2, 6\}$, $\{3, 6\}$, $\{4, 7\}$. Since $\{3, 6\}$ cannot be placed in $V_4$ with $\{2, 6\}$ by (3.1) and $\{3, 6\}$ is adjacent to $\{2, 4\}$ and $\{4, 7\}$, it follows that $V_4 = \{2, 4\}$, $\{2, 6\}$ or $V_4 = \{2, 4\}$, $\{4, 7\}$, which in turn forces $V_5 = \{2, 5\}$ or $V_5 = \{2, 7\}$. Similarly, $V_8 = \{2, 5\}$, $\{5, 7\}$ or $V_8 = \{3, 7\}$, $\{5, 7\}$, which in turn forces $V_7 = \{2, 7\}$ or $V_7 = \{4, 7\}$. We see that nowhere can $\{3, 6\}$ be placed.

**Case 2:** For all $i$, $|V_i| = 1$.

By a similar argument used at the beginning of Case 2 in the proof of Theorem 3.16, we have $d \leq 3$.

In this case, $k = |V(KG_2(m,2))|$. If $d = 2$, then $2m - 5 = k = m(m-3)/2$, i.e., $m^2 - 7m + 10 = 0$. No integer $m \geq 6$ satisfies the last identity. If $d = 3$, then we have could have: (i) $k/3 = m - 2 - 1/3$ or (ii) $k/3 = m - 2 - 2/3$. For (i),
the derived identity is \( m^2 - 9m + 14 = 0 \) and \( m = 7 \) is the only possible solution. For (ii), the derived identity \( m^2 - 9m + 16 = 0 \) has no integer solutions.

Assume that \( m=7 \) and there is a \((14,3)\)-partition \((V_0, V_1, \ldots, V_{13})\) of \( KG_2(7,2) \). Assume that \( V_0 = \{\{x, y\}\} \) and \( V_{13} = \{\{x, z\}\} \). Since the unique vertex of \( V_1 \) is adjacent to the unique vertex of \( V_{12} \), we may then suppose that \( V_{12} = \{\{y, z\}\} \) and \( V_1 = \{\{x, w\}\} \) for distinct number \( x, y, z \) and \( w \). This forces \( V_2 = \{\{y, w\}\} \).

Note that, for each number \( x \in [7] \), there are exactly four vertices in \( KG_2(7,2) \) that contain \( x \): \( \{x, x \pm 2\}, \{x, x \pm 3\} \) (addition modulo 7). If \( y = x \pm 3 \), then both \( z \) and \( w \) would be forced to equal \( x \pm 2 \), a contradiction. It follows that \( y = x \pm 2 \).

Indeed, the above argument also shows that, if there are three consecutive color classes occupied by three vertices with a number \( s \) in common, and if \( \{s, t\} \) belongs to the middle of these three classes, then we must have \( s = t \pm 2 \).

Because \( V_0 = \{x, y\} \), we see that \( V_{11}, V_{12}, \) and \( V_{13} \) are occupied by vertices with \( z \) as a common number, and \( \{y, z\} \) belongs to the middle of the three classes; \( V_1, V_2, \) and \( V_3 \) are occupied by vertices with \( w \) as a common number, and \( \{y, w\} \) belongs to the middle of the three classes. Therefore, by
the previous paragraph, we have \( z = y \pm 2 \) and \( w = y \pm 2 \). However, in view of \( y = x \pm 2 \), \( z \) must equal \( w \) since \( x \) is different from \( z \) and \( w \). Then, we have a contradiction because \( z \) and \( w \) were chosen to be distinct.

We state the following useful lemma without proof.

**Lemma 3.18:** Suppose \( G \) is an \( n \)-vertex graph with \( \chi(G) = k \). If for some \( s \geq 1 \), any independent set \( X \) of size \( \geq \frac{sn}{(s+1)k} \) is contained in a unique maximal independent set, then \( \chi_c(G) = p/q \) for some \( q \leq s \). In particular, if any independent set of size \( \geq \frac{n}{2k} \) is contained in a unique maximal independent set, then \( \chi_c(G) = \chi(G) \).

**Theorem 3.19** (Hajiabolhassan and Zhu [HZ3]): For any fixed positive integer \( n \), if \( m \) is large enough, then \( \chi_c(KG_2(m,n)) = \chi(KG_2(m,n)) \).

**Proof:** We denote by \( V \) the vertex set of \( KG_2(m,n) \). By Lemma 3.18, it suffices to show that any independent set \( X \) of \( KG_2(m,n) \) of size \( \geq |V|/2(m-2n+2) \) is contained in a unique maximal independent set (when \( m \) is sufficiently large). The vertex set \( V \) has cardinality
\[
\left(\frac{m-n-1}{n-1}\right)^n = \Omega(m^n)
\]
(Each 2-stable n-subset of \([m]\) containing 1 corresponds to an integral solution of the equation \(x_1 + x_2 + \ldots + x_n = m\) with \(x_i \geq 2\). So there are \(\left(\frac{m-n-1}{n-1}\right)^n\) 2-stable n-subsets of \([m]\) containing 1.)

Thus \(|V|/2(m-2n+2) = \Omega(m^{n-1})\). It was proved by Hilton and Milner [HM] that if \(X\) is an independent set \(KG(m,n)\) of size

\[
\left(\frac{m-1}{n-1}\right) - \left(\frac{m-n-1}{n-1}\right) + 2,
\]

then \(\bigcap_{A \in X} A = \{i\}\), for some \(i \in [m]\). Note that \(\left(\frac{m-1}{n-1}\right) - \left(\frac{m-n-1}{n-1}\right) + 2 = O(m^{n-2})\).

Therefore, there exists an integer \(t(n)\) such that if \(m \geq t(n)\), then

\[
\left(\frac{m-1}{n-1}\right) - \left(\frac{m-n-1}{n-1}\right) + 2 \leq |V|/2(m-2n+2).\]

If \(X\) is an independent set of \(KG_1(m,n)\) of size \(\geq |V|/2(m-2n+2)\), then \(X\) is an independent set of \(KG(m,n)\) (as \(KG_2(m,n)\) is an induced subgraph of \(KG(m,n)\)) and

\[
|X| \geq |V|/2(m-2n+2) \geq \left(\frac{m-1}{n-1}\right) - \left(\frac{m-n-1}{n-1}\right) + 2.\]

Hence \(\bigcap_{A \in X} A = \{i\}\) for some \(i \in [m]\).

Any independent set \(Y\) containing \(X\) also has the property \(\bigcap_{A \in Y} A = \{i\}\).

Therefore \(X\) is contained in the unique maximal independent set \(Y = \{A \in V: i \in A\}\) of \(KG_1(m,n)\).
Theorem 3.20 (Hajiabolhassan and Zhu [HZ3]) For any positive integer n, if $m \geq 2n^2(n-1)$, then $\chi_e(KG(m,n)) = \chi(KG(m,n))$.

Proof: We only need to consider $n \geq 3$. Assume that $n \geq 3$ and $m \geq 2n^2(n-1)$. By Proof of Theorem 3.19, for the equality $\chi_e(KG(m,n)) = \chi(KG(m,n))$ to hold, it suffices that

$$\left(\frac{m}{n-1}\right) - \left(\frac{m-n-1}{n-1}\right) + 2 \leq \frac{\binom{m}{n}}{2(m-2(n-1))}$$

Or equivalently,

$$\left(1 - \frac{m-n-1}{m-1} \cdots \frac{m-2n+1}{m-n+1}\right) \frac{m-2(n-1)}{m} \leq \frac{1}{2n} - \frac{2(m-2(n-1))}{\binom{m-1}{n-1}m}$$

For $3 \leq n \leq 4$, the inequality can be verified by straightforward calculations.

Assume $n \geq 5$. In the following, we shall use the inequality that for any $x > -1$, $e^{x/(x+1)} \leq 1 + x \leq e^x$. By using the inequality above, for $i = 1, 2, \ldots, n-1$,

$$\frac{m-n-i}{m-i} \geq e^{-\frac{n}{m-n-i}}.$$ As $n \geq 5$ and $m \geq 2n^2(n-1)$, easy calculation shows that for $i = 1, 2, \ldots, n-1$,
\[
\frac{1}{m-n-i} + \frac{1}{m-2n+i} \leq \frac{2}{m-2n+2} \cdot \frac{4n(n-1)}{(m-n)^2(m-2n+2)}.
\]

So
\[
\sum_{i=1}^{n-1} \frac{1}{m-n-i} \leq \frac{n-1}{m-2n+2} \cdot \frac{2n(n-1)^2}{(m-n)^2(m-2n+2)}.
\]

Hence
\[
\frac{m-n-1}{m-1} \cdots \frac{m-2n+1}{m-n+1} \geq e^{-\sum_{i=n}^{\infty} \frac{n}{m-i}} \geq e^{-\frac{(n-1)}{m-2n+2} \cdot \frac{2n^2(n-1)^2}{(m-n)^2(m-2n+2)}}.
\]

Therefore,
\[
\left(1 - \frac{m-n-1}{m-1} \cdots \frac{m-2n+1}{m-n+1}\right) \frac{m-2(n-1)}{m} \leq \left(1 - e^{-\frac{(n-1)}{m-2n+2} \cdot \frac{2n^2(n-1)^2}{(m-n)^2(m-2n+2)}}\right) \frac{m-2(n-1)}{m}
\]
\[
\leq \left(\frac{n(n-1)}{m-2n+2} - \frac{2n^2(n-1)^2}{(m-n)^2(m-2n+2)}\right) \frac{m-2(n-1)}{m}
\]
\[
= \frac{n(n-1)}{m} - \frac{2n^2(n-1)^2}{m(m-n)^2}
\]
\[
\leq \frac{1}{2n} \cdot \frac{2(m-2(n-1))}{\binom{m-1}{n-1}^m}.
\]