Chapter 2

Continuous Monotonic Decomposition of Tensor Product of Simple Graphs

In this Chapter, we define continuous monotonic decomposition of any graph and determine CMD of tensor product of some simple graphs with $K_2$. We find CMD of tensor product of a Path $P_n$, the Cycle $C_n$, the Star $K_{1,n}$ and the Wheel $W_n$ with $K_2$. We also obtain CMD of tensor product of a specified Caterpillar, the Comb $Cb_n$ and particular case of a Lobster. Each decomposition is explained with an illustration and the table of values for which the corresponding graphs have decompositions is also given.

2.1 Introduction

Let $G = (V, E)$ be a simple graph of order $n$ and size $m$. If $H_1, H_2, \ldots, H_k$ \forall $k \in \mathbb{N}$ are edge-disjoint subgraphs of $G$ such that $E(G) = E(H_1) \cup E(H_2) \cup \ldots \cup E(H_k)$, then $\{H_1, H_2, \ldots, H_k\}$ is said to be a decomposition of $G$. Different types
of decomposition of $G$ have been studied in the literature by imposing suitable conditions on the subgraphs $H_i$. Alavi et al. [1] introduced Ascending Subgraph Decomposition (ASD) as a decomposition of $G$ into subgraphs $H_i$ (not necessarily connected) such that $|E(H_i)| = i$ and each $H_i$ is isomorphic to a proper subgraph of $H_{i+1}$. Gnana Dhas and Paulraj Joseph modified ASD and introduced the concept known as Continuous Monotonic Decomposition of graphs [4] for connected graphs. A decomposition, $\{H_1, H_2, \ldots, H_k\}$ $\forall k \in \mathbb{N}$, is said to be Continuous Monotonic Decomposition (CMD) if each $H_i$ is connected and $|E(H_i)| = i \forall i \in \mathbb{N}$. 

![Diagram of G, H1, and H4]

$G$: $u_1$ $u_2$ $u_3$ $u_4$ $u_6$ $u_7$ $u_8$ $u_9$ $u_5$

$H_1$: $u_1$ $u_2$

$H_4$: $u_2$ $u_3$ $u_4$ $u_5$ $u_8$

$H_4$: $u_2$ $u_3$ $u_4$ $u_5$ $u_8$
Fig 2.1 CMD Of $G$ into $H_1, H_2, H_3$ and $H_4$

If $G$ admits a CMD, \{\(H_1, H_2, \ldots, H_k\) \(\forall k \in \mathbb{N}\), where each $H_i$ is a cycle of length $i$ in $G$, then we say that $G$ admits Continuous Monotonic Cycle Decomposition (CMCD) [5]. CMD of a wide variety of graphs had been studied by Gnana Dhas, Paulraj Joseph, Navaneetha Krishnan and Nagarajan [4], [5], [13]. A graph $G$ admits a CMD $\{H_1, H_2, \ldots, H_k\} \forall k \in \mathbb{N}$ if and only if $m = \binom{k+1}{2}$ [4].

Joseph Varghese and A. Antonysamy discussed CMD of some Complete Tripartite Graphs [11] and Double Continuous Monotonic Decomposition of Graphs [12]. They also defined Modified Continuous Monotonic Decomposition for disconnected graphs. A decomposition, $\{H_1, H_2, \ldots, H_k\} \forall k \in \mathbb{N}$, of a graph $G$, not necessarily connected, is said to be a Modified Continuous Monotonic Decomposition (MCMD) if each $H_i$ is connected and $|E(H_i)| = i \forall i \in \mathbb{N}$.

Since the connectedness of a graph does not affect the existence of a Continuous Monotonic Decomposition (the components can be decomposed separately) we don’t differentiate between Continuous Monotonic Decomposition and Modified Continuous Monotonic Decomposition. Hence we use the same terminology Continuous Monotonic Decomposition (CMD) for any graph.

**Definition 2.1.1.** A graph $G$ of size $m = \binom{k+1}{2}$ is said to have a Continuous
Monotonic Decomposition (CMD) if $G$ can be decomposed into $k$-subgraphs $G_1, G_2,\ldots, G_k$ such that each $G_i$ is connected and $|E(G_i)| = i$ for each $i = 1, 2, 3, \ldots, k$.

**Definition 2.1.2.** Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The tensor product $G = G_1 \wedge G_2$ is defined as a graph with vertex set $V_1 \times V_2$. Edge set is defined as follows: If $w_1 = (u_1, v_1)$ and $w_2 = (u_2, v_2)$ are two vertices of $G$ with $u_i \in V_1$ and $v_i \in V_2 (i = 1, 2)$ then $w_1 w_2 \in E(G)$ if and only if $u_1 u_2 \in E_1$ and $v_1 v_2 \in E_2$.

### 2.2 CMD of some class of graphs

#### 2.2.1 CMD of $P_n \wedge K_2$

In this section, we prove $P_n \wedge K_2$ admits CMD under certain conditions. We prove two lemmas to exhibit a CMD of $P_n \wedge K_2$ in theorem 2.2.3.

**Lemma 2.2.1.** Let $k \equiv 0 \pmod{4}$. The set $\{1, 2, \ldots, k\}$ can be partitioned into two sets $S_1$ and $S_2$ such that $\sum_{x \in S_1} x = \sum_{y \in S_2} y = n - 1$. Here $\frac{k(k+1)}{2} = 2n - 2$.

**Proof.** Let $k = 4r$, $r \geq 1, r \in \mathbb{Z}$. Proof is by induction on $r$. When $r = 1, k = 4$.

Now $n = \frac{1}{2}[2 + \frac{k(k+1)}{2}] = 6$. If $S_1 = \{1, 4\}$ and $S_2 = \{2, 3\}$ then $\sum_{x \in S_1} x = 1 + 4 = 5 = 6 - 1$ and $\sum_{y \in S_2} y = 2 + 3 = 5 = 6 - 1$. Hence the result is true if $r = 1$. Assume that the result is true for $r - 1$. Hence the set $\{1, 2, \ldots, 4(r-1)\}$ can be partitioned into two sets $S_1$ and $S_2$ such that $\sum_{x \in S_1} x = \sum_{y \in S_2} y = n - 1 = (4r - 3)(r - 1)$. To prove the result is true for $r$. The set $\{1, 2, \ldots, 4r\}$ can be partitioned into two
sets $S'_1$ and $S'_2$ where $S'_1 = S_1 \cup \{4r-3,4r\}$ and $S'_2 = S_2 \cup \{4r-2,4r-1\}$.

Clearly $\sum_{x \in S'_1} x = \sum_{x \in S_1} x + 4r - 3 + 4r$

$= (4r - 3)(r - 1) + 4r - 3 + 4r$

$= 4r^2 + r$

$= r(4r + 1)$

$= n - 1$

Similarly $\sum_{y \in S'_2} y = n - 1$

Hence the lemma is proved for all $r$.

**Lemma 2.2.2.** Let $k + 1 \equiv 0 \pmod{4}$. The set $\{1, 2, \ldots, k\}$ can be partitioned into two sets $S_1$ and $S_2$ such that $\sum_{x \in S_1} x = \sum_{y \in S_2} y = n - 1$. Here $\frac{k(k+1)}{2} = 2n - 2$.

**Proof.** Let $k + 1 = 4r$, $r \geq 1, r \in \mathbb{Z}$ and so $k = 4r - 1$. Proof is by induction on $r$. When $r = 1, k = 3$. Now $n = 4$. If $S_1 = \{1,2\}$ and $S_2 = \{3\}$ then $\sum_{x \in S_1} x = 3 = 4 - 1$ and $\sum_{y \in S_2} y = 3 = 4 - 1$. Hence the result is true if $r = 1$. Assume that the result is true for $r - 1$. Hence the set $\{1, 2, \ldots, (4(r-1) - 1)\}$ can be partitioned into two sets $S_1$ and $S_2$ such that $\sum_{x \in S_1} x = \sum_{y \in S_2} y = n-1 = (r-1)(4r-5)$. To prove the result is true for $r$. The set $\{1, 2, \ldots, 4r - 1\}$ can be partitioned into
two sets $S'_1$ and $S'_2$ where $S'_1 = S_1 \cup \{4r - 4, 4r - 1\}$ and $S'_2 = S_2 \cup \{4r - 3, 4r - 2\}$.

Now $\sum_{x \in S'_1} x = \sum_{x \in S_1} x + 4r - 4 + 4r - 1$

$= (4r - 5)(r - 1) + 4r - 4 + 4r - 1$

$= 4r^2 - r$

$= r(4r - 1)$

$= n - 1$

Also $\sum_{y \in S'_2} y = \sum_{y \in S_2} y + 4r - 3 + 4r - 2$

$= (4r - 5)(r - 1) + 8r - 5$

$= 4r^2 - r$

$= r(4r - 1)$

$= n - 1$

Hence by induction, the lemma is true for all $r$. \qed

**Theorem 2.2.3.** For any integer $n$, $P_n \land K_2$ has a CMD $\{H_1, H_2, \ldots, H_k\}$ if and only if there exists an integer $k$ satisfying the following properties:

(i) $k = 4r$ or $4r - 1$ ($r \geq 1, r \in \mathbb{Z}$)

(ii) $\frac{k(k+1)}{2} = 2n - 2$.

**Proof.** Let $G = P_n \land K_2$. By definition, $m = 2n - 2$. Assume $P_n \land K_2$ has a CMD $\{H_1, H_2, \ldots, H_k\}$. Now by definition, $m = \binom{k+1}{2}$. Hence $2n - 2 = \binom{k+1}{2}$, i.e.,
\[2n - 2 = \frac{k(k+1)}{2}\]. Since \(P_n \land K_2\) has a CMD,

\[
2n - 2 = 1 + 2 + \ldots + k
\]

\[
\Rightarrow 2(n - 1) = \frac{k(k + 1)}{2}
\]

\[
\Rightarrow k(k + 1) = 4(n - 1)
\]

Hence \(k(k + 1) \equiv 0 (\text{mod} \ 4)\)

\[
\Rightarrow k(k + 1) = 4r \ (r \geq 1, \ r \in \mathbb{Z})
\]

Now either \(k = 4r\) or \(k + 1 = 4r\). Thus \(k = 4r\) or \(k = 4r - 1\), where \(r \geq 1, r \in \mathbb{Z}\).

Conversely assume \(k(k+1) \equiv 0 (\text{mod} \ 4)\). Let \(G = P_n \land K_2\). Let \(P_n = (u_1, u_2, \ldots u_n)\) and \(K_2 = (v_1, v_2)\). Define \(w_{ij} = (u_i, v_j)\) where \(1 \leq i \leq n, 1 \leq j \leq 2\). Now \(V(G) = \{w_{ij} : 1 \leq i \leq n, 1 \leq j \leq 2\}\) and \(m = 2n - 2\).

**Case (i):** Suppose \(n\) is even.

Define \(T_1 = \{w_{i1}, w_{(i+1)2} : 1 \leq i \leq n, i-\text{odd}\}\) \(\cup\{(w_{i2}, w_{(i+1)1} : 1 \leq i \leq n-1, i-\text{even}\}\)

and \(T_2 = \{w_{i2}, w_{(i+1)1} : 1 \leq i \leq n, i-\text{odd}\}\) \(\cup\{(w_{i1}, w_{(i+1)2} : 1 \leq i \leq n-1, i-\text{even}\}\}\).

Here \(|T_1| = n - 1\) and \(|T_2| = n - 1\). Also, \(|T_1| + |T_2| = 1 + 2 + \ldots + k = \binom{k+1}{2}\). By lemmas 2.2.1 and 2.2.2, \(\{1, 2, 3 \ldots k\} = S_1 \cup S_2\) where \(\sum x = n - 1\) and \(\sum y = n - 1\). Decompose \(T_1, T_2\) into trees \(\{H_i\}\) as follows: \(T_1 = \bigcup_{i \in S_1} H_i, T_2 = \bigcup_{i \in S_2} H_i\) and \(|E(H_i)| = i, 1 \leq i \leq k\). Clearly \(\{H_1, H_2, \ldots H_k\}\) forms a CMD of \(P_n \land K_2\).

**Case (ii):** Suppose \(n\) is odd.

Define \(T_1 = \{w_{i1}, w_{(i+1)2} : 1 \leq i \leq n-1, i-\text{odd}\}\) \(\cup\{(w_{i2}, w_{(i+1)1} : 1 \leq i \leq n, i-\text{even}\}\)

and \(T_2 = \{w_{i2}, w_{(i+1)1} : 1 \leq i \leq n-1, i-\text{odd}\}\) \(\cup\{(w_{i1}, w_{(i+1)2} : 1 \leq i \leq n, i-\text{even}\}\}\).

Here \(|T_1| = n - 1\) and \(|T_2| = n - 1\). Also, \(|T_1| + |T_2| = 1 + 2 + \ldots + k = \binom{k+1}{2}\). By lemmas 2.2.1 and 2.2.2, \(\{1, 2, 3 \ldots k\} = S_1 \cup S_2\) where \(\sum x = n - 1\) and \(\sum y = n - 1\). Decompose \(T_1, T_2\) into trees \(\{H_i\}\) as follows: \(T_1 = \bigcup_{i \in S_1} H_i\) and \(T_2 = \bigcup_{i \in S_2} H_i\). \(|E(H_i)| = i, 1 \leq i \leq k\). Clearly \(\{H_1, H_2, \ldots H_k\}\) forms a CMD of \(P_n \land K_2\). \(\square\)
Illustration 2.2.4. As an illustration let us decompose $P_6 \wedge K_2$. Let $V(P_6) = \{u_1, u_2, \ldots, u_6\}$. Let $V(K_2) = \{v_1, v_2\}$. $P_6 \wedge K_2$ is given in figure 2.2.

![Figure 2.2. $P_6 \wedge K_2$](image)

Here $|E(G)| = 10$ and $m = 4$. Let $e_{ij} = ((u_i, v_1), (u_j, v_2))$ where $1 \leq i, j \leq 6$. $T_1 = \{e_{12}, e_{32}, e_{54}, e_{56}\}$ and $T_2 = \{e_{21}, e_{23}, e_{43}, e_{45}, e_{65}\}$. $|T_1| = 5 = |T_2|$. Hence $|T_1| + |T_2| = 10 = 1 + 2 + 3 + 4 = \binom{4}{2}$. $S_1 = \{1, 4\}$ and $S_2 = \{2, 3\}$. $T_1$ is decomposed as, $T_1 = H_1 \cup H_4$ where $H_1 = \{e_{12}\}$ and $H_4 = \{e_{32}, e_{34}, e_{54}, e_{56}\}$. $T_2$ is decomposed as, $T_2 = H_2 \cup H_3$ where $H_2 = \{e_{21}, e_{23}\}$ and $H_3 = \{e_{43}, e_{45}, e_{65}\}$. \{H_1, H_2, H_3, H_4\} forms a CMD of $P_6 \wedge K_2$. 

\[ \]
Table 2.1. List of first 10 \( P_n \land K'_2 S \) which accept CMD and their Decompositions

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<th>( n )</th>
<th>CMD</th>
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<td>4</td>
<td>( H_1, H_2, H_3 )</td>
</tr>
<tr>
<td>6</td>
<td>( H_1, H_2, H_3, H_4 )</td>
</tr>
<tr>
<td>15</td>
<td>( H_1, H_2, \ldots, H_7 )</td>
</tr>
<tr>
<td>19</td>
<td>( H_1, H_2, \ldots, H_8 )</td>
</tr>
<tr>
<td>34</td>
<td>( H_1, H_2, \ldots, H_{11} )</td>
</tr>
<tr>
<td>40</td>
<td>( H_1, H_2, \ldots, H_{12} )</td>
</tr>
<tr>
<td>61</td>
<td>( H_1, H_2, \ldots, H_{15} )</td>
</tr>
<tr>
<td>69</td>
<td>( H_1, H_2, \ldots, H_{16} )</td>
</tr>
<tr>
<td>96</td>
<td>( H_1, H_2, \ldots, H_{19} )</td>
</tr>
<tr>
<td>106</td>
<td>( H_1, H_2, \ldots, H_{20} )</td>
</tr>
</tbody>
</table>

2.2.2 CMD of \( C_n \land K_2 \) and \( K_{1,n} \land K_2 \)

Lemma 2.2.5. Let \( k \equiv 0 \ (mod \ 4) \). The set \( \{1, 2, 3, \ldots, k\} \) can be partitioned into two sets \( S_1, S_2 \) such that \( \sum_{x \in S_1} x = \sum_{y \in S_2} y = n \). Here \( \frac{k(k+1)}{2} = 2n \).

**Proof.** Let \( k = 4r, \ r \geq 1, r \in \mathbb{Z} \). Proof is by induction on \( r \). When \( r = 1, k = 4 \) and \( n = 5 \). Let \( S_1 = \{1, 4\} \) and \( S_2 = \{2, 3\} \). Now \( \sum_{x \in S_1} x = 1 + 4 = 5 = n \) and \( \sum_{y \in S_2} y = 2 + 3 = 5 = n \) so that the result is true if \( r = 1 \). Assume that the result is true for \( r - 1 \). Hence the set \( \{1, 2, 3, \ldots, 4(r-1)\} \) can be partitioned into two sets \( S_1 \) and \( S_2 \) such that \( \sum_{x \in S_1} x = \sum_{y \in S_2} y = n = (r-1)(4r-3) \). To prove the result...
is true for $r$. The set $\{1, 2, 3 \ldots, 4r\}$ can be partitioned into two sets $S'_1$ and $S'_2$ where $S'_1 = S_1 \cup \{4r - 3, 4r\}$ and $S'_2 = S_2 \cup \{4r - 2, 4r - 1\}$.

Now
$$\sum_{x \in S'_1} x = \sum_{x \in S_1} x + 4r - 3 + 4r = (r - 1)(4r - 3) + 4r - 3 + 4r = 4r^2 + r = n.$$ Also
$$\sum_{y \in S'_2} y = \sum_{y \in S_2} y + 4r - 2 + 4r - 1 = (r - 1)(4r - 3) + 4r - 2 + 4r - 1 = 4r^2 + r = n.$$ Hence by induction the lemma is true for all $r$. $\square$

**Lemma 2.2.6.** Let $k + 1 \equiv 0 \pmod{4}$. The set $\{1, 2, 3, \ldots, k\}$ can be partitioned into two sets $S_1$ and $S_2$ such that $\sum_{x \in S_1} x = \sum_{y \in S_2} y = n$. Here $\frac{k(k+1)}{2} = 2n$.

**Proof.** Let $k + 1 = 4r$, $r \geq 1, r \in \mathbb{Z}$ so that $k = 4r - 1$. Proof is by induction on $r$. When $r = 1, k = 3$ and $n = 3$. Let $S_1 = \{1, 2\}$ and $S_2 = \{3\}$. Now $\sum_{x \in S_1} x = 1 + 2 = 3 = n$ and $\sum_{y \in S_2} y = 3 = n$. Hence the result is true if $r = 1$. Assume that the result is true for $r - 1$. Hence the set $\{1, 2, 3, \ldots, 4(r - 1) - 1\}$ can be partitioned into two sets $S_1$ and $S_2$ such that $\sum_{x \in S_1} x = \sum_{y \in S_2} y = n = (r - 1)(4r - 5)$. To prove the result is true for $r$. The set $\{1, 2, 3, \ldots, 4r - 1\}$ can be partitioned into
two sets \( S_1' \) and \( S_2' \) where \( S_1' = S_1 \cup \{4r - 4, 4r - 1\} \) and \( S_2' = S_2 \cup \{4r - 3, 4r - 2\} \).

Now \( \sum_{x \in S_1'} x = \sum_{x \in S_1} x + 4r - 4 + 4r - 1 \)
\[ = (r - 1)(4r - 5) + 4r - 4 + 4r - 1 \]
\[ = 4r^2 - r \]
\[ = n. \]

Also \( \sum_{y \in S_2'} y = \sum_{y \in S_2} y + 4r - 3 + 4r - 2 \)
\[ = (r - 1)(4r - 5) + 8r - 5 \]
\[ = 4r^2 - r \]
\[ = n. \]

Hence by induction the lemma is true for all \( r \). \( \square \)

**Theorem 2.2.7.** For any integer \( n, C_n \wedge K_2 \) has a CMD \( \{H_1, H_2, \ldots, H_k\} \) if and only if there exists an integer \( k \) satisfying the following properties:

(i) \( k = 4r \) or \( 4r - 1 \) \( (r \geq 1, r \in \mathbb{Z}) \)

(ii) \( \frac{k(k+1)}{2} = 2n \).

**Proof.** Let \( G = C_n \wedge K_2 \). By definition, \( m = 2n \). Assume \( C_n \wedge K_2 \) has a CMD.

By definition, \( m = \binom{k+1}{2} \). Hence \( 2n = \binom{k+1}{2} = \frac{k(k+1)}{2} \).
Since \( C_n \land K_2 \) has a CMD,

\[
2n = 1 + 2 + 3 + \ldots + k
\]

\[
\Rightarrow 2n = \frac{k(k+1)}{2}
\]

\[
\Rightarrow k(k+1) = 4n
\]

Hence \( k(k+1) \equiv 0 (mod \ 4) \)

\[
\Rightarrow k(k+1) = 4r
\]

\[
\Rightarrow k = 4r \text{ or } k+1 = 4r
\]

\[
\Rightarrow k = 4r \text{ or } k = 4r - 1, \ \text{where } r \geq 1, \ r \in \mathbb{Z}
\]

Conversely, assume \( k(k+1) \equiv 0 (mod \ 4) \). Let \( G = C_n \land K_2 \). Let \( C_n = (u_1, u_2, \ldots, u_n, u_1) \) and \( K_2 = (v_1, v_2) \). Define \( w_{ij} = (u_i, v_j) \) where \( 1 \leq i \leq n, 1 \leq j \leq 2 \). Now \( V(G) = \{w_{ij} : 1 \leq i \leq n, 1 \leq j \leq 2\} \) and \( m = 2n \).

**Case (i):** Suppose \( n \) is even.

Define \( T_1 = \{(w_{i1}, w_{i(i+1)}): 1 \leq i \leq n, \ i - odd\} \cup \{(w_{i2}, w_{i(i+1)}): 1 \leq i \leq n-1, \ i - even\} \cup \{(w_{i1}, w_{11}): i = n\} \) and \( T_2 = \{(w_{i2}, w_{i(i+1)}): 1 \leq i \leq n, \ i - odd\} \cup \{(w_{i1}, w_{i(i+1)}): 1 \leq i \leq n-1, \ i - even\} \cup \{(w_{i1}, w_{12}): i = n\} \). Here \( |T_1| = n \) and \( |T_2| = n \). Also, \( |T_1| + |T_2| = 1 + 2 + 3 \ldots + k = \left(\frac{k+1}{2}\right) \). By lemmas 2.2.5 and 2.2.6, \( \{1, 2, 3, \ldots, k\} = S_1 \cup S_2 \) where \( \sum_{x \in S_1} x = n \) and \( \sum_{y \in S_2} y = n \). Decompose \( T_1 \) and \( T_2 \) into trees \( \{H_i\} \) as follows: \( T_1 = \bigcup_{i \in S_1} H_i, \ T_2 = \bigcup_{i \in S_2} H_i \) and \( |E(H_i)| = i, 1 \leq i \leq k \).

Clearly \( \{H_1, H_2, \ldots, H_k\} \) forms a CMD of \( C_n \land K_2 \).

**Case (ii):** Suppose \( n \) is odd.

Define \( T_1 = \{(w_{i1}, w_{i(i+1)}): 1 \leq i \leq n-1, \ i - odd\} \cup \{(w_{i2}, w_{i(i+1)}): 1 \leq i \leq n, \ i - even\} \cup \{(w_{i1}, w_{12}): i = n\} \) and \( T_2 = \{(w_{i2}, w_{i(i+1)}): 1 \leq i \leq n-1, \ i - odd\} \cup \{(w_{i1}, w_{i(i+1)}): 1 \leq i \leq n, \ i - even\} \cup \{(w_{i1}, w_{12}): i = n\} \).

Here \( |T_1| = n \) and \( |T_2| = n \). Also, \( |T_1| + |T_2| = 1 + 2 + 3 \ldots + k = \left(\frac{k+1}{2}\right) \).

By lemmas 2.2.5 and 2.2.6, \( \{1, 2, 3, \ldots, k\} = S_1 \cup S_2 \) where \( \sum_{x \in S_1} x = n \) and \( \sum_{y \in S_2} y = n \).
n. Decompose $T_1$ and $T_2$ into trees $\{H_i\}$ as follows: $T_1 = \bigcup_{i \in S_1} H_i$, $T_2 = \bigcup_{i \in S_2} H_i$ and $|E(H_i)| = i, 1 \leq i \leq k$. Clearly $\{H_1, H_2, \ldots, H_k\}$ forms a CMD of $C_n \wedge K_2$.

Illustration 2.2.8. As an illustration, let us decompose $C_5 \wedge K_2$. Let $V(C_5) = \{u_1, u_2, \ldots, u_5\}$ and $V(K_2) = \{v_1, v_2\}$. $C_5 \wedge K_2$ is given in Figure 2.3.

Here $|E(G)| = 10$ and $m = 4$. Let $e_{ij} = ((u_i, v_1), (u_j, v_2))$, where $1 \leq i, j \leq 5$.

$T_1 = \{e_{12}, e_{32}, e_{34}, e_{54}, e_{51}\}$, $T_2 = \{e_{21}, e_{23}, e_{43}, e_{45}, e_{15}\}$. $|T_1| = |T_2| = 5$.

Hence $|T_1| + |T_2| = 10 = 1 + 2 + 3 + 4 = \binom{5}{2}$. $S_1 = \{1, 4\}$ and $S_2 = \{2, 3\}$. $T_1$ is decomposed as $T_1 = H_1 \cup H_4$ where $H_1 = \{e_{12}\}$ and $H_2 = \{e_{32}, e_{34}, e_{54}, e_{51}\}$.

$T_2$ is decomposed as $T_2 = H_2 \cup H_3$ where $H_2 = \{e_{21}, e_{23}\}$ and $H_3 = \{e_{43}, e_{45}, e_{15}\}$.

$\{H_1, H_2, H_3, H_4\}$ forms a CMD of $C_5 \wedge K_2$.

Figure 2.3. $C_5 \wedge K_2$
Table 2.2. List of first 10 $C_n \wedge K_2'$s which accept CMD and their Decompositions

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<th>$n$</th>
<th>CMD</th>
</tr>
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<tbody>
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<td>3</td>
<td>$H_1, H_2, H_3$</td>
</tr>
<tr>
<td>5</td>
<td>$H_1, H_2, H_3, H_4$</td>
</tr>
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<td>$H_1, H_2, \ldots, H_{15}$</td>
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<tr>
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<tr>
<td>105</td>
<td>$H_1, H_2, \ldots, H_{20}$</td>
</tr>
</tbody>
</table>

**Theorem 2.2.9.** For any integer $n$, $K_{1,n} \wedge K_2$ has a CMD if and only if there exists an integer $k$ satisfying the following properties:

(i) $k = 4r$ or $4r - 1 (r \geq 1, r \in \mathbb{Z})$

(ii) $\frac{k(k+1)}{2} = 2n$

**Proof.** Let $G = K_{1,n} \wedge K_2$. By definition, $m = 2n$. Assume $K_{1,n} \wedge K_2$ has a CMD. By definition, $m = \binom{k+1}{2}$. Hence $2n = \binom{k+1}{2} = \frac{k(k+1)}{2}$. Since $K_{1,n} \wedge K_2$ has
a CMD,

\[ 2n = 1 + 2 + 3 + \ldots + k \]
\[ \Rightarrow 2n = \frac{k(k + 1)}{2} \]
\[ \Rightarrow k(k + 1) = 4n \]

Hence \( k(k + 1) \equiv 0 \pmod{4} \)
\[ \Rightarrow k(k + 1) = 4r \]
\[ \Rightarrow k = 4r \text{ or } k + 1 = 4r \]
\[ \Rightarrow k = 4r \text{ or } k = 4r - 1, \text{ where } r \geq 1, r \in \mathbb{Z} \]

Conversely assume \( k(k + 1) \equiv 0 \pmod{4} \). Let \( G = K_{1,n} \wedge K_2 \). Let \( K_{1,n} \) be the \( n \)-star with \( u_1 \) as the center and pendants denoted by \( u_2, u_3, \ldots u_{n+1} \). Let \( K_2 = (v_1, v_2) \). Define \( w_{ij} = (u_i, v_j) \) where \( 1 \leq i \leq n + 1, 1 \leq j \leq 2 \). Now \( V(G) = \{w_{ij} : 1 \leq i \leq n + 1, 1 \leq j \leq 2\} \) and \( m = 2n \). For every integer \( n \in \mathbb{Z} \), define \( T_1 = \{w_{11}, w_{(i+1)2} : 1 \leq i \leq n\} \) and \( T_2 = \{w_{12}, w_{(i+1)1} : 1 \leq i \leq n\} \). Here \( |T_1| = n \) and \( |T_2| = n \). Also, \( |T_1| + |T_2| = 1 + 2 + \ldots + k = \binom{k+1}{2} \).

By lemmas 2.2.5 and 2.2.6, \( \{1, 2, \ldots, k\} = S_1 \cup S_2 \), where \( \sum_{x \in S_1} x = n \) and \( \sum_{y \in S_2} y = n \).

Decompose \( T_1 \) and \( T_2 \) into trees \( \{H_i\} \) as follows: \( T_1 = \bigcup_{i \in S_1} H_i, T_2 = \bigcup_{i \in S_2} H_i \) and \( |E(H_i)| = i, 1 \leq i \leq k \). Clearly \( \{H_1, H_2, \ldots, H_k\} \) forms a CMD of \( K_{1,n} \wedge K_2 \). \( \square \)

**Illustration 2.2.10.** As an illustration let us decompose \( K_{1,5} \wedge K_2 \). Let \( V(K_{1,5}) = \{u_1, u_2, \ldots u_6\} \) where \( u_1 \) is the center of \( K_{1,5} \). Let \( K_2 = \{v_1, v_2\} \). \( K_{1,5} \wedge K_2 \) is given in figure 2.4.

![Diagram](image-url)
Here $E(G) = 10$ and $m = 4$. Let $e_{ij} = ((u_i, v_1), (u_j, v_2))$, where $1 \leq i, j \leq 6$. 
$T_1 = \{e_{12}, e_{13}, e_{14}, e_{15}, e_{16}\}$, $T_2 = \{e_{21}, e_{31}, e_{41}, e_{51}, e_{61}\}$.  
$|T_1| = |T_2| = 5$. Hence $|T_1| = |T_2| = 10 = 1 + 2 + 3 + 4 = \binom{5}{2}$. $S_1 = \{1, 4\}$ and $S_2 = \{2, 3\}$. $T_1$ is decomposed as, $T_1 = H_1 \cup H_4$ where $H_1 = \{e_{12}\}$ and $H_4 = \{e_{13}, e_{14}, e_{15}, e_{16}\}$. $T_2$ is decomposed as, $T_2 = H_2 \cup H_3$ where $H_2 = \{e_{21}, e_{31}\}$ and $H_3 = \{e_{41}, e_{51}, e_{61}\}$.  
$\{H_1, H_2, H_3, H_4\}$ forms a CMD of $K_{1,5} \wedge K_2$. 

Figure 2.4. $K_{1,5} \wedge K_2$
Table 2.3. List of first 10 $K_{1,n} \wedge K'_2$s which accept CMD and their Decompositions

<table>
<thead>
<tr>
<th>$n$</th>
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</tr>
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<tbody>
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<td>3</td>
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<td>5</td>
<td>$H_1, H_2, H_3, H_4$</td>
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<tr>
<td>14</td>
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</tr>
<tr>
<td>18</td>
<td>$H_1, H_2, \ldots, H_8$</td>
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<tr>
<td>33</td>
<td>$H_1, H_2, \ldots, H_{11}$</td>
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<tr>
<td>39</td>
<td>$H_1, H_2, \ldots, H_{12}$</td>
</tr>
<tr>
<td>60</td>
<td>$H_1, H_2, \ldots, H_{15}$</td>
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<tr>
<td>68</td>
<td>$H_1, H_2, \ldots, H_{16}$</td>
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<tr>
<td>94</td>
<td>$H_1, H_2, \ldots, H_{19}$</td>
</tr>
<tr>
<td>104</td>
<td>$H_1, H_2, \ldots, H_{20}$</td>
</tr>
</tbody>
</table>

2.2.3 CMD of $W_{n+1} \wedge K_2$

Lemma 2.2.11. Let $k \equiv 0$(mod $8$). The set $\{1, 2, \ldots, k\}$ can be partitioned into two sets $S_1, S_2$ such that $\sum_{x \in S_1} x = \sum_{y \in S_2} y = 2n$. Here $\frac{k(k+1)}{2} = 4n$.

Proof. Let $k = 8r$, $r \geq 1$, $r \in \mathbb{Z}$.

Proof is by induction on $r$. When $r = 1, k = 8$ and $n = 9$. Let $S_1 = \{1, 4, 5, 8\}$ and $S_2 = \{2, 3, 6, 7\}$. Now $\sum_{x \in S_1} x = 1+4+5+8 = 18 = 2n$ and $\sum_{y \in S_2} y = 2+3+6+7 = 2n$. Hence the result is true if $r = 1$. Assume the result is true for $r - 1$. Hence the set $\{1, 2, \ldots, 8(r - 1)\}$ can be partitioned into two sets $S_1$ and $S_2$ such that...
\begin{align*}
\sum_{x \in S_1} x + \sum_{y \in S_2} y &= 2n = (2r - 2)(8r - 7) \\
\text{To prove the result is true for } r.
\end{align*}

The set \( \{1, 2, 3, \ldots, 8r\} \) can be partitioned into two sets \( S'_1 \) and \( S'_2 \) where \( S'_1 = S_1 \cup \{8r, 8r - 2, 8r - 5, 8r - 7\} \) and \( S'_2 = S_2 \cup \{8r - 1, 8r - 3, 8r - 4, 8r - 6\} \).

Now \( \sum_{x \in S'_1} x = \sum_{x \in S_1} x + 8r + 8r - 2 + 8r - 5 + 8r - 7 = (2r - 2)(8r - 7) + 32r - 14 = 16r^2 + 2r = 2n. \)

Also \( \sum_{y \in S'_2} y = \sum_{y \in S_2} y + 8r - 1 + 8r - 3 + 8r - 4 + 8r - 6 = (2r - 2)(8r - 7) + 32r - 14 = 16r^2 + 2r = 2n. \)

Hence the lemma is proved for all \( r \). \( \blacksquare \)

**Lemma 2.2.12.** Let \( k + 1 \equiv 0(\text{mod} 8) \). The set \( \{1, 2, \ldots, k\} \) can be partitioned into two sets \( S_1 \) and \( S_2 \) such that \( \sum_{x \in S_1} x = \sum_{y \in S_2} y = 2n. \) Here \( \frac{k(k+1)}{2} = 4n \).

**Proof.** Let \( k + 1 = 8r, r \geq 1, r \in \mathbb{Z} \) so that \( k = 8r - 1 \).

Proof is by induction on \( r \). When \( r = 1, k = 7 \) and \( n = 7 \). Let \( S_1 = \{1, 2, 4, 7\} \) and \( S_2 = \{3, 5, 6\} \). Now \( \sum_{x \in S_1} x = 1 + 2 + 4 + 7 = 14 = 2n \) and \( \sum_{y \in S_2} y = 3 + 5 + 6 = 14 = 2n \). Hence the result is true if \( r = 1 \). Assume the result is true for \( r - 1 \).

Hence the set \( \{1, 2, \ldots, (8(r - 1) - 1)\} \) can be partitioned into two sets \( S_1 \) and \( S_2 \) such that \( \sum_{x \in S_1} x = \sum_{y \in S_2} y = 2n = (2r - 2)(8r - 9). \)

To prove the result is true for \( r \).

The set \( \{1, 2, 3, \ldots, 8r - 1\} \) can be partitioned into sets \( S'_1 \) and \( S'_2 \) where \( S'_1 = \ldots \).
Now \( \sum_{x \in S_1'} x = \sum_{x \in S_1} x + 8r - 1 + 8r - 3 + 8r - 6 + 8r - 8 \)
\[ = (2r - 2)(8r - 9) + 32r - 18 \]
\[ = 16r^2 - 2r \]
\[ = 2n. \]

Also \( \sum_{y \in S_2'} y = \sum_{y \in S_2} y + 8r - 2 + 8r - 4 + 8r - 5 + 8r - 7 \)
\[ = (2r - 2)(8r - 9) + 32r - 18 \]
\[ = 16r^2 - 2r \]
\[ = 2n. \]

Hence the lemma is proved for all \( r \). \qed

**Theorem 2.2.13.** For any integer \( n, W_{n+1} \land K_2 \) has a CMD \( \{H_1, H_2, \ldots, H_k\} \) if and only if there exists an integer \( k \) satisfying the following properties:

(i) \( k = 8r \) or \( 8r - 1 \) (\( r \geq 1, r \in \mathbb{Z} \))

(ii) \( \frac{k(k+1)}{2} = 4n \)

**Proof.** Let \( G = W_{n+1} \land K_2 \). By definition, \( m = 4n \). Assume \( W_{n+1} \land K_2 \) has a CMD. By definition, \( m = \binom{k+1}{2} \). Hence \( 4n = \binom{k+1}{2} = \frac{k(k+1)}{2} \). Since \( W_{n+1} \land K_2 \) has
a CMD,

\[ 4n = 1 + 2 + \ldots + k \]
\[ \Rightarrow 4n = \frac{k(k + 1)}{2} \]
\[ \Rightarrow k(k + 1) = 8n. \]

Hence \( k(k + 1) \equiv 0 \pmod{8} \)
\[ \Rightarrow k(k + 1) = 8r \]
\[ \Rightarrow k = 8r \text{ or } k + 1 = 8r \]
\[ \Rightarrow k = 8r \text{ or } k = 8r - 1 \] where \( r \geq 1, r \in \mathbb{Z} \)

Conversely, assume \( k(k + 1) \equiv 0 \pmod{8} \). Let \( G = W_{n+1} \wedge K_2 \). Here \( W_{n+1} \) is a wheel on \( n+1 \) vertices with \( u_1, u_2, \ldots, u_n \) representing the vertices on the cycle and \( u_{n+1} \) representing the vertex of degree \( n \). Let \( K_2 = (v_1, v_2) \). Define \( w_{ij} = (u_i, v_j) \) where \( 1 \leq i \leq n+1, 1 \leq j \leq 2 \). Now \( V(G) = \{w_{ij} : 1 \leq i \leq n+1, 1 \leq j \leq 2\} \) and \( m = 4n \). For every integer \( n \in \mathbb{Z} \), define \( T_1 = \{w_{i1}, w_{(i+1)2} : 1 \leq i \leq n, i-odd\} \cup \{(w_{i1}, w_{(n+1)2} : 1 \leq i \leq n, i-odd) \cup \{w_{(i+1)1}, w_{i2} : 1 \leq i < n, i-odd\} \cup \{(w_{(i+1)1}, w_{i2} : 1 \leq i < n, i-even\} \cup \{(w_{i1}, w_{(n+1)2} : 1 \leq i \leq n, i-even\} \cup \{w_{(i+1)1}, w_{i2} : 1 \leq i < n, i-even\} \cup \{(w_{(n+1)1}, w_{i2} : 1 \leq i \leq n, i-even\} \cup \{w_{i1}, w_{i2} : i = n\} \cup \{w_{11}, w_{21} : i = n\}. \) Here \( |T_1| = 2n \) and \( |T_2| = 2n \). Also, \( |T_1| + |T_2| = 1 + 2 + \ldots + k = \binom{k+1}{2} \). By lemmas 2.2.11 and 2.2.12, \( \{1, 2, 3 \ldots k\} = S_1 \cup S_2 \) where \( \sum_{x \in S_1} x = 2n \) and \( \sum_{y \in S_2} y = 2n \). Decompose \( T_1 \) and \( T_2 \) into trees \( \{H_i\} \) as follows: \( T_1 = \bigcup_{i \in S_1} H_i, T_2 = \bigcup_{i \in S_2} H_i \) and \( |E(H_i)| = i, 1 \leq i \leq k \). Clearly \( \{H_1, H_2, \ldots, H_k\} \) forms a CMD of \( W_{n+1} \wedge K_2 \).

**Illustration 2.2.14.** As an illustration let us decompose \( W_8 \wedge K_2 \). Let \( V(W_8) = \{u_1, u_2, \ldots, u_8\} \) where \( u_1, u_2, \ldots, u_7 \) are the vertices on the cycle and \( u_8 \) represents the vertex of degree 7. Let \( K_2 = \{v_1, v_2\} \). \( W_8 \wedge K_2 \) is given in Figure 2.5.
Here \(|E(G)| = 28\) and \(m = 7\). Let \(e_{ij} = ((u_i, v_1), (u_j, v_2))\). \(T_1 = \{e_{12}, e_{34}, e_{56}, e_{18}, e_{28}, e_{58}, e_{78}, e_{21}, e_{43}, e_{65}, e_{81}, e_{83}, e_{85}, e_{87}\}\). \(T_2 = \{e_{17}, e_{23}, e_{28}, e_{32}, e_{45}, e_{48}, e_{54}, e_{67}, e_{68}, e_{71}, e_{76}, e_{82}, e_{84}, e_{86}\}\). \(|T_1| = |T_2| = 14\). Hence \(|T_1| = |T_2| = 28 = 1 + 2 + 3 + 4 + 5 + 6 + 7 = (\binom{8}{2})\). Here \(S_1 = \{1, 2, 4, 7\}\) and \(S_2 = \{3, 5, 6\}\). \(T_1\) is decomposed as \(T_1 = H_1 \cup H_2 \cup H_4 \cup H_7\) where \(H_1 = \{e_{56}\}\), \(H_2 = \{e_{12}, e_{18}\}\), \(H_4 = \{e_{34}, e_{38}, e_{58}, e_{78}\}\) and \(H_7 = \{e_{21}, e_{43}, e_{65}, e_{81}, e_{83}, e_{85}, e_{87}\}\). \(T_2\) is decomposed as \(T_2 = H_3 \cup H_5 \cup H_6\) where \(H_3 = \{e_{67}, e_{71}, e_{17}\}\), \(H_5 = \{e_{23}, e_{28}, e_{45}, e_{48}, e_{68}\}\) and \(H_6 = \{e_{32}, e_{54}, e_{76}, e_{82}, e_{84}, e_{86}\}\). Clearly \(\{H_1, H_2, \ldots, H_7\}\) forms a CMD of \(W_n \land K_2\).
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</tr>
<tr>
<td>205</td>
<td>$H_1, H_2, \ldots, H_{40}$</td>
</tr>
</tbody>
</table>

Table 2.4. List of first 10 $W_{n+1} \land K_2$'s which accept CMD and their Decompositions

2.2.4 CMD of $G \land K_2$ where $G$ is a specified Caterpillar on \( n \) vertices

In this section, we investigate tensor product of two types of Caterpillars with $K_2$.

**Definition 2.2.15.** [17] Caterpillar is a tree with the property that the removal of its end points leave a path.

The first type of caterpillar we consider is $G_n$ which is a particular caterpillar that is defined recursively as follows:
$G_3$ is $K_{1,2}$

$G_4$ is

Continuing in this way $G_n$ is a caterpillar in which every vertex of the base path $u_1, u_4, u_7, \ldots, u_l$ is of degree 3 except probably $u_l$ which can be of degree 1, 2 or 3. Here $u_l$ is the last vertex in the base path.

The second type of caterpillar is the comb.

**Theorem 2.2.16.** For any integer $n$, $G_n \wedge K_2$ has a CMD if and only if there exists an integer $k$ satisfying the following properties:

(i) $k = 4r$ or $4r - 1$ $(r \geq 1, r \in \mathbb{Z})$

(ii) $\frac{k(k+1)}{2} = 2n - 2$.

**Proof.** Let $G = G_n \wedge K_2$. By definition, $m = 2n - 2$. Assume $G_n \wedge K_2$ has a CMD. By definition, $m = \binom{k+1}{2}$. Hence $2n - 2 = \binom{k+1}{2}$ i.e., $2n - 2 = \frac{k(k+1)}{2}$. Since $G_n \wedge K_2$ has a CMD, $2n - 2 = 1 + 2 + \ldots + k$ i.e., $2(n - 1) = \frac{k(k+1)}{2}$. So $k(k+1) = 4(n - 1)$. Hence $k(k+1) \equiv 0 \pmod{4}$

$\Rightarrow k(k+1) = 4r$

$\Rightarrow k = 4r$ or $k + 1 = 4r$. 

27
Hence $k = 4r$ or $k = 4r - 1$ where $r \geq 1, r \in \mathbb{Z}$.

Conversely, assume $k(k + 1) \equiv 0 \pmod{4}$. Let $G = G_n \land K_2$. Define $w_{ij} = (u_i, v_j)$ where $1 \leq i \leq n, 1 \leq j \leq 2$. Define $V(G) = \{w_{ij} : 1 \leq i \leq n, 1 \leq j \leq 2\}$ and $m = 2n - 2$.

**Case (i): $n$ is odd**

**Subcase (a): $deg(u_i) = 3$.**

Define $T_1 = \{(w_{i1}, w_{(i+1)2}) : i = 6j - 5, j = 1, 2, \ldots, \left\lfloor \frac{n}{6} \right\rfloor\}$

$\cup\{(w_{i1}, w_{(i+2)2}) : i = 6j - 5, j = 1, 2, \ldots, \left\lfloor \frac{n}{6} \right\rfloor\}$

$\cup\{(w_{i1}, w_{(i+3)2}) : i = 6j - 5, j = 1, 2, \ldots, \left\lfloor \frac{n}{6} \right\rfloor\}$

and $T_2 = \{(w_{i1}, w_{i2}) : i = 6j - 5, j = 1, 2, \ldots, \left\lfloor \frac{n}{6} \right\rfloor\}$

$\cup\{(w_{i1}, w_{(i+1)2}) : i = 6j - 5, j = 1, 2, \ldots, \left\lfloor \frac{n}{6} \right\rfloor\}$

$\cup\{(w_{i1}, w_{(i+2)2}) : i = 6j - 5, j = 1, 2, \ldots, \left\lfloor \frac{n}{6} \right\rfloor\}$

$\cup\{(w_{i1}, w_{(i+3)2}) : i = 6j - 5, j = 1, 2, \ldots, \left\lfloor \frac{n}{6} \right\rfloor\}$

Here $|T_1| = n - 1$ and $|T_2| = n - 1$. Also, $|T_1| + |T_2| = 1 + 2 + \ldots + k = \left(\frac{k+1}{2}\right)$.

By lemmas 2.2.1 and 2.2.2, $\{1, 2, \ldots, k\} = S_1 \cup S_2$ where $\sum_{x \in S_1} x = n - 1 = \sum_{y \in S_2} y$.

Decompose $T_1$ and $T_2$ into trees $\{H_i\}$ as follows: $T_1 = \cup_{i \in S_1} H_i$, $T_2 = \cup_{i \in S_2} H_i$ and $|E(H_i)| = i, 1 \leq i \leq k$. Clearly $\{H_1, H_2, \ldots, H_k\}$ forms a CMD of $G_n \land K_2$.

**Subcase (b): $deg(u_i) = 2$**

Then it is not possible to have $|E(H_i)| = i, 1 \leq i \leq k$, since $m \neq \left(\frac{k+1}{2}\right)$ for any $k$.

Hence $G_n \land K_2$ cannot have a CMD.

**Subcase (c): $deg(u_i) = 1$**

Define $T_1 = \{(w_{i1}, w_{(i+1)2}) : i = 6j - 5, j = 1, 2, \ldots, \left\lfloor \frac{n}{6} \right\rfloor\}$

$\cup\{(w_{i1}, w_{(i+2)2}) : i = 6j - 5, j = 1, 2, \ldots, \left\lfloor \frac{n}{6} \right\rfloor\}$

28
Here $|T_1| = n - 1$ and $|T_2| = n - 1$. Also, $|T_1| + |T_2| = 1 + 2 + \ldots + k = \binom{k+1}{2}$. By lemmas 2.2.1 and 2.2.2, $\{1, 2, \ldots, k\} = S_1 \cup S_2$ where $\sum_{x \in S_1} x = n - 1 = \sum_{y \in S_2} y$. Decompose $T_1$ and $T_2$ into trees $\{H_i\}$ as follows: $T_1 = \cup_{i \in S_1} H_i$, $T_2 = \cup_{i \in S_2} H_i$ and $|E(H_i)| = i, 1 \leq i \leq k$. Clearly $\{H_1, H_2, \ldots, H_k\}$ forms a CMD of $G_n \land K_2$.

Case(ii): $n$ is even.

Subcase(a): $\text{deg}(u_i) = 3$.

Define $T_1 = \{(w_{i1}, w_{i(i+1)2}) : i = 6j - 5, j = 1, 2, \ldots, \left\lfloor \frac{n}{6} \right\rfloor\}$

$\cup \{(w_{i1}, w_{i(i+2)2}) : i = 6j - 5, j = 1, 2, \ldots, \left\lfloor \frac{n}{6} \right\rfloor\}$

$\cup \{(w_{i1}, w_{i(i+3)2}) : i = 6j - 5, j = 1, 2, \ldots, \left\lfloor \frac{n}{6} \right\rfloor\}$

$\cup \{(w_{i1}, w_{i(i+1)1}) : i = 6j - 2, j = 1, 2, \ldots, \left\lfloor \frac{n}{6} \right\rfloor\}$

$\cup \{(w_{i1}, w_{i(i+2)1}) : i = 6j - 2, j = 1, 2, \ldots, \left\lfloor \frac{n}{6} \right\rfloor\}$

$\cup \{(w_{i1}, w_{i(i+3)1}) : i = 6j - 2, j = 1, 2, \ldots, \left\lfloor \frac{n}{6} \right\rfloor\}$

and $T_2 = \{(w_{i+11}, w_{i+22}) : i = 6j - 5, j = 1, 2, \ldots, \left\lfloor \frac{n}{6} \right\rfloor\}$

$\cup \{(w_{i+11}, w_{i+32}) : i = 6j - 5, j = 1, 2, \ldots, \left\lfloor \frac{n}{6} \right\rfloor\}$

$\cup \{(w_{i+11}, w_{i+21}) : i = 6j - 2, j = 1, 2, \ldots, \left\lfloor \frac{n}{6} \right\rfloor\}$

$\cup \{(w_{i+11}, w_{i+31}) : i = 6j - 2, j = 1, 2, \ldots, \left\lfloor \frac{n}{6} \right\rfloor\}$

$\cup \{(w_{i+11}, w_{i+23}) : i = 6j - 2, j = 1, 2, \ldots, \left\lfloor \frac{n}{6} \right\rfloor\}$

$\cup \{(w_{i+11}, w_{i+33}) : i = 6j - 2, j = 1, 2, \ldots, \left\lfloor \frac{n}{6} \right\rfloor\}$
Here $|T_1| = n - 1$ and $|T_2| = n - 1$. Also, $|T_1| + |T_2| = 1 + 2 + \ldots + k = \binom{k+1}{2}$.

By lemmas 2.2.1 and 2.2.2, $\{1, 2, \ldots, k\} = S_1 \cup S_2$ where $\sum_{x \in S_1} x = n - 1 = \sum_{y \in S_2} y$.

Decompose $T_1$ and $T_2$ into trees $\{H_i\}$ as follows: $T_1 = \bigcup_{i \in S_1} H_i$, $T_2 = \bigcup_{i \in S_2} H_i$ and $|E(H_i)| = i, 1 \leq i \leq k$. Clearly $\{H_1, H_2, \ldots, H_k\}$ forms a CMD of $G_n \wedge K_2$.

**Subcase (b):** $\deg(u_i) = 2$

Then it is not possible to have $|E(H_i)| = i, 1 \leq i \leq k$, since $m \neq \binom{k+1}{2}$ for any $k$.

Hence $G_n \wedge K_2$ cannot have a CMD.

**Subcase (c):** $\deg(u_i) = 1$

Then it is not possible to have $|E(H_i)| = i, 1 \leq i \leq k$, since $m \neq \binom{k+1}{2}$ for any $k$.

Hence $G_n \wedge K_2$ cannot have a CMD.

Illustration 2.2.17. As an illustration, let us decompose $G_{15} \wedge K_2$. Let $V(G_{15}) = \{u_1, u_2, \ldots, u_{15}\}$ and let $V(K_2) = \{v_1, v_2\}$. $G_{15} \wedge K_2$ is given in figure 2.6.
Figure 2.6. $G_{15} \land K_2$
Here \( m = 28 \) and \( k = 7 \). Let \( e_{ij} = ((u_i, v_1), (u_j, v_2)) \) where \( 1 \leq i \leq 15; 1 \leq j \leq 15 \). In this illustration \( d(u_i) = 3 \). Hence \( T_1 = \{e_{12}, e_{78}, e_{13,14}, e_{79}, e_{13,15}, e_{14}, e_{7,10}, e_{54}, e_{11,10}, e_{64}, e_{12,10}, e_{74}, e_{13,10}\} \) and \( T_2 = \{e_{21}, e_{87}, e_{14,13}, e_{31}, e_{97}, e_{15,13}, e_{41}, e_{10,7}, e_{45}, e_{10,11}, e_{46}, e_{10,12}, e_{47}, e_{10,13}\} \). Here \( |T_1| = 14 \) and \( |T_2| = 14 \). Hence \( |T_1| + |T_2| = 28 = 1 + 2 + \ldots + 7 = \binom{8}{2} \). \( S_1 = \{1, 2, 4, 7\} \) and \( S_2 = \{3, 5, 6\} \). \( T_1 \) is decomposed as \( T_1 = H_1 \cup H_2 \cup H_4 \cup H_7 \) where \( H_1 = \{e_{12}\}, H_2 = \{e_{13}, e_{14}\}, H_4 = \{e_{54}, e_{64}, e_{74}, e_{78}\} \) and \( H_7 = \{e_{79}, e_{7,10}, e_{11,10}, e_{12,10}, e_{13,10}, e_{13,14}, e_{13,15}\} \). \( T_2 \) is decomposed as \( T_2 = H_3 \cup H_5 \cup H_6 \) where \( H_3 = \{e_{21}, e_{31}, e_{41}\}, H_5 = \{e_{45}, e_{46}, e_{47}, e_{87}, e_{97}\} \) and \( H_6 = \{e_{10,7}, e_{10,11}, e_{10,12}, e_{10,13}, e_{14,13}, e_{15,13}\} \). Therefore \( \{H_1, H_2, \ldots, H_7\} \) forms a CMD of \( G_{15} \land K_2 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \text{CMD} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>( H_1, H_2, \ldots, H_7 )</td>
</tr>
<tr>
<td>9</td>
<td>( H_1, H_2, \ldots, H_8 )</td>
</tr>
<tr>
<td>30</td>
<td>( H_1, H_2, \ldots, H_{15} )</td>
</tr>
<tr>
<td>34</td>
<td>( H_1, H_2, \ldots, H_{16} )</td>
</tr>
<tr>
<td>69</td>
<td>( H_1, H_2, \ldots, H_{23} )</td>
</tr>
<tr>
<td>75</td>
<td>( H_1, H_2, \ldots, H_{24} )</td>
</tr>
<tr>
<td>124</td>
<td>( H_1, H_2, \ldots, H_{31} )</td>
</tr>
<tr>
<td>132</td>
<td>( H_1, H_2, \ldots, H_{32} )</td>
</tr>
<tr>
<td>195</td>
<td>( H_1, H_2, \ldots, H_{39} )</td>
</tr>
<tr>
<td>205</td>
<td>( H_1, H_2, \ldots, H_{40} )</td>
</tr>
</tbody>
</table>

Table 2.5 List of first 10 \( G_n \land K_2' \)s which accept CMD and their decompositions
Definition 2.2.18. [16] Comb is a particular type of caterpillar denoted by $Cb_n$. It is defined as $P_2 \odot K_1$ where $\odot$ represents corona of two graphs.

Lemma 2.2.19. Let $k + 1 \equiv 0 \pmod{4}$. The set $\{1, 2, \ldots, k\}$ can be partitioned into two sets $S_1$ and $S_2$ such that $\sum x = n - 1 = \sum y$. Here $\frac{k(k+1)}{2} = 2n - 2$.

Proof. Let $k + 1 = 4r, r > 2, r \in \mathbb{Z}$ ie., $k = 4r - 1$. Proof is by induction on $r$. When $r = 3, k = 11$. Now $n = 34$. Then $S_1 = \{1, 2, 4, 7, 8, 11\}$ and $S_2 = \{3, 5, 6, 9, 10\}$. $\sum x = 33 = 34 - 1$. $\sum y = 33 = 34 - 1$. Hence the result is true if $r = 3$. Assume the result is true for $r - 1$. Hence the set $\{1, 2, \ldots, 4(r-1)-1\}$ can be partitioned into two sets $S_1$ and $S_2$ such that $\sum x = \sum y = n - 1 = (r-1)(4r - 5)$. To prove the result is true for $r$. The set $\{1, 2, \ldots, 4r-1\}$ can be partitioned into sets $S'_1$ and $S'_2$ where $S'_1 = S_1 \cup \{4r - 4, 4r - 1\}$ and $S'_2 = S_2 \cup \{4r - 3, 4r - 2\}$.

Now $\sum x = \sum x + 4r - 4 + 4r - 1 = (4r - 5)(r - 1) + 4r - 4 + 4r - 1 = 4r^2 - r - n - 1$. Similarly $\sum y = 4r^2 - r = n - 1$. Hence by induction hypothesis, the lemma is true for all $r$. 

Theorem 2.2.20. For any integer $n$, where $n$ is the total number of vertices, $Cb_n \land K_2$ has a CMD if and only if there exists an integer $k$ satisfying the following properties:

(i) $k = 4r$ or $k = 4r - 1 (r \geq 1, r \in \mathbb{Z})$

(ii) $\frac{k(k+1)}{2} = 2n - 2$.

Proof. Let $G = Cb_n \land K_2$. By definition $m = 2n - 2$. Assume $Cb_n \land K_2$ has a CMD. By definition, $m = \binom{k+1}{2}$. Hence $2n - 2 = \binom{k+1}{2}$ ie., $2n - 2 = \frac{k(k+1)}{2}$. Since $Cb_n \land K_2$ has a CMD, $2n - 2 = 1 + 2 + \ldots + k$. Therefore $2(n - 1) = \frac{k(k+1)}{2}$ ie., $k(k+1) = 4(n - 1)$. Hence $k(k+1) \equiv 0 \pmod{4}$

$\Rightarrow k(k+1) = 4r$

$\Rightarrow k = 4r$ or $k + 1 = 4r$.
$\Rightarrow k = 4r \text{ or } k = 4r - 1 \text{ where } r \geq 1, r \in \mathbb{Z}$. Conversely, assume that $k(k + 1) \equiv 0 (mod \ 4)$. Let $G = Cb_n \land K_2$. Define $w_{ij} = (u_i, v_j)$ where $1 \leq i \leq n$, $1 \leq j \leq 2$. Define $V(G) = \{w_{ij} : 1 \leq i \leq n, 1 \leq j \leq 2\}$ and $m = 2n - 2$. For every positive integer $n$, define $T_1 = \{(w_{i1}, w_{(i+1)2}) : i = 4j - 3, j = 1, 2, \ldots, \left\lceil \frac{n}{4} \right\rceil\}$
$\cup\{(w_{i1}, w_{(i+2)2}) : i = 4j - 3, j = 1, 2, \ldots, \left\lceil \frac{n}{4} \right\rceil\}$
$\cup\{(w_{(i+1)1}, w_{i2}) : i = 4j - 1, j = 1, 2, \ldots, \left\lceil \frac{n}{4} \right\rceil\}$
$\cup\{(w_{(i+2)1}, w_{i2}) : i = 4j - 1, j = 1, 2, \ldots, \left\lceil \frac{n}{4} \right\rceil\}$ and $T_2 = \{(w_{(i+1)1}, w_{i2}) : i = 4j - 3, j = 1, 2, \ldots, \left\lceil \frac{n}{4} \right\rceil\}$
$\cup\{(w_{(i+2)1}, w_{i2}) : i = 4j - 3, j = 1, 2, \ldots, \left\lceil \frac{n}{4} \right\rceil\}$
$\cup\{(w_{i1}, w_{(i+1)2}) : i = 4j - 1, j = 1, 2, \ldots, \left\lceil \frac{n}{4} \right\rceil\}$
$\cup\{(w_{i1}, w_{(i+2)2}) : i = 4j - 1, j = 1, 2, \ldots, \left\lceil \frac{n}{4} \right\rceil\}$.

Here $|T_1| = n - 1$ and $|T_2| = n - 1$. Also, $|T_1| + |T_2| = 1 + 2 + \ldots + k = \binom{k+1}{2}$.

By lemmas 2.2.1 and 2.2.19, $\{1, 2, \ldots, k\} = S_1 \cup S_2$ where $\sum_{x \in S_1} x = n - 1 = \sum_{y \in S_2} y$.

Decompose $T_1$ and $T_2$ into trees $\{H_i\}$ as follows: $T_1 = \bigcup_{i \in S_1} H_i$, $T_2 = \bigcup_{i \in S_2} H_i$ and $|E(H_i)| = i, 1 \leq i \leq k$. Clearly $\{H_1, H_2, \ldots, H_k\}$ forms a CMD of $Cb_n \land K_2$. $\square$

**Illustration 2.2.21.** As an illustration, let us decompose $Cb_6 \land K_2$. Let $V(Cb_6) = \{u_1, u_2, \ldots, u_6\}$ and let $V(K_2) = \{v_1, v_2\}$. $Cb_6 \land K_2$ is given in figure 2.7.
Figure 2.7. $C_{b_6} \land K_2$
Here \( m = 10 \) and \( k = 4 \). Let \( e_{ij} = ((u_i, v_1), (u_j, v_2)) \). Here \( T_1 = \{e_{12}, e_{13}, e_{43}, e_{53}, e_{56}\} \) and \( T_2 = \{e_{21}, e_{31}, e_{34}, e_{35}, e_{65}\} \). \(|T_1| = 5 = |T_2|\). Hence \(|T_1| + |T_2| = 10 = 1 + 2 + 3 + 4 = \binom{5}{2}\). \( S_1 = \{1, 4\} \) and \( S_2 = \{2, 3\} \). \( T_1 \) is decomposed as \( T_1 = H_1 \cup H_4 \) where \( H_1 = \{e_{12}\} \) and \( H_4 = \{e_{13}, e_{43}, e_{53}, e_{56}\} \). \( T_2 \) is decomposed as \( T_2 = H_2 \cup H_3 \) where \( H_2 = \{e_{21}, e_{31}\} \) and \( H_3 = \{e_{34}, e_{35}, e_{65}\} \). Hence \( \{H_1, H_2, H_3, H_4\} \) forms a CMD of \( Cb_6 \land K_2 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( CMD )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>( H_1, H_2, \ldots, H_4 )</td>
</tr>
<tr>
<td>34</td>
<td>( H_1, H_2, \ldots, H_{11} )</td>
</tr>
<tr>
<td>40</td>
<td>( H_1, H_2, \ldots, H_{12} )</td>
</tr>
<tr>
<td>96</td>
<td>( H_1, H_2, \ldots, H_{19} )</td>
</tr>
<tr>
<td>106</td>
<td>( H_1, H_2, \ldots, H_{20} )</td>
</tr>
<tr>
<td>190</td>
<td>( H_1, H_2, \ldots, H_{27} )</td>
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<td>204</td>
<td>( H_1, H_2, \ldots, H_{28} )</td>
</tr>
<tr>
<td>316</td>
<td>( H_1, H_2, \ldots, H_{35} )</td>
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<td>334</td>
<td>( H_1, H_2, \ldots, H_{36} )</td>
</tr>
<tr>
<td>474</td>
<td>( H_1, H_2, \ldots, H_{43} )</td>
</tr>
</tbody>
</table>

Table 2.6 List of first 10 \( Cb_n \land K'_2 \)'s which accept CMD and their decompositions.
2.2.5 CMD of tensor product of Lobster with $K_2$

In this section, we prove $L_n \wedge K_2$ admits CMD under certain conditions. We prove two lemmas to exhibit a CMD of $L_n \wedge K_2$ in theorem 2.2.25.

**Definition 2.2.22.** [15] Lobster is a tree with the property that the removal of the endpoints leave a caterpillar. Here $L_n$ is a particular Lobster that is defined recursively as follows:

$L_4:
\begin{array}{c}
\mathbf{u}_1 \\
\mathbf{u}_2 & \mathbf{u}_4 \\
\mathbf{u}_3
\end{array}$

$L_5:
\begin{array}{c}
\mathbf{u}_1 \\
\mathbf{u}_2 & \mathbf{u}_4 \\
\mathbf{u}_3 & \mathbf{u}_5
\end{array}$
Continuing in this way, \( L_n \) is a Lobster in which vertices \( u_1, u_6, u_{11}, \ldots, u_l \) are said to be at the root level. \( d(u_1) = 3; d(u_6) = d(u_{11}) = \ldots = 4; \) and \( d(u_l) \) is at most \( 3( u_l \) is the last vertex in the root level). Vertices \( u_2, u_4, u_7, \ldots, u'_l \) are said to be at level 1. \( d(u_2) = d(u_4) = \ldots = 2; \) and \( d(u'_l) \) is at most 2. Vertices \( u_3, u_5, u_8, \ldots, u''_l \) are said to be at level 2 and all are of degree 1.

**Lemma 2.2.23.** Let \( k \equiv 0(\text{mod } 4) \). The set \( \{1, 2, \ldots, k\} \) can be partitioned into two sets \( S_1, S_2 \) such that \( \sum_{x \in S_1} x = n - 1 = \sum_{y \in S_2} y \). Here \( \frac{k(k+1)}{2} = 2n - 2 \).

**Proof.** Let \( k = 4r \), where \( r > 1, r \in \mathbb{Z} \). Proof is by induction on \( r \). When \( r = 2, \) \( k = 8 \). Now \( n = 19 \). Then \( S_1 = \{1, 4, 5, 8\} \) and \( S_2 = \{2, 3, 6, 7\} \).
\[ \sum_{x \in S_1} x = 18 = 19 - 1. \sum_{y \in S_2} y = 18 = 19 - 1. \] Hence the result is true if \( r = 2 \). Assume that the result is true for \( r - 1 \). Hence the set \( \{1, 2, \ldots, 4(r-1)\} \) can be partitioned into two sets \( S_1 \) and \( S_2 \) such that \( \sum_{x \in S_1} x = \sum_{y \in S_2} y = n - 1 = (r - 1)(4r - 3) \).

To prove the result is true for \( r \). The set \( \{1, 2, \ldots, 4r\} \) can be partitioned into sets \( S'_1 \) and \( S'_2 \) where \( S'_1 = S_1 \cup \{4r - 3, 4r\} \) and \( S'_2 = S_2 \cup \{4r - 2, 4r - 1\} \). Now
\[ \sum_{x \in S'_1} x = \sum_{x \in S_1} x + 4r - 3 + 4r = (4r - 3)(r - 1) + 8r - 3 = 4r^2 + r = n - 1. \] Similarly
\[ \sum_{y \in S'_2} y = 4r^2 + r = n - 1. \] Hence by induction hypothesis, the lemma is true for all \( r \). \( \square \)
Lemma 2.2.24. Let \( k + 1 \equiv 0(\text{mod} \ 4) \). The set \( \{1, 2, \ldots, k\} \) can be partitioned into two sets \( S_1 \) and \( S_2 \) such that \( \sum_{x \in S_1} x = n - 1 = \sum_{y \in S_2} y \). Here \( \frac{k(k+1)}{2} = 2n - 2 \).

Proof. Let \( k + 1 = 4r \), where \( r > 1, r \in \mathbb{Z} \) ie., \( k = 4r - 1 \). Proof is by induction on \( r \). When \( r = 2, k = 7 \). Now \( n = 15 \). Then \( S_1 = \{1, 2, 4, 7\} \) and \( S_2 = \{3, 5, 6\} \).

\[ \sum_{x \in S_1} x = 14 = 15 - 1 \text{ and } \sum_{y \in S_2} y = 14 = 15 - 1 \] Hence the result is true if \( r = 2 \). Assume the result is true for \( r - 1 \). Hence the set \( \{1, 2, \ldots, 4(r - 1) - 1\} \) can be partitioned into two sets \( S_1 \) and \( S_2 \) such that \( \sum_{x \in S_1} x = \sum_{y \in S_2} y = n - 1 = (r - 1)(4r - 5) \). To prove the result is true for \( r \). The set \( \{1, 2, \ldots, 4r - 1\} \) can be partitioned into sets \( S'_1 \) and \( S'_2 \) where \( S'_1 = S_1 \cup \{4r - 4, 4r - 1\} \) and \( S'_2 = S_2 \cup \{4r - 3, 4r - 2\} \).

Now \( \sum_{x \in S'_1} x = \sum_{x \in S_1} x + 4r - 4 + 4r - 1 = (4r - 5)(r - 1) + 8r - 5 = 4r^2 - r = n - 1 \).

Similarly \( \sum_{y \in S'_2} y = 4r^2 + r = n - 1 \). Hence by induction hypothesis, the lemma is true for all \( r \).

\[ \square \]

Theorem 2.2.25. For any integer \( n \), \( L_n \cap K_2 \) has a CMD if and only if there exists an integer \( k \) satisfying the following properties:

(i) \( k = 4r \) or \( k = 4r - 1, r > 1, r \in \mathbb{Z} \)

(ii) \( \frac{k(k+1)}{2} = 2n - 2 \).

Proof. Let \( G = L_n \cap K_2 \). By definition \( m = 2n - 2 \). Assume \( L_n \cap K_2 \) has CMD.

By definition, \( m = \binom{k+1}{2} \). Hence \( 2n - 2 = \binom{k+1}{2} \) ie., \( 2n - 2 = \frac{k(k+1)}{2} \). Since \( L_n \cap K_2 \) has a CMD, \( 2n - 2 = 1 + 2 + \ldots + k \). Therefore \( 2(n - 1) = \frac{k(k+1)}{2} \) ie., \( k(k+1) = 4(n - 1) \). Hence \( k(k+1) \equiv 0(\text{mod} \ 4) \)

\[ \Rightarrow k(k+1) = 4r \]

\[ \Rightarrow k = 4r \text{ or } k + 1 = 4r \]

\[ \Rightarrow k = 4r \text{ or } k = 4r - 1 \] where \( r > 1, r \in \mathbb{Z} \). Conversely, assume that \( k(k+1) \equiv 0(\text{mod} \ 4) \). Let \( G = L_n \cap K_2 \). Define \( w_{ij} = (u_i, v_j) \) where \( 1 \leq i \leq n, 1 \leq j \leq 2 \).

Define \( V(G) = \{w_{ij} : 1 \leq i \leq n, 1 \leq j \leq 2\} \) and \( m = 2n - 2 \).

Case(i): \( d(u_i) = 1 \), where \( u_i \) is as in definition 2.2.22.
Then it is not possible to have \(|E(H_i)| = i; \, 1 \leq i \leq n\), since \(m \neq \binom{k+1}{2}\) for any \(k\).

Hence \(L_n \land K_2\) cannot have a CMD.

**Case (ii):** \(d(u_t) = 2\).

Then it is not possible to have \(|E(H_i)| = i; \, 1 \leq i \leq n\), since \(m \neq \binom{k+1}{2}\) for any \(k\).

Hence \(L_n \land K_2\) cannot have a CMD.

**Case (iii):** \(d(u_t) = 3\).

**Subcase (a):** \(d(u_t') = 1\), where \(u_t'\) is as in definition 2.2.22.

For every positive integer \(n\), define

\[
T_1 = \{(w_{i1}, w_{(i+1)2}) : i = 10j - 9, j = 1, 2, \ldots, \lceil \frac{n}{10} \rceil \}
\]

\[\cup\{(w_{i1}, w_{(i+3)2}) : i = 10j - 9, j = 1, 2, \ldots, \lceil \frac{n}{10} \rceil \}
\]

\[\cup\{(w_{i1}, w_{(i+5)2}) : i = 10j - 9, j = 1, 2, \ldots, \lceil \frac{n}{10} \rceil \}
\]

\[\cup\{(w_{i+11}, w_{i+2}) : i = 10j - 4, j = 1, 2, \ldots, \lceil \frac{n}{10} \rceil \}
\]

\[\cup\{(w_{i+31}, w_{i+2}) : i = 10j - 4, j = 1, 2, \ldots, \lceil \frac{n}{10} \rceil \}
\]

\[\cup\{(w_{i+51}, w_{i+2}) : i = 10j - 4, j = 1, 2, \ldots, \lceil \frac{n}{10} \rceil \}
\]

\[\cup\{(w_{i+21}, w_{i+12}) : i = 10j - 9, j = 1, 2, \ldots, \lceil \frac{n}{10} \rceil \}
\]

\[\cup\{(w_{i+41}, w_{i+12}) : i = 10j - 9, j = 1, 2, \ldots, \lceil \frac{n}{10} \rceil \}
\]

\[\cup\{(w_{i+11}, w_{i+22}) : i = 10j - 4, j = 1, 2, \ldots, \lceil \frac{n}{10} \rceil \}
\]

\[\cup\{(w_{i+31}, w_{i+22}) : i = 10j - 4, j = 1, 2, \ldots, \lceil \frac{n}{10} \rceil \}
\]

\[\cup\{(w_{i+51}, w_{i+22}) : i = 10j - 4, j = 1, 2, \ldots, \lceil \frac{n}{10} \rceil \}
\]

\[\text{and } T_2 = \{(w_{i+11}, w_{i+2}) : i = 10j - 9, j = 1, 2, \ldots, \lceil \frac{n}{10} \rceil \}.
\]

\[\cup\{(w_{i+31}, w_{i+2}) : i = 10j - 9, j = 1, 2, \ldots, \lceil \frac{n}{10} \rceil \}
\]

\[\cup\{(w_{i+51}, w_{i+2}) : i = 10j - 9, j = 1, 2, \ldots, \lceil \frac{n}{10} \rceil \}
\]

\[\cup\{(w_{i+1}, w_{i+12}) : i = 10j - 4, j = 1, 2, \ldots, \lceil \frac{n}{10} \rceil \}
\]

\[\cup\{(w_{i+1}, w_{i+32}) : i = 10j - 4, j = 1, 2, \ldots, \lceil \frac{n}{10} \rceil \}
\]

\[\cup\{(w_{i+1}, w_{i+52}) : i = 10j - 4, j = 1, 2, \ldots, \lceil \frac{n}{10} \rceil \}
\]

\[\cup\{(w_{i+11}, w_{i+22}) : i = 10j - 9, j = 1, 2, \ldots, \lceil \frac{n}{10} \rceil \}
\]

\[\cup\{(w_{i+31}, w_{i+22}) : i = 10j - 9, j = 1, 2, \ldots, \lceil \frac{n}{10} \rceil \}
\]

\[\cup\{(w_{i+51}, w_{i+22}) : i = 10j - 9, j = 1, 2, \ldots, \lceil \frac{n}{10} \rceil \}
\]

\[\cup\{(w_{i+21}, w_{i+32}) : i = 10j - 4, j = 1, 2, \ldots, \lceil \frac{n}{10} \rceil \}
\]

\[\cup\{(w_{i+41}, w_{i+32}) : i = 10j - 4, j = 1, 2, \ldots, \lceil \frac{n}{10} \rceil \}
\]

\[\cup\{(w_{i+21}, w_{i+52}) : i = 10j - 4, j = 1, 2, \ldots, \lceil \frac{n}{10} \rceil \}
\]

\[\cup\{(w_{i+41}, w_{i+52}) : i = 10j - 4, j = 1, 2, \ldots, \lceil \frac{n}{10} \rceil \}.
\]
Here \(|T_1| = n - 1\) and \(|T_2| = n - 1\). Also, \(|T_1| + |T_2| = 1 + 2 + \ldots + k = \binom{k+1}{2}\).

By lemmas 2.2.23 and 2.2.24, \(\{1, 2, \ldots, k\} = S_1 \cup S_2\) where \(\sum_{x \in S_1} x = n - 1 = \sum_{y \in S_2} y\).

Decompose \(T_1\) and \(T_2\) into trees \(\{H_i\}\) as follows: \(T_1 = \bigcup_{i \in S_1} H_i\), \(T_2 = \bigcup_{i \in S_2} H_i\) and \(|E(H_i)| = i, 1 \leq i \leq k\). Clearly \(\{H_1, H_2, \ldots, H_k\}\) forms a CMD of \(L_n \land K_2\).

**Subcase (b):** \(d(u'_i) = 2\).

For every positive integer \(n\), define

\[T_1 = \{(w_{i1}, w_{(i+1)2}) : i = 10j - 9, j = 1, 2, \ldots, \left\lfloor \frac{n}{10} \right\rfloor\}\]

\[\bigcup \{(w_{i1}, w_{(i+3)2}) : i = 10j - 9, j = 1, 2, \ldots, \left\lfloor \frac{n}{10} \right\rfloor\}\]

\[\bigcup \{(w_{i1}, w_{(i+5)2}) : i = 10j - 9, j = 1, 2, \ldots, \left\lfloor \frac{n}{10} \right\rfloor\}\]

\[\bigcup \{(w_{i1}, w_{(i+1)2}) : i = 10j - 4, j = 1, 2, \ldots, \left\lfloor \frac{n}{10} \right\rfloor\}\]

\[\bigcup \{(w_{i1}, w_{(i+3)2}) : i = 10j - 4, j = 1, 2, \ldots, \left\lfloor \frac{n}{10} \right\rfloor\}\]

\[\bigcup \{(w_{i1}, w_{(i+5)2}) : i = 10j - 4, j = 1, 2, \ldots, \left\lfloor \frac{n}{10} \right\rfloor\}\]

and \(T_2 = \{(w_{(i+1)1}, w_{i2}) : i = 10j - 9, j = 1, 2, \ldots, \left\lfloor \frac{n}{10} \right\rfloor\}\)

\[\bigcup \{(w_{i1}, w_{(i+1)2}) : i = 10j - 9, j = 1, 2, \ldots, \left\lfloor \frac{n}{10} \right\rfloor\}\]

\[\bigcup \{(w_{i1}, w_{(i+3)2}) : i = 10j - 9, j = 1, 2, \ldots, \left\lfloor \frac{n}{10} \right\rfloor\}\]

Here \(|T_1| = n - 1\) and \(|T_2| = n - 1\). Also, \(|T_1| + |T_2| = 1 + 2 + \ldots + k = \binom{k+1}{2}\).

By lemmas 2.2.23 and 2.2.24, \(\{1, 2, \ldots, k\} = S_1 \cup S_2\) where \(\sum_{x \in S_1} x = n - 1 = \sum_{y \in S_2} y\).
Decompose $T_1$ and $T_2$ into trees $\{H_i\}$ as follows: $T_1 = \bigcup_{i \in S_1} H_i$, $T_2 = \bigcup_{i \in S_2} H_i$ and $|E(H_i)| = i, 1 \leq i \leq k$. Clearly $\{H_1, H_2, \ldots, H_k\}$ forms a CMD of $L_n \wedge K_2$. \hfill \Box$

**Illustration 2.2.26.** As an illustration, let us decompose $L_{19} \wedge K_2$. Let $V(L_{19}) = \{u_1, u_2, \ldots, u_{19}\}$ and let $V(K_2) = \{v_1, v_2\}$. $L_{19} \wedge K_2$ is given in figure 2.8.
Figure 2.8. $L_{19} \wedge K_2$
Here \( m = 36 \) and \( k = 8 \). Let \( e_{ij} = ((u_i, v_1), (u_j, v_2)) \) where \( 1 \leq i \leq 15, \ 1 \leq j \leq 15 \). Here \( d(u_i) = 3 \) and \( d(u'_i) = 1 \). \( T_1 = \{e_{12}, e_{11,12}, e_{14}, e_{11,14}, e_{16}, e_{11,16}, e_{17,16}, e_{19,16}, e_{54}, e_{15,14}, e_{17,18}, e_{9,10}\} \) and \( T_2 = \{e_{21}, e_{23}, e_{41}, e_{51}, e_{45}, e_{67}, e_{69}, e_{10,9}, e_{12,11}, e_{14,11}, e_{16,11}, e_{12,13}, e_{16,17}, e_{16,19}, e_{18,17}, e_{87}, e_{14,15}\} \). \( |T_1| = 18 = |T_2| \). Hence \( |T_1| + |T_2| = 36 = 1 + 2 + \ldots + 8 = \binom{8}{2} \). \( T_1 \) is decomposed as \( T_1 = \{H_1 \cup H_4 \cup H_5 \cup H_8\} \) where \( H_1 = \{e_{9,10}\} \), \( H_4 = \{e_{11,16}, e_{17,16}, e_{19,16}, e_{17,18}\} \), \( H_5 = \{e_{11,6}, e_{11,12}, e_{13,12}, e_{11,14}, e_{15,14}\} \) and \( H_8 = \{e_{12}, e_{14}, e_{16}, e_{32}, e_{54}, e_{76}, e_{78}, e_{96}\} \). \( T_2 \) is decomposed as \( T_2 = \{H_2 \cup H_3 \cup H_6 \cup H_7\} \) where \( H_2 = \{e_{10,9}, e_{69}\} \), \( H_3 = \{e_{16,17}, e_{16,19}, e_{18,17}\} \), \( H_6 = \{e_{6,11}, e_{16,11}, e_{12,11}, e_{12,13}, e_{14,11}, e_{14,15}\} \) and \( H_7 = \{e_{21}, e_{41}, e_{61}, e_{23}, e_{45}, e_{67}, e_{87}\} \). Hence \( \{H_1, H_2, \ldots, H_8\} \) forms a CMD of \( L_{19} \land K_2 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>CMD</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>( H_1, H_2, \ldots, H_{28} )</td>
</tr>
<tr>
<td>19</td>
<td>( H_1, H_2, \ldots, H_{36} )</td>
</tr>
<tr>
<td>34</td>
<td>( H_1, H_2, \ldots, H_{66} )</td>
</tr>
<tr>
<td>40</td>
<td>( H_1, H_2, \ldots, H_{78} )</td>
</tr>
<tr>
<td>61</td>
<td>( H_1, H_2, \ldots, H_{120} )</td>
</tr>
<tr>
<td>69</td>
<td>( H_1, H_2, \ldots, H_{136} )</td>
</tr>
<tr>
<td>96</td>
<td>( H_1, H_2, \ldots, H_{190} )</td>
</tr>
<tr>
<td>106</td>
<td>( H_1, H_2, \ldots, H_{210} )</td>
</tr>
<tr>
<td>139</td>
<td>( H_1, H_2, \ldots, H_{276} )</td>
</tr>
<tr>
<td>151</td>
<td>( H_1, H_2, \ldots, H_{300} )</td>
</tr>
</tbody>
</table>

Table 2.7 List of first 10 \( L_n \land K_2' \)s which accept CMD and their decompositions.