# CHAPTER 4

# $H^1$-GALERKIN MIXED FINITE ELEMENT METHOD FOR EXTENDED FISHER-KOLMOGOROV EQUATION

## 4.1 INTRODUCTION

In this chapter, we discuss an $H^1$-Galerkin mixed finite element cubic spline approximation method for the following extended Fisher-Kolmogorov equation:

$$u_t + \gamma u_{xxxx} - u_{xx} + f(u) = 0, \quad 0 < t < T, \ \gamma > 0, \ x \in I = (0,1); \tag{4.1}$$

subject to the boundary and initial conditions

$$u(0,t) = 0, \ u(1,t) = 0, \ u_{xx}(0,t) = 0, \ u_{xx}(1,t) = 0; \tag{4.2}$$

$$u(x,0) = g(x); \tag{4.3}$$

where $f(u) = u^3 - u$. When $\gamma = 0$ in (4.1), it becomes the standard Fisher-Kolmogorov equation. The above equation is a nonlinear time dependent fourth order partial differential equation. The aim of this work is to analyze the $H^1$-Galerkin mixed finite element method for the above extended Fisher-Kolmogorov equation after splitting (4.1) with an introduction of an intermediate function. Let us define the splitting of the above equation as follows:
Set

$$u_{xx} = v, \ x \in I. \tag{4.4}$$

Then (4.1) becomes

$$u_t + \gamma v_{xx} - v + f(u) = 0, \ x \in I. \tag{4.5}$$

In the present study, an $H^1$-Galerkin method with cubic splines as trial functions is formulated, analysed for the coupled equations (4.4) and (4.5) and error bounds are obtained for both semi discrete and fully discrete schemes.

### 4.2 SEMI DISCRETE $H^1$-GALERKIN FORMULATION

For $\varphi, \psi \in H^2(I) \bigcap \overset{0}{H^1}(I)$, let us define the following bilinear form

$$A(\lambda : \varphi, \psi) = (\gamma\varphi_{xx}, \psi_{xx}) - (\varphi, \psi_{xx}) + \lambda(\varphi, \psi),$$

where $\lambda > 0$ is chosen appropriately later so that $A(\lambda : \varphi, \psi)$ is coercive. This can easily be seen from the following:

For $\varphi$ satisfying the boundary conditions $\varphi(0) = 0$, $\varphi(1) = 0$, we have

$$\begin{aligned} A(\lambda : \varphi, \varphi) &= (\gamma\varphi_{xx}, \varphi_{xx}) - (\varphi, \varphi_{xx}) + \lambda(\varphi, \varphi) \\ &= (\gamma\varphi_{xx}, \varphi_{xx}) + (\varphi_x, \varphi_x) + \lambda(\varphi, \varphi) \\ &= \gamma\|\varphi_{xx}\|^2 + \|\varphi_x\|^2 + \lambda\|\varphi\|^2 \\ &\geq \alpha_0\|\varphi\|_2^2, \end{aligned}$$

where we have used integration by parts. Further $\lambda > 0$ is chosen in such a way that $\alpha_0 = \min(\gamma, 1, \lambda)$.

It can also be shown that the bilinear form $A(\lambda : \varphi, \psi)$ is continuous,

$$i.e., \ |A(\lambda : \varphi, \psi)| \leq K\|\varphi\|_2\|\psi\|_2,$$

where $K$ depends only on $\gamma$ and $\lambda$.

We now see the weak formulation and $H^1$-Galerkin mixed finite element formulation for the split up equations (4.4) and (4.5) of the main equation (4.1). We obtain weak formulation for $u$ by multiplying both sides of (4.4) with $\phi_{xx}$, where $\phi \in H^2(I) \bigcap \overset{0}{H^1}(I)$ and then integrating the resulting expression with respect to $x$ over the interval $[0, 1]$. In a similar manner, multiplying both sides of (4.5) by $\phi_{xx}$ with $\phi \in H^2(I) \bigcap \overset{0}{H^1}(I)$, integrating with respect to $x$ over the interval $[0, 1]$, using integration by parts twice for the first term twice and then adding and subtracting $\lambda(v, \phi)$ in the resulting equation, we obtain the following weak formulation for $v$. Thus weak formulation corresponding to the split up equations (4.4) and (4.5) is given below:

**Weak Formulation:** Find $u, v : [0, T] \to H^2(I) \bigcap \overset{0}{H^1}(I)$, such that

$$\begin{aligned} (u_{xx}, \phi_{xx}) &= (v, \phi_{xx}), \phi \in H^2(I) \bigcap \overset{0}{H^1}(I), \qquad (4.6) \\ u(x, 0) &= g(x); \end{aligned}$$

$$(v_t, \phi) + A(\lambda : v, \phi) - \lambda(v, \phi) \quad + \quad (f(u), \phi_{xx}) = 0,$$
$$\phi \in H^2(I) \bigcap \overset{0}{H^1}(I), \qquad (4.7)$$
$$v(x, 0) \quad = \quad g_{xx}(x).$$

The Semi discrete $H^1$-Galerkin mixed finite element formulation corresponding to the above weak formulation (4.6) and (4.7) is defined respectively as follows:

**The Semi discrete $H^1$-Galerkin mixed finite element formulation:**

Find $U, V \in \overset{0}{S}_{h,3}$ such that

$$(U_{xx}, \phi_{h_{xx}}) = (V, \phi_{h_{xx}}), \qquad \phi_h \in \overset{0}{S}_{h,3}, \qquad (4.8)$$
$$U(x, 0) \quad = \quad g(x);$$
$$(V_t, \phi_h) + A(\lambda : V, \phi_h) - \lambda(V, \phi_h) + (f(U), \phi_{hxx}) \quad = \quad 0, \phi_h \in \overset{0}{S}_{h,3}, \qquad (4.9)$$
$$V(x, 0) \quad = \quad g_{xx}(x).$$

The above formulation leads to a system of coupled equations. This method may be regarded as a Petrov-Galerkin method with cubic spline space as trial space and piecewise linear space as test space, since the second derivative of a cubic spline is a piecewise linear spline.

### 4.3 AUXILIARY PROJECTION

The error analysis is generally carried over with the help of a comparison function. In this analysis, the comparison function is the usual auxiliary projection $\widehat{v} \in \overset{0}{S}_{h,3}$ of the weak solution $v$ onto the finite dimensional subspace $\overset{0}{S}_{h,3}$ through the elliptic part appearing in the weak formulation.

Let $\widehat{v} : [0, T] \rightarrow \overset{0}{S}_{h,3}$ be the auxiliary projection of $v$ defined by

$$A(\lambda : v - \widehat{v}, \phi_h) = 0, \quad \phi_h \in \overset{0}{S}_{h,3} \qquad (4.10)$$

and $\widehat{v}(x, 0) = g_{xx}$.

We initially find the error involved in the auxiliary projection, *i.e.*, the error between the weak solution $v$ and the intermediate comparison function $\widehat{v}$. In the next section, we compute the error between the comparison function $\widehat{v}$ and the mixed $H^1$-Galerkin approximation $V$. Let $v - \widehat{v} = \rho$. Error estimates for a similar auxiliary projection with complicated non linear terms are proved in (Jones and Pani 1995) and (Pani and Das 1991). In the following lemma, we obtain the error estimates for the auxiliary projection.

**Lemma 4.3.1** *There exists a constant $C$ such that for sufficiently small $h$ and $i = 0, 1, 2$*

$$\|\rho\|_i \leq Ch^{4-i}\|v\|_4;$$
$$\|\rho_t\|_i \leq Ch^{4-i}\|v_t\|_4.$$

**Proof:** We have from (4.10), that

$$A(\lambda : \rho, \phi_h) = 0, \;\; \phi_h \in \overset{0}{S}_{h,3}. \tag{4.11}$$

Choosing $\phi_h = \rho - (v - \chi)$ for $\chi \in \overset{0}{S}_{h,3}$, we have $A(\lambda : \rho, \rho) = A(\lambda : \rho, v - \chi)$. Using coercivity and continuity of $A(\lambda : \varphi, \psi)$, we obtain

$$\alpha_0\|\rho\|_2^2 \leq C\|\rho\|_2\|v - \chi\|_2.$$

Taking infimum over $\chi \in \overset{0}{S}_{h,3}$ first and then applying the approximation property, we obtain

$$\alpha_0\|\rho\|_2 \leq C \inf_{\chi \in \overset{0}{S}_{h,3}} \|v - \chi\|_2 \leq Ch^2\|v\|_4.$$

Hence, we obtain

$$\|\rho\|_2 \leq Ch^2\|v\|_4. \tag{4.12}$$

We now compute the error estimate of $\rho$ in $L_2$ norm using the following duality argument.

Let $\psi \in H^4(I)$ be the solution of the auxiliary problem

$$L\psi := \gamma\psi_{xxxx} - \psi_{xx} + \lambda\psi = \rho,$$

with boundary conditions

$$\psi_{xxx}(0) = \psi_{xxx}(1) = 0$$
$$\psi_{xx}(0) = \psi_{xx}(1) = 0.$$

The above problem has the regularity property

$$\|\psi\|_4 \leq \|\rho\|.$$

It is easy to see that

$$\begin{aligned}
A(\lambda:\rho,\psi) &= \gamma(\rho_{xx},\psi_{xx}) - (\rho,\psi_{xx}) + \lambda(\rho,\psi)\\
&= -\gamma(\rho_x,\psi_{xxx}) + \gamma\rho_x\psi_{xx}|_0^1 - (\rho,\psi_{xx}) + \lambda(\rho,\psi)\\
&= \gamma(\rho,\psi_{xxxx}) - \gamma\psi_{xxx}\rho|_0^1 + \gamma\rho_x\psi_{xx}|_0^1 - (\rho,\psi_{xx}) + \lambda(\rho,\psi)\\
&= (\rho,\gamma\psi_{xxxx} - \psi_{xx}) + \lambda(\rho,\psi) - \gamma\psi_{xxx}\rho|_0^1 + \gamma\rho_x\psi_{xx}|_0^1\\
&= (\rho, L\psi),
\end{aligned}$$

where we have used integration by parts and the boundary conditions of the auxiliary problem.
Using the above and (4.11), we have that

$$(\rho,\rho) = (\rho,L\psi) = A(\lambda:\rho,\psi) = A(\lambda:\rho,\psi-\chi) \ \text{ for } \ \chi \in \overset{0}{S}_{h,3}.$$

Hence, applying the continuity of $A(\lambda:\varphi,\psi)$, the approximation property and regularity of $\psi$, we obtain

$$\|\rho\|^2 \leq C\|\rho\|_2\|\psi-\chi\|_2,$$

$$i.e., \ \|\rho\|^2 \leq C\|\rho\|_2 \inf_{\chi\in\overset{0}{S}_{h,3}} \|\psi-\chi\|_2 \leq C\|\rho\|_2 h^2\|\psi\|_4 \leq Ch^2\|\rho\|_2\|\|\rho\|.$$

Hence,

$$\|\rho\| \leq Ch^2\|\rho\|_2. \tag{4.13}$$

On an application of (4.12) in the above, we have

$$\|\rho\| \leq Ch^4\|v\|_4. \tag{4.14}$$

Using (4.12) and (4.14) in (1.17) with $m=2$ and $i=1$, we have

$$\begin{aligned}
\|\rho\|_1 &\leq C\left[h^{-1}\|\rho\| + h\|\rho\|_2\right],\\
i.e., \ \|\rho\|_1 &\leq Ch^3\|v\|_4. \qquad (4.15)
\end{aligned}$$

To obtain similar error estimates for the temporal derivative of $\rho$, we differentiate the projection equation with respect to time variable $t$. Hence, we obtain

$$A(\lambda : \rho_t, \phi_h) = 0, \ \ \phi_h \in \overset{0}{S}_{h,3}.$$

It can be easily verified from the above analysis that

$$\begin{aligned} \|\rho_t\| &\leq Ch^4 \|v_t\|_4; \\ \|\rho_t\|_1 &\leq Ch^3 \|v_t\|_4; \\ \|\rho_t\|_2 &\leq Ch^2 \|v_t\|_4. \end{aligned}$$

Thus, we obtain the error estimates for $\rho$ and $\rho_t$ in $L_2$, $H^1$ and $H^2$ norms. ∎

## 4.4 ERROR ANALYSIS OF SEMI DISCRETE SCHEME

In this section, we obtain *a priori* error estimate for the error between the comparison function $\widehat{v}$ and the $H^1$-Galerkin solution $V$. We also discuss the error analysis of the error between the weak solution $u$ and its corresponding $H^1$-Galerkin approximation $U$. For this, we first write the error equation related to the $H^1$-Galerkin approximation. Subtracting (4.9) from (4.7), we obtain the following error equation:

$$\begin{aligned} (v_t - V_t, \phi_h) + A(\lambda : v - V, \phi_h) &- \lambda(v - V, \phi_h) + (f(u) - f(U), \phi_{hxx}) = 0, \\ & \phi_h \in \overset{0}{S}_{h,3}. \end{aligned} \tag{4.16}$$

In a similar manner, subtracting (4.8) from (4.6), we obtain

$$(u_{xx} - U_{xx}, \phi_{hxx}) = (v - V, \phi_{hxx}), \ \ \phi_h \in \overset{0}{S}_{h,3}. \tag{4.17}$$

Let $e_1$ be the error between $u$ and $U$ and $e_2$ be that between $v$ and $V$. Then we have $e_1 = u - U$ and $e_2 = v - V = v - \widehat{v} + \widehat{v} - V = \rho + \zeta$, where $\zeta = \widehat{v} - V$. Then, (4.16) can also be written as

$$(\rho_t + \zeta_t, \phi_h) + A(\lambda : \rho + \zeta, \phi_h) - \lambda(\rho + \zeta, \phi_h) + (f(u) - f(U), \phi_{hxx}) = 0.$$

Using projection (4.11) in the above equation, we obtain

$$\begin{aligned} (\rho_t + \zeta_t, \phi_h) + A(\lambda : \zeta, \phi_h) - \lambda(\rho + \zeta, \phi_h) &+ (f(u) - f(U), \phi_{hxx}) = 0 \\ & \text{for} \ \ \phi_h \in \overset{0}{S}_{h,3}, \end{aligned}$$

$$i.e., \; (\zeta_t, \phi_h) + A(\lambda : \zeta, \phi_h) \;=\; -(\rho_t, \phi_h) + \lambda(\rho, \phi_h) + \lambda(\zeta, \phi_h)$$
$$-\;(f(u) - f(U), \phi_{hxx}), \phi_h \in \overset{0}{S}_{h,3}. \qquad (4.18)$$

In a similar manner, (4.17) can also be written as

$$(u_{xx} - U_{xx}, \phi_{hxx}) = (\rho + \zeta, \phi_{hxx}) \text{ for } \phi_h \in \overset{0}{S}_{h,3}. \qquad (4.19)$$

In the following lemma, We compute $|e_1(\overline{x})|$, where $\overline{x}$ is an arbitrary point in $[0, 1]$. This result is required for proving the main theorem in this section.

**Lemma 4.4.1** *Let $u$ and $v$ be the weak solutions of* (4.4) *and* (4.5) *defined through* (4.6) *and* (4.7) *respectively. Further, let $U$ and $V$ be the corresponding semi discrete $H^1$-Galerkin mixed finite element formulation defined through* (4.8) *and* (4.9) *. Then the error $e_1 = u - U$ satisfies*

$$|e_1(\overline{x})| \leq C \left[h^2 \|e_{1xx}\| + \|e_2\|\right],$$

*where $\overline{x}$ is an arbitrary point in* $[0, 1]$.

**Proof**: For a given $\overline{x} \in [0, 1]$, let $\Phi$ be an element of $L_2(I) \bigcap C(I)$ satisfying the following auxiliary problem:

$$\Phi_{xx} \;=\; 0, \; x \in I - \{\overline{x}\},$$
$$\Phi(0) \;=\; \Phi(1) = 0, \; \Phi_x^-(\overline{x}) - \Phi_x^+(\overline{x}) = -1.$$

The above problem has a solution. For example,

$$\Phi(x) = \begin{cases} (\overline{x} - 1)x, & 0 \leq x \leq \overline{x}, \\ \overline{x}(x - 1), & \overline{x} \leq x \leq 1 \end{cases}$$

satisfies the above differential equation and the boundary conditions.
Let us define $\Psi$ as follows:

$$\Psi(x) = \begin{cases} \Phi_{xx}, & x \in I - \{\overline{x}\}, \\ 0, & x = \overline{x}. \end{cases}$$

Then $\Psi = 0$ *a.e.* on $I$. We first multiply $e_1$ with $\Psi$ and then integrate over $I$. On applying integration by parts twice, using the fact that $e_1(0) = e_1(1) = 0$ and then using given jump condition, we now obtain

$$
\begin{aligned}
0 = (e_1, \Psi) = \int_0^{\overline{x}} e_1 \Psi + \int_{\overline{x}}^1 e_1 \Psi = \int_0^{\overline{x}} e_1 \Phi_{xx} + \int_{\overline{x}}^1 e_1 \Phi_{xx} \\
&= [e_1 \Phi_x]_0^{\overline{x}} - \int_0^{\overline{x}} e_{1x} \Phi_x + [e_1 \Phi_x]_{\overline{x}}^1 - \int_{\overline{x}}^1 e_{1x} \Phi_x = e_1(\overline{x}) \left[ \Phi_x^-(\overline{x}) - \Phi_x^+(\overline{x}) \right] \\
&- \int_0^{\overline{x}} e_{1x} \Phi_x - \int_{\overline{x}}^1 e_{1x} \Phi_x \\
&= -e_1(\overline{x}) - \left\{ [e_{1x} \Phi]_0^{\overline{x}} - \int_0^{\overline{x}} e_{1xx} \Phi + [e_{1x} \Phi]_{\overline{x}}^1 - \int_{\overline{x}}^1 e_{1xx} \Phi \right\} \\
&= -e_1(\overline{x}) + (e_{1xx}, \Phi), \\
& \quad i.e., \; e_1(\overline{x}) = (e_{1xx}, \Phi).
\end{aligned}
$$

We have used the zero boundary conditions and the continuity of $\Phi$ for evaluating $[e_{1x}\Phi]_0^{\overline{x}}$ and $[e_{1x}\Phi]_{\overline{x}}^1$.

Let $\Phi_h$ be the linear interpolant of $\Phi$. Then we have,

$$
\begin{aligned}
e_1(\overline{x}) &= (e_{1xx}, \Phi - \Phi_h) + (e_{1xx}, \Phi_h) \\
|e_1(\overline{x})| &\leq |(e_{1xx}, \Phi - \Phi_h)| + |(e_{1xx}, \Phi_h)| \\
&\leq T_{4,A} + T_{4,B}. \qquad (4.20)
\end{aligned}
$$

We know that

$$
\|\Phi_h\|_1 \leq \|\Phi - \Phi_h\|_1 + \|\Phi\|_1 \leq Ch \|\Phi\|_2 + \|\Phi\|_2 \leq C \|\Phi\|_2 . \qquad (4.21)
$$

We now compute the estimates for the terms $T_{4,A}$ and $T_{4,B}$ as follows:

$$
T_{4,A} = |(e_{1xx}, \Phi - \Phi_h)| \leq \|e_{1xx}\| \||\|\Phi - \Phi_h\| \leq Ch^2 \|e_{1xx}\| \|\Phi\|_2 .
$$

Since $\Phi_h$ is a linear interpolant of $\Phi$ vanishing at boundaries, $\Phi_h$ can be considered as the second derivative of some $\chi_h \in \overset{0}{S}_{h,3}$. Using (4.21) and (4.17), we have

$$
T_{4,B} = |(e_{1xx}, \Phi_h)| = |(e_2, \Phi_h)| \leq C \|e_2\| \|\Phi_h\| \leq C \|e_2\| \|\Phi\|_2 .
$$

For $\Phi$ satisfying the auxiliary problem, it is easy to verify that $\|\Phi\|_2 \leq K$, where $K$ is a constant not depending on $h$.

Using $T_{4,A}$ and $T_{4,B}$ in (4.20), we have

$$
|e_1(\overline{x})| \leq C \left[ h^2 \|e_{1xx}\| + \|e_2\| \right] .
$$

This completes the proof. ■

Below, we shall discuss *a priori* error estimates for the semi discrete mixed $H^1$-Galerkin procedure. Initially we obtain the error $\|u-U\|$ in the following lemma:

**Lemma 4.4.2** *Let $u$ and $v$ be the weak solutions of* (4.4) *and* (4.5) *defined through* (4.6) *and* (4.7) *respectively. Further, let $U$ and $V$ be the corresponding semi discrete $H^1$-Galerkin mixed finite element formulation defined through* (4.8) *and* (4.9). *Also, let $\widehat{v}$ be the auxiliary projection of $v$ as in (4.10). Then, we have*

$$\|u-U\|^2 \leq C\left[h^8\|u\|_4^2+\|\rho\|^2+\|\zeta\|^2\right], \quad \textit{where } \rho = v-\widehat{v},\ \zeta=\widehat{v}-V.$$

**Proof:** Choose $\phi_h = (u-U)-(u-\chi)$ in (4.17) for some $\chi \in \overset{0}{S}_{h,3}$. Then, it becomes

$$\begin{aligned}
(u_{xx}-U_{xx}, u_{xx}-U_{xx}) &= (u_{xx}-U_{xx}, u_{xx}-\chi_{xx}) + (v-V, u_{xx}-U_{xx}) \\
&- (v-V, u_{xx}-\chi_{xx}), \\
\|u_{xx}-U_{xx}\|^2 &\leq \|u_{xx}-U_{xx}\|\|u_{xx}-\chi_{xx}\| + \|v-V\|\|u_{xx}-U_{xx}\| \\
&+ \|v-V\|\|u_{xx}-\chi_{xx}\|.
\end{aligned}$$

Taking infimum over $\chi \in \overset{0}{S}_{h,3}$ first and then applying the approximation property, we obtain

$$\begin{aligned}
\|u_{xx}-U_{xx}\|^2 &\leq \|u_{xx}-U_{xx}\| \inf_{\chi\in \overset{0}{S}_{h,3}} \|u_{xx}-\chi_{xx}\| + \|v-V\|\|u_{xx}-U_{xx}\| \\
&+ \|v-V\| \inf_{\chi\in \overset{0}{S}_{h,3}} \|u_{xx}-\chi_{xx}\| \\
&\leq C\left[h^2\|u_{xx}-U_{xx}\|\|u\|_4 + \|v-V\|\|u_{xx}-U_{xx}\| \right.\\
&+ \left. h^2\|v-V\|\|u\|_4\right].
\end{aligned}$$

Using Young's inequality, we have

$$\begin{aligned}
\|u_{xx}-U_{xx}\|^2 &\leq C\left[\frac{\epsilon}{2}\|u_{xx}-U_{xx}\|^2 + \frac{1}{2\epsilon}h^4\|u\|_4^2 + \frac{\epsilon}{2}\|u_{xx}-U_{xx}\|^2 \right.\\
&+ \left. \frac{1}{2\epsilon}\|v-V\|^2 + \frac{\epsilon}{2}\|v-V\|^2 + h^4\frac{1}{2\epsilon}\|u\|_4^2\right], \\
(1-C\epsilon)\|u_{xx}-U_{xx}\|^2 &\leq C\left[\frac{1}{\epsilon}h^4\|u\|_4^2 + \left(\frac{1}{2\epsilon}+\frac{\epsilon}{2}\right)\|v-V\|^2\right].
\end{aligned}$$

Choosing $\epsilon > 0$ (for example $\epsilon = 1/2C$) properly so that inequality is maintained, we obtain that

$$\|e_{1xx}\|^2 = \|u_{xx} - U_{xx}\|^2 \leq C\left[h^4\|u\|_4^2 + \|v - V\|^2\right]. \tag{4.22}$$

Now we compute the estimate of $e_1$ in $L_2$ norm. For that, we apply the following duality argument:
Let $\Phi \in H^4(I)$ be the solution of the auxiliary problem

$$\Phi_{xxxx} = u - U = e_1, \;\; x \in I$$

satisfying the boundary conditions

$$\Phi_{xxx}(0) = \Phi_{xxx}(1) = 0, \; \Phi_{xx}(0) = \Phi_{xx}(1) = 0.$$

Then, using integration by parts and (4.19), we obtain

$$\begin{aligned}
(e_1, e_1) = (\Phi_{xxxx}, e_1) &= (\Phi_{xx}, e_{1xx}) = (e_{1xx}, \Phi_{xx}) \\
&= (e_{1xx}, \Phi_{xx} - \chi_{xx}) + (e_{1xx}, \chi_{xx}) \\
&= (e_{1xx}, \Phi_{xx} - \chi_{xx}) + (\rho + \zeta, \chi_{xx}) \\
&= (e_{1xx}, \Phi_{xx} - \chi_{xx}) + (\rho + \zeta, \chi_{xx} - \Phi_{xx}) \\
&+ (\rho + \zeta, \Phi_{xx}) \;\text{ for } \chi \in \overset{0}{S}_{h,3}\,.
\end{aligned}$$

$$\|e_1\|^2 \leq \left[\|e_{1xx}\|\|\Phi_{xx} - \chi_{xx}\| + \|\rho + \zeta\|\|\chi_{xx} - \Phi_{xx}\| + \|\rho + \zeta\|\|\Phi_{xx}\|\right].$$

Taking infimum over $\chi \in \overset{0}{S}_{h,3}$ first and then applying the approximation property, we obtain

$$\begin{aligned}
\|e_1\|^2 &\leq C\left[h^2\|e_{1xx}\|\|\Phi\|_4 + h^2\|\rho + \zeta\|\|\Phi\|_4 + \|\rho + \zeta\|\|\Phi\|_4\right] \\
&\leq C\left[h^2\|e_{1xx}\| + h^2\|\rho + \zeta\| + \|\rho + \zeta\|\right]\|\Phi\|_4.
\end{aligned}$$

Using the regularity condition $\|\Phi\|_4 \leq \|e_1\|$ of the auxiliary problem, we have that

$$\begin{aligned}
\|e_1\|^2 &\leq C\left[h^2\|e_{1xx}\| + h^2\|\rho + \zeta\| + \|\rho + \zeta\|\right]\|e_1\|, \\
i.e., \;\; \|e_1\| &\leq C\left[h^2\|e_{1xx}\| + h^2\|\rho + \zeta\| + \|\rho + \zeta\|\right], \\
i.e., \;\; \|e_1\| &\leq C\left[h^2\|e_{1xx}\| + \|v - V\|\right].
\end{aligned}$$

Squaring both sides and using Hölder's inequality, we obtain

$$\|e_1\|^2 \le C\left[h^4\|e_{1xx}\|^2 + \|v-V\|^2\right].$$

Using (4.22) in the above equation, we obtain

$$\begin{aligned} \|e_1\|^2 &\le C\left[h^4(h^4\|u\|_4^2 + \|v-V\|^2) + \|v-V\|^2\right], \\ i.e.,\ \|u-U\|^2 = \|e_1\|^2 &\le C\left[h^8\|u\|_4^2 + \|v-V\|^2\right], \qquad (4.23) \\ i.e.,\ \|u-U\|^2 = \|e_1\|^2 &\le C\left[h^8\|u\|_4^2 + \|\rho\|^2 + \|\zeta\|^2\right]. \end{aligned}$$

Hence the proof is completed. ■

We have now computed the error bound for $\|u-U\|$ in terms of the estimates of $\rho$ and $\zeta$. Below, we shall obtain *a priori* bound for $\|\zeta\|$ in terms of $\rho$ and $\rho_t$. This will help us in computing error bounds for $u-U$ and $v-V$ in terms of $h$, since the estimates of $\rho$ and $\rho_t$ have already been obtained.

**Theorem 4.4.1** *Let $u$ and $v$ be the weak solutions of* (4.4) *and* (4.5) *defined through* (4.6) *and* (4.7) *respectively. Further, let $U$ and $V$ be the corresponding semi discrete $H^1$-Galerkin mixed finite element formulation defined through* (4.8) *and* (4.9) *. Let $u,\ v,\ v_t \in L_2(H^4)$. Then, for a sufficiently small $h$, we have*

$$\begin{aligned} \|v-V\|^2_{L_\infty(L_2)} &\le Ch^8\left[\|u\|^2_{L_2(H^4)} + \|v\|^2_{L_2(H^4)} + \|v_t\|^2_{L_2(H^4)} + \|v\|^2_{L_\infty(H^4)}\right]; \\ \|v-V\|^2_{L_2(H^2)} &\le Ch^4\left[\|u\|^2_{L_2(H^4)} + \|v\|^2_{L_2(H^4)} + \|v_t\|^2_{L_2(H^4)}\right]; \\ \|u_{xx}-U_{xx}\|^2_{L_\infty(L_2)} &\le Ch^4\left[\|u\|^2_{L_2(H^4)} + \|v\|^2_{L_2(H^4)} + \|v_t\|^2_{L_2(H^4)} + \|v\|^2_{L_\infty(H^4)}\right]; \\ \|u-U\|^2_{L_\infty(L_2)} &\le Ch^8\left[\|u\|^2_{L_2(H^4)} + \|v\|^2_{L_2(H^4)} + \|v_t\|^2_{L_2(H^4)} + \|v\|^2_{L_\infty(H^4)}\right]. \end{aligned}$$

**Proof:** Set $\phi_h = \zeta$ in (4.18). Then, we have

$$\begin{aligned} (\zeta_t,\zeta) + A(\lambda:\zeta,\zeta) &= -(\rho_t,\zeta) + \lambda(\rho,\zeta) + \lambda(\zeta,\zeta) - (f(u)-f(U),\zeta_{xx}) \\ &\le |-(\rho_t,\zeta)| + |\lambda(\rho,\zeta)| + |\lambda(\zeta,\zeta)| \\ &+ |(f(u)-f(U),\zeta_{xx})| \\ &\le T_{4,1} + T_{4,2} + T_{4,3} + T_{4,4}. \end{aligned}$$

Using coercivity of $A(\lambda:\varphi,\psi)$ and since $(\zeta_t,\zeta) = \dfrac{1}{2}\dfrac{d}{dt}(\zeta,\zeta)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\zeta\|^2 + \alpha_0\|\zeta\|_2^2 \le T_{4,1} + T_{4,2} + T_{4,3} + T_{4,4}. \qquad (4.24)$$

We now estimate the terms on the right hand side of the above equation. Using Young's inequality for $T_{4,1}$ and $T_{4,2}$, we obtain

$$\begin{aligned} T_{4,1} &= |-(\rho_t,\zeta)| \le C\left[\frac{1}{2\epsilon}\|\rho_t\|^2 + \frac{\epsilon}{2}\|\zeta\|^2\right]; \\ T_{4,2} &= |\lambda(\rho,\zeta)| \le C\left[\frac{1}{2\epsilon}\|\rho\|^2 + \frac{\epsilon}{2}\|\zeta\|^2\right]. \end{aligned}$$

It is easy to see that

$$T_{4,3} = |\lambda(\zeta,\zeta)| \le C\|\zeta\|^2.$$

For the computation of $T_{4,4}$ we now temporarily assume that $\|U\|_{0,\infty} \le K^*$ . After having obtained the relevant error estimate, we can prove that this assumption is no longer a strong condition. With this temporary assumption and using Young's inequality, we now estimate $T_{4,4}$ as follows:

$$\begin{aligned} T_{4,4} &= |(f(u)-f(U),\zeta_{xx})| \le \|f(u)-f(U)\|\|\zeta_{xx}\| \\ &\le C\left[\frac{1}{2\epsilon}\|f(u)-f(U)\|^2 + \frac{\epsilon}{2}\|\zeta_{xx}\|^2\right] \\ &\le C\left[\frac{1}{2\epsilon}\|(u^3-u)-(U^3-U)\|^2 + \frac{\epsilon}{2}\|\zeta_{xx}\|^2\right] \\ &\le C\left[\frac{1}{2\epsilon}\|(u-U)(u^2+uU+U^2-1)\|^2 + \frac{\epsilon}{2}\|\zeta_{xx}\|^2\right] \\ &\le C(K^*)\frac{1}{2\epsilon}\|(u-U)\|^2 + C\frac{\epsilon}{2}\|\zeta_{xx}\|^2 \\ &\le C(K^*)\frac{1}{2\epsilon}\|e_1\|^2 + C\frac{\epsilon}{2}\|\zeta_{xx}\|^2 \end{aligned} \tag{4.25}$$

Using the estimates of $T_{4,1}$, $T_{4,2}$, $I_{4,3}$ and $T_{4,4}$ in (4.24),

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\zeta\|^2 + \alpha_0\|\zeta\|_2^2 &\le C\left[\frac{1}{2\epsilon}\left(\|\rho_t\|^2 + \|\rho\|^2\right) + \|\zeta\|^2 + \epsilon\|\zeta\|^2\right] \\ &+ C(K^*)\frac{1}{2\epsilon}\|e_1\|^2 + C\frac{\epsilon}{2}\|\zeta_{xx}\|^2. \end{aligned}$$

Using the inequalities $\|\zeta\| \le \|\zeta\|_2$ and $\|\zeta_{xx}\| \le \|\zeta\|_2$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\zeta\|^2 + \left(\alpha_0 - \frac{3C\epsilon}{2}\right)\|\zeta\|_2^2 &\le C\left[\frac{1}{2\epsilon}(\|\rho_t\|^2 + \|\rho\|^2)\right] \\ &+ C(K^*)\frac{1}{2\epsilon}\|e_1\|^2 + C\|\zeta\|^2. \end{aligned}$$

Using the result of Lemma 4.4.2,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\zeta\|^2 + \left(\alpha_0 - \frac{3C\epsilon}{2}\right)\|\zeta\|_2^2 &\le C\left[\frac{1}{2\epsilon}(\|\rho_t\|^2 + \|\rho\|^2)\right] \\ &+ C(K^*)\frac{1}{2\epsilon}(h^8\|u\|_4^2 + \|\rho\|^2 + \|\zeta\|^2) + C\|\zeta\|^2, \end{aligned}$$

$$\frac{1}{2}\frac{d}{dt}\|\zeta\|^2 + \left(\alpha_0 - \frac{3C\epsilon}{2}\right)\|\zeta\|_2^2 \;\leq\; C(K^*)\left[\frac{1}{2\epsilon}(h^8\|u\|_4^2 + \|\rho_t\|^2 + \|\rho\|^2 + \|\zeta\|^2) + \|\zeta\|^2\right].$$

Now, $\epsilon > 0$ can be chosen appropriately (for example $\epsilon < \frac{2\alpha_0 - 1}{3C}$) in such a way that the above inequality is maintained. Then, we have

$$\frac{d}{dt}\|\zeta\|^2 + \|\zeta\|_2^2 \leq C(K^*)\left[h^8\|u\|_4^2 + \|\rho_t\|^2 + \|\rho\|^2 + \|\zeta\|^2\right] + C(K^*)\|\zeta\|^2. \quad (4.26)$$

Integrate (4.26) with respect to time variable $t$ from 0 to $\tau$ with $\tau \leq T$, to obtain

$$\|\zeta\|^2 + \int_0^\tau \|\zeta\|_2^2 dt \leq C\|\zeta(x,0)\|^2 + C(K^*)\int_0^\tau (h^8\|u\|_4^2 + \|\rho_t\|^2 + \|\rho\|^2)dt + C(K^*)\int_0^\tau \|\zeta\|^2 dt. \quad (4.27)$$

From the definition of auxiliary projection, we have $\zeta(x,0) = \widehat{v}(x,0) - V(x,0) = 0$. Then an application of Gronwall's lemma in (4.27) and usage of estimates for $\|\rho\|, \|\rho_t\|$ from Lemma 4.3.1, we have that

$$\|\zeta\|_{L_\infty(L_2)}^2 + \|\zeta\|_{L_2(H^2)}^2 \leq C(K^*)h^8\left[\|u\|_{L_2(H^4)}^2 + \|v\|_{L_2(H^4)}^2 + \|v_t\|_{L_2(H^4)}^2\right]. \quad (4.28)$$

Using the triangle inequality, estimate for $\|\rho\|$ from Lemma 4.3.1 and the above expression, we obtain

$$\begin{aligned}\|v - V\|_{L_\infty(L_2)}^2 &= \|e_2\|_{L_\infty(L_2)}^2 \leq C\left[\|\rho\|_{L_\infty(L_2)}^2 + \|\zeta\|_{L_\infty(L_2)}^2\right] \\ &\leq C(K^*)h^8\left[\|u\|_{L_2(H^4)}^2 + \|v\|_{L_2(H^4)}^2 + \|v_t\|_{L_2(H^4)}^2 \right. \\ &\quad + \left. \|v\|_{L_\infty(H^4)}^2\right] \end{aligned} \quad (4.29)$$

and

$$\begin{aligned}\|v - V\|_{L_2(H^2)}^2 &= \|e_2\|_{L_2(H^2)}^2 \leq C(K^*)\left[\|\rho\|_{L_2(H^2)}^2 + \|\zeta\|_{L_2(H^2)}^2\right] \\ &\leq C(K^*)h^4\left[\|u\|_{L_2(H^4)}^2 + \|v\|_{L_2(H^4)}^2 + \|v_t\|_{L_2(H^4)}^2\right] \end{aligned} \quad (4.30)$$

Applying (4.29) in (4.22), we obtain

$$\begin{aligned}\|u_{xx} - U_{xx}\|^2 = \|e_{1xx}\|^2 &\leq C(K^*)h^4\left[\|u\|_{L_2(H^4)}^2 + \|v\|_{L_2(H^4)}^2 + \|v_t\|_{L_2(H^4)}^2 \right. \\ &\quad + \left. \|v\|_{L_\infty(H^4)}^2\right]. \end{aligned} \quad (4.31)$$

Applying (4.29) in (4.23), we obtain

$$\begin{aligned}\|u-U\|^2 = \|e_1\|^2 &\leq C(K^*)h^8\left[\|u\|^2_{L_2(H^4)} + \|v\|^2_{L_2(H^4)} + \|v_t\|^2_{L_2(H^4)}\right.\\ &+ \left.\|v\|^2_{L_\infty(H^4)}\right]. \qquad (4.32)\end{aligned}$$

From (4.29), (4.30), (4.31)and (4.32), we obtain the required result.

To complete our argument, we have to show that $C$ can be chosen independent of $K^*$ for sufficiently small $h$. For that, using Lemma 4.4.1, we have

$$\begin{aligned}\|U\|_{0,\infty} &\leq \|U-u\|_{0,\infty} + \|u\|_{0,\infty} \leq \|e_1\|_{0,\infty} + \|u\|_{0,\infty}\\ &\leq C[h^2\|e_{1xx}\| + \|e_2\|] + \|u\|_{0,\infty}\\ &\leq C[h^2\|u_{xx}-U_{xx}\| + \|v-V\|] + \|u\|_{0,\infty}\\ &\leq C(K^*)h^4\|u\|_4 + \|u\|_{0,\infty} \leq C(K^*)h^4 + C.\end{aligned}$$

Now, for sufficiently small $h$, the above expression can be written as

$$\|U\|_{0,\infty} \leq K^*.$$

Once we obtain an error estimate for $\|u_{xx}-U_{xx}\|$ and $\|v-V\|$, the boundedness of $\|U\|_{0,\infty}$ can automatically be obtained. Therefore, the temporary assumption is not a strange or strong condition. Hence, $C$ can be chosen independent of $K^*$. This completes the proof. ■

**Remark 4.1**:

Using the interpolation inequality (1.17) with $m = 2$ and $i = 1$, and the estimates of $v - V$ in $L_2$ norm and in $H^2$ norm, we can easily obtain the following result.

$$\|v-V\|_1^2 \leq Ch^6\left[\|u\|^2_{L_2(H^4)} + \|v\|^2_{L_2(H^4)} + \|v_t\|^2_{L_2(H^4)} + \|v\|^2_{L_\infty(H^4)}\right].$$

### 4.5 ERROR ANALYSIS OF FULLY DISCRETE EULER BACKWARD SCHEME

In this section, we see the fully discrete approximation for the split up equations (4.4) and (4.5). We retain the $H^1$-Galerkin mixed method in the spatial direction and replace the time derivative by backward finite difference in the time direction. For a time step $k = T/M$ with $M$ a positive integer, let

$t^n = nk$ be the time levels for $n = 0, 1, 2...M$. For a given continuous function $\psi$, let $d_t\psi^{n+1} = (\psi^{n+1} - \psi^n)/k$.

**Linearised fully discrete scheme:**

For the weak solutions $v$ mentioned in (4.7) and $u$ in (4.6), we consider a linearised fully discrete Euler backward approximations $W$ and $Z$: $\{t^0, t^1, ...t^M\} \rightarrow \overset{0}{S}_{h,3}$ respectively defined as follows:

$$(d_tW^{n+1}, \phi_h) + A(\lambda : W^{n+1}, \phi_h) - \lambda(W^{n+1}, \phi_h) + (f(Z^n), \phi_{hxx}) = 0, \quad \phi_h \in \overset{0}{S}_{h,3}, \ \ n = 0, 1, 2...M-1 \qquad (4.33)$$

with $W^0 = g_{xx}$ and

$$(Z^{n+1}_{xx}, \phi_{hxx}) = (W^{n+1}, \phi_{hxx}), \ \ \phi_h \in \overset{0}{S}_{h,3}, \ \ n = 0, 1, 2...M-1 \qquad (4.34)$$

with $Z^0 = g$.

Let us describe the numerical scheme as follows:

Express $W^{n+1} = \sum_{j=0}^{N} \alpha_j^{n+1}\phi_j$ and $Z^{n+1} = \sum_{j=0}^{N} \beta_j^{n+1}\phi_j$, where $\{\phi_j\}_{j=0}^{N}$ is a basis of $\overset{0}{S}_{h,3}$.

Step 1: Knowing $Z^n$ and $W^n$, we compute $W^{n+1}$ ($i.e.,\ \ \alpha_j^{n+1},\ j = 0,\ 1,\ ...N$) using (4.33) as follows:

$$\sum_{j=0}^{N} \left[(\phi_j, \phi_i) + kA(\lambda, \phi_j, \phi_i) - k\lambda(\phi_j, \phi_i)\right] \alpha_j^{n+1} = (W^n, \phi_i) - k(f(Z^n), \phi_{ixx}), \quad i = 0,\ 1,\ ...N.$$

Step 2: With the value of $W^{n+1}$, we solve for $Z^{n+1}$ ($i.e.,\ \ \beta_j^{n+1},\ j = 0,\ 1,\ ...N$) using (4.34) as follows:

$$\sum_{j=0}^{N} (\phi_{jxx}, \phi_{ixx})\beta_j^{n+1} = (W^{n+1}, \phi_{ixx}), \ \ i = 0,\ 1,\ ...N.$$

The above two steps can equivalently be posed as solving the following systems of linear equations:

We first solve the system

$$B\overline{\alpha} = \overline{C}$$

for $\overline{\alpha} = (\alpha_0^{n+1}, \alpha_1^{n+1}, ...\alpha_N^{n+1})^T$, where $B = \{b_{ij}\}_{i,j=0}^N$ with

$$b_{ij} = (\phi_j, \phi_i) + kA(\lambda, \phi_j, \phi_i) - k\lambda(\phi_j, \phi_i)$$

and $\overline{C} = (c_0, c_1...c_N)^T$ with

$$c_i = (W^n, \phi_i) - k(f(Z^n), \phi_{ixx}).$$

We then solve the second system

$$D\overline{\beta} = \overline{\delta}$$

for $\overline{\beta} = (\beta_0^{n+1}, \beta_1^{n+1}, ...\beta_N^{n+1})^T$, where $D = \{d_{ij}\}_{i,j=0}^N$ with $d_{ij} = (\phi_{jxx}, \phi_{ixx})$ and $\overline{\delta} = (\delta_0, \delta_1...\delta_N)^T$ with $\delta_i = (W^{n+1}, \phi_{ixx})$.

We observe that the above linearised fully discrete scheme gives rise to a system of decoupled equations. Before discussing the error estimates we first obtain the error equations for the approximations of $u$ and $v$. Evaluating (4.7) at $t^{n+1}$ and then subtracting (4.33) from the resulting equation, we obtain

$$\begin{aligned}(v_t^{n+1} - d_tW^{n+1}, \phi_h) + A(\lambda : v^{n+1} - W^{n+1}, \phi_h) &- \lambda(v^{n+1} - W^{n+1}, \phi_h) \\ + \Big(f(u^{n+1}) - f(Z^n), \phi_{hxx}\Big) &= 0, \quad \phi_h \in \overset{0}{S}_{h,3}. \end{aligned} \tag{4.35}$$

In a similar manner, evaluating (4.6) at $t^{n+1}$ and then subtracting (4.34) from the resulting equation, we obtain

$$(u_{xx}^{n+1} - Z_{xx}^{n+1}, \phi_{hxx}) = (v^{n+1} - W^{n+1}, \phi_{hxx}), \quad \phi_h \in \overset{0}{S}_{h,3}. \tag{4.36}$$

Let us denote the error between $u^n$ and $Z^n$ by $\varepsilon_1^n$ and that between $v^n$ and $W^n$ by $\varepsilon_2^n$, *i.e.*, $\varepsilon_1^n = u^n - Z^n$ and $\varepsilon_2^n = v^n - W^n$.

Let $v^n - W^n = v^n - \widehat{v}^n + \widehat{v}^n - W^n = \rho^n + \zeta^n$, where $\widehat{v}^n$ is defined earlier as in (4.10). Then (4.35) can be written as

$$\begin{aligned}(v_t^{n+1} - d_tW^{n+1}, \phi_h) &+ A(\lambda : \rho^{n+1} + \zeta^{n+1}, \phi_h) - \lambda(\rho^{n+1} + \zeta^{n+1}, \phi_h) \\ &+ \Big(f(u^{n+1}) - f(Z^n), \phi_{hxx}\Big) = 0, \quad \phi_h \in \overset{0}{S}_{h,3}. \end{aligned} \tag{4.37}$$

But the term $v_t^{n+1} - d_tW^{n+1}$ can be written as

$$\begin{aligned} v_t^{n+1} - d_tW^{n+1} &= v_t^{n+1} - d_tv^{n+1} + d_tv^{n+1} - d_tW^{n+1} \\ &= \sigma_{n+1} + d_t(\rho^{n+1} + \zeta^{n+1}), \end{aligned}$$

where $\sigma_{n+1} = v_t^{n+1} - d_t v^{n+1}$.

Using the projection (4.11) and the above expression in (4.37), we obtain the following error equation:

$$\begin{aligned}(\sigma_{n+1}, \phi_h) &+ (d_t[\rho^{n+1} + \zeta^{n+1}], \phi_h) + A(\lambda : \zeta^{n+1}, \phi_h) - \lambda(\rho^{n+1} + \zeta^{n+1}, \phi_h) \\ &+ \Big(f(u^{n+1}) - f(Z^n), \phi_{hxx}\Big) = 0, \;\; \phi_h \in \overset{0}{S}_{h,3};\end{aligned}$$

*i.e.*,

$$\begin{aligned}(d_t\zeta^{n+1}, \phi_h) + A(\lambda : \zeta^{n+1}, \phi_h) &= -(d_t\rho^{n+1}, \phi_h) - (\sigma_{n+1}, \phi_h) \\ + \;\; \lambda(\rho^{n+1} + \zeta^{n+1}, \phi_h) &- \Big(f(u^{n+1}) - f(Z^n), \phi_{hxx}\Big), \;\; \phi_h \in \overset{0}{S}_{h,3}. \quad (4.38)\end{aligned}$$

In a similar way, (4.36) can be written as

$$(u_{xx}^{n+1} - Z_{xx}^{n+1}, \phi_{hxx}) = (\rho^{n+1} + \zeta^{n+1}, \phi_{hxx}), \;\; \phi_h \in \overset{0}{S}_{h,3}. \quad (4.39)$$

In the following lemma, we compute $|\varepsilon_1^n(\overline{x})|$, where $\overline{x}$ is an arbitrary point in $[0,1]$, the proof of which is similar to the proof of Lemma 4.4.1. This result is required for the proof of the main theorem in this section.

**Lemma 4.5.1** *Let $u$ and $v$ be the weak solutions of* (4.4) *and* (4.5) *defined through* (4.6) *and* (4.7) *respectively. Further, let $Z^n$ and $W^n$ $\in \overset{0}{S}_{h,3}$ be the fully discrete $H^1$-Galerkin mixed finite element formulation defined through* (4.34) *and* (4.33) *respectively. Then the error $\varepsilon_1^n = u^n - Z^n$ satisfies*

$$|\varepsilon_1^n(\overline{x})| \le C\left[h^2 \|\varepsilon_{1xx}^n\| + \|\varepsilon_2^n\|\right],$$

*where $\overline{x}$ is an arbitrary point in* $[0,1]$.

Below, we discuss the analysis of the error involved in the fully discrete scheme. We first compute the error estimate $\|u^n - Z^n\|$ in the following lemma, the proof of which is similar to the proof of Lemma 4.4.2.

**Lemma 4.5.2** *Let $u$ and $v$ be the weak solution of* (4.4) *and* (4.5) *defined as in (4.6) and (4.7). Further let $Z^n$ and $W^n$ $\in \overset{0}{S}_{h,3}$ be the fully discrete $H^1$-Galerkin* *approximation of u and v as defined in (4.34) and (4.33) respectively. Also, let* $\widehat{v}^n$ *be the auxiliary projection of* $v^n$ *as in (4.10). Then, we have*

$$\|u^n - Z^n\|^2 \leq C\left[h^8\|u^n\|_4^2 + \|\rho^n\|^2 + \|\zeta^n\|^2\right],$$
$$\textit{where }\ \rho^n = v^n - \widehat{v}^n,\ \zeta^n = \widehat{v}^n - W^n,\ \ n = 0,\,1,\,2,\,...,M.$$

**Proof:** In (4.36), choose $\phi_h = (u^{n+1} - Z^{n+1}) - (u^{n+1} - \chi)$ for some $\chi \in \overset{0}{S}_{h,3}$. Then it becomes

$$\begin{aligned} & (u_{xx}^{n+1} - Z_{xx}^{n+1}, u_{xx}^{n+1} - Z_{xx}^{n+1}) = (u_{xx}^{n+1} - Z_{xx}^{n+1}, u_{xx}^{n+1} - \chi_{xx}) \\ + \quad & (v^{n+1} - W^{n+1}, u_{xx}^{n+1} - Z_{xx}^{n+1}) - (v^{n+1} - W^{n+1}, u_{xx}^{n+1} - \chi_{xx}). \end{aligned}$$

$$\begin{aligned} \|u_{xx}^{n+1} - Z_{xx}^{n+1}\|^2 \quad & \leq \quad \|u_{xx}^{n+1} - Z_{xx}^{n+1}\|\|u_{xx}^{n+1} - \chi_{xx}\| \\ & + \quad \|v^{n+1} - W^{n+1}\|\|u_{xx}^{n+1} - Z_{xx}^{n+1}\| \\ & + \quad \|v^{n+1} - W^{n+1}\|\|u_{xx}^{n+1} - \chi_{xx}\|. \end{aligned}$$

Taking infimum over $\chi \in \overset{0}{S}_{h,3}$ first and then applying the approximation property, we obtain

$$\begin{aligned} \|u_{xx}^{n+1} - Z_{xx}^{n+1}\|^2 \quad & \leq \quad \|u_{xx}^{n+1} - Z_{xx}^{n+1}\| \inf_{\chi \in \overset{0}{S}_{h,3}} \|u_{xx}^{n+1} - \chi_{xx}\| \\ & + \quad \|v^{n+1} - W^{n+1}\|\|u_{xx}^{n+1} - Z_{xx}^{n+1}\| \\ & + \quad \|v^{n+1} - W^{n+1}\| \inf_{\chi \in \overset{0}{S}_{h,3}} \|u_{xx}^{n+1} - \chi_{xx}\| \\ & \leq \quad C\Big[\|u_{xx}^{n+1} - Z_{xx}^{n+1}\|\, h^2\, \|u^{n+1}\|_4 \\ & + \quad \|v^{n+1} - W^{n+1}\|\|u_{xx}^{n+1} - Z_{xx}^{n+1}\| \\ & + \quad \|v^{n+1} - W^{n+1}\|\, h^2\, \|u^{n+1}\|_4\Big]. \end{aligned}$$

Using Young's inequality, we have

$$\begin{aligned} \|u_{xx}^{n+1} - Z_{xx}^{n+1}\|^2 \quad & \leq \quad C\left[\frac{\epsilon}{2}\|u_{xx}^{n+1} - Z_{xx}^{n+1}\|^2 + \frac{1}{2\epsilon}h^4\|u^{n+1}\|_4^2\right. \\ & + \quad \frac{\epsilon}{2}\|u_{xx}^{n+1} - Z_{xx}^{n+1}\|^2 + \frac{1}{2\epsilon}\|v^{n+1} - W^{n+1}\|^2 \\ & + \quad \left.\frac{\epsilon}{2}\|v^{n+1} - W^{n+1}\|^2 + h^4\frac{1}{2\epsilon}\|u\|_4^2\right], \end{aligned}$$

$$(1 - C\epsilon)\|u_{xx}^{n+1} - Z_{xx}^{n+1}\|^2 \leq C\left[\frac{1}{\epsilon}h^4\|u^{n+1}\|_4^2 + \left(\frac{1}{2\epsilon} + \frac{\epsilon}{2}\right)\|v^{n+1} - W^{n+1}\|^2\right].$$

Choosing $\epsilon > 0$ properly (for example, $\epsilon = \frac{1}{2C}$) so that inequality is maintained , we obtain

$$\|\epsilon_{1xx}^{n+1}\|^2 = \|u_{xx}^{n+1} - Z_{xx}^{n+1}\|^2 \leq C\left[h^4\|u^{n+1}\|_4^2 + \|v^{n+1} - W^{n+1}\|^2\right]. \tag{4.40}$$

We now apply the following duality argument to obtain the estimate of $\varepsilon_1^{n+1}$ in $L_2$ norm.

Let $\Phi$ be the solution of the auxiliary problem

$$\Phi_{xxxx} = u^{n+1} - Z^{n+1} = \varepsilon_1^{n+1}, \quad x \in I$$

with the boundary conditions

$$\Phi_{xxx}(0) = \Phi_{xxx}(1) = 0, \ \Phi_{xx}(0) = \Phi_{xx}(1) = 0.$$

Then, using integration by parts, the boundary conditions and (4.39), we obtain

$$\begin{aligned}
(\varepsilon_1^{n+1}, \varepsilon_1^{n+1}) &= (\Phi_{xxxx}, \varepsilon_1^{n+1}) = (\Phi_{xx}, \varepsilon_{1xx}^{n+1}) = (\varepsilon_{1xx}^{n+1}, \Phi_{xx}) \\
&= (\varepsilon_{1xx}^{n+1}, \Phi_{xx} - \chi_{xx}) + (\varepsilon_{1xx}^{n+1}, \chi_{xx}) \\
&= (\varepsilon_{1xx}^{n+1}, \Phi_{xx} - \chi_{xx}) + (\rho^{n+1} + \zeta^{n+1}, \chi_{xx}) \\
&= (\varepsilon_{1xx}^{n+1}, \Phi_{xx} - \chi_{xx}) + (\rho^{n+1} + \zeta^{n+1}, \chi_{xx} - \Phi_{xx}) \\
&+ (\rho^{n+1} + \zeta^{n+1}, \Phi_{xx}), \ \text{ for } \ \chi \in \overset{0}{S}_{h,3} \\
&\leq \|\varepsilon_{1xx}^{n+1}\|\|\Phi_{xx} - \chi_{xx}\| + \|\rho^{n+1} + \zeta^{n+1}\|\|\chi_{xx} - \Phi_{xx}\| \\
&+ \|\rho^{n+1} + \zeta^{n+1}\|\|\Phi_{xx}\|
\end{aligned}$$

Taking infimum over $\chi \in \overset{0}{S}_{h,3}$ first and then applying the approximation property, we obtain

$$\begin{aligned}
\|\varepsilon_1^{n+1}\|^2 &\leq C\left[h^2\|\varepsilon_{1xx}^{n+1}\|\|\Phi\|_4 + h^2\|\rho^{n+1} + \zeta^{n+1}\|\|\Phi\|_4 + \|\rho^{n+1} + \zeta^{n+1}\|\|\Phi\|_4\right] \\
&\leq C\left[h^2\|\varepsilon_{1xx}^{n+1}\| + h^2\|\rho^{n+1} + \zeta^{n+1}\| + \|\rho^{n+1} + \zeta^{n+1}\|\right]\|\Phi\|_4.
\end{aligned}$$

Using the regularity of the auxiliary problem $\|\Phi\|_4 \leq \|\varepsilon_1^{n+1}\|$, we have

$$\begin{aligned}
\|\varepsilon_1^{n+1}\|^2 &\leq C\left[h^2\|\varepsilon_{1xx}^{n+1}\| + h^2\|\rho^{n+1} + \zeta^{n+1}\| + \|\rho^{n+1} + \zeta^{n+1}\|\right]\|\varepsilon_1^{n+1}\|, \\
i.e., \ \|\varepsilon_1^{n+1}\| &\leq C\left[h^2\|\varepsilon_{1xx}^{n+1}\| + h^2\|\rho^{n+1} + \zeta^{n+1}\| + \|\rho^{n+1} + \zeta^{n+1}\|\right], \\
i.e., \ \|\varepsilon_1^{n+1}\| &\leq C\left[h^2\|\varepsilon_{1xx}^{n+1}\| + \|\rho^{n+1} + \zeta^{n+1}\|\right].
\end{aligned}$$

Squaring both sides and using Hölder's inequality, we obtain

$$\|\varepsilon_1^{n+1}\|^2 \leq C\left[h^4\|\varepsilon_{1xx}^{n+1}\|^2 + \|\rho^{n+1}\|^2 + \|\zeta^{n+1}\|^2\right].$$

Using (4.40) in the above equation, we obtain

$$\|\varepsilon_1^{n+1}\|^2 \leq C\left[h^4(h^4\|u^{n+1}\|_4^2 + \|v^{n+1} - W^{n+1}\|^2) + \|\rho^{n+1}\|^2 + \|\zeta^{n+1}\|^2\right],$$

*i.e.*,

$$\begin{aligned}\|u^{n+1} - Z^{n+1}\|^2 = \|\varepsilon_1^{n+1}\|^2 &\leq C\left[h^8\|u^{n+1}\|_4^2 + \|\rho^{n+1}\|^2 + \|\zeta^{n+1}\|^2\right],\\ &\qquad n = 0,\, 1,\, 2,\, ...,\, M-1.\end{aligned}$$

Now, since $u^0 = Z^0$, we have that

$$\begin{aligned}\|u^n - Z^n\|^2 = \|\varepsilon_1^n\|^2 &\leq C\left[h^8\|u^n\|_4^2 + \|\rho^n\|^2 + \|\zeta^n\|^2\right],\\ n = 0,\, 1,\, 2,\, ...,\, M. &\end{aligned} \tag{4.41}$$

Hence the proof is completed. ■

We have now computed the error bound of $\|u^n - Z^n\|$ in terms of the estimates of $\rho^n$ and $\zeta^n$ at time level $n$. Below, we shall obtain *a priori* bound for $\|\zeta^n\|$ in terms of $\|\rho^n\|$. This will help us in computing the error bounds of $(u^n - Z^n)$ and $(v^n - W^n)$ in terms of $h$ and the regularity condition, since the estimates of $\rho^n$ has already been obtained.

**Theorem 4.5.1** *Let $u$ and $v$ be the weak solutions of* (4.4) *and* (4.5) *defined through* (4.6) *and* (4.7) *respectively. Further, let $Z^n$ and $W^n$ be the corresponding semi discrete $H^1$-Galerkin mixed finite element formulation defined through* (4.34) *and* (4.33) . *Then, for a sufficiently small $h$, the error in the fully discrete approximation of $u$ and $v$ by the backward Euler scheme is given by*

$$\begin{aligned}\|u^{J+1} - Z^{J+1}\| &\leq C\left[k^{\frac{3}{2}} + h^4\right]\\ \|u_{xx}^{J+1} - Z_{xx}^{J+1}\| &\leq C\left[k^{\frac{3}{2}} + h^2\right];\\ \|v^{J+1} - W^{J+1}\| &\leq C\left[k^{\frac{3}{2}} + h^4\right] \ \textit{for}\ J = 0, 1, 2, ..., M-1,\end{aligned}$$

*where $C$ is a generic constant depending only on $u$ and $v$.*

**Proof:** Substituting $\phi_h = \zeta^{n+1}$ in (4.38) and using coercivity of $A(\lambda : \varphi, \psi)$, we obtain

$$\begin{aligned}(d_t\zeta^{n+1}, \zeta^{n+1}) + \alpha_0\|\zeta^{n+1}\|_2^2 &\leq |(d_t\rho^{n+1}, \zeta^{n+1})| + |(\sigma_{n+1}, \zeta^{n+1})| \\ &+ |\lambda(\rho^{n+1}, \zeta^{n+1})| + |\lambda(\zeta^{n+1}, \zeta^{n+1})| \\ &+ \left|\left(f(u^{n+1}) - f(Z^n), \zeta_{xx}^{n+1}\right)\right|\end{aligned}$$

$$\begin{aligned}(d_t\zeta^{n+1}, \zeta^{n+1}) + \alpha_0\|\zeta^{n+1}\|_2^2 &\leq |(d_t\rho^{n+1}, \zeta^{n+1})| + |(\sigma_{n+1}, \zeta^{n+1})| \\ &+ |\lambda(\rho^{n+1}, \zeta^{n+1})| + |\lambda(\zeta^{n+1}, \zeta^{n+1})| \\ &+ \left|\left(f(u^{n+1}) - f(u^n), \zeta_{xx}^{n+1}\right)\right| \\ &+ \left|\left(f(u^n) - f(Z^n), \zeta_{xx}^{n+1}\right)\right| \\ &\leq T_{4,1}^{n+1} + T_{4,2}^{n+1} + T_{4,3}^{n+1} \\ &+ T_{4,4}^{n+1} + T_{4,5}^{n+1} + T_{4,6}^{n+1}. \end{aligned} \tag{4.42}$$

For the first term on the left hand side, we have

$$(d_t\zeta^{n+1}, \zeta^{n+1}) = \frac{1}{k}(\zeta^{n+1} - \zeta^n, \zeta^{n+1}) = \frac{1}{k}\{\|\zeta^{n+1}\|^2 - (\zeta^{n+1}, \zeta^n)\} \geq \frac{1}{2}d_t\|\zeta^{n+1}\|^2,$$

where we have used Young's inequality with $\epsilon = 1$ for $(\zeta^{n+1}, \zeta^n)$. Hence, (4.42) becomes

$$\begin{aligned}\frac{\|\zeta^{n+1}\|^2 - \|\zeta^n\|^2}{2k} + \alpha_0\|\zeta^{n+1}\|_2^2 &\leq T_{4,1}^{n+1} + T_{4,2}^{n+1} + T_{4,3}^{n+1} \\ &+ T_{4,4}^{n+1} + T_{4,5}^{n+1} + T_{4,6}^{n+1}.\end{aligned}$$

Summing the above from n=0, 1, 2, 3,...J after multiplying both sides by $2k$, we obtain

$$\begin{aligned}&\|\zeta^{J+1}\|^2 - \|\zeta^0\|^2 + 2k\alpha_0\sum_{n=0}^{J}\|\zeta^{n+1}\|_2^2 \\ \leq\ & 2k\sum_{n=0}^{J}\left(T_{4,1}^{n+1} + T_{4,2}^{n+1} + T_{4,3}^{n+1} + T_{4,4}^{n+1} + T_{4,5}^{n+1} + T_{4,6}^{n+1}\right).\end{aligned} \tag{4.43}$$

We now estimate the terms on the right hand side of the above expression.
For the term $T_{4,1}^{n+1} = |(d_t\rho^{n+1}, \zeta^{n+1})|$, we use Young's inequality to obtain

$$kT_{4,1}^{n+1} \leq C\left[\frac{1}{2\epsilon}k\|d_t\rho^{n+1}\|^2 + \frac{\epsilon}{2}k\|\zeta^{n+1}\|^2\right]. \tag{4.44}$$

We compute the error bound for $T_{4,2}^{n+1} = |(\sigma_{n+1}, \zeta^{n+1})|$ as follows. Recall that

$$\sigma_{n+1} = v_t^{n+1} - d_t v^{n+1}.$$

From the Taylor series expansion

$$v^{n+1} = v^n + \frac{k}{1!}v_t^n + \frac{k^2}{2!}v_{tt}^n(x,\theta_1) \text{ for } t_n < \theta_1 < t_{n+1}.$$

we have that

$$d_t v^{n+1} = \frac{v^{n+1} - v^n}{k} = v_t^n + \frac{k}{2!}v_{tt}^n(x,\theta_1) \text{ for } t_n < \theta_1 < t_{n+1}.$$

Using Taylor series expansion $v_t^{n+1} = v_t^n + \frac{k}{1!}v_{tt}^n(x,\theta_2)$ for $t_n < \theta_2 < t_{n+1}$ for the first term, we obtain

$$\sigma_{n+1} = kv_{tt}^n(x,\theta_2) - \frac{k}{2}v_{tt}^n(x,\theta_1) \text{ for } t_n < \theta_1 < t_{n+1},\ t_n < \theta_2 < t_{n+1}.$$

Therefore,

$$\|\sigma_{n+1}\| \leq k\|v_{tt}^n(x,\theta_2)\| + \frac{k}{2}\|v_{tt}^n(x,\theta_1)\| \text{ for } t_n < \theta_1 < t_{n+1},\ t_n < \theta_2 < t_{n+1}.$$

Hence, we have that

$$\|\sigma_{n+1}\|^2 \leq Ck^2\|v_{tt}^n\|_{L_\infty(L_2)}^2. \tag{4.45}$$

For the term $T_{4,2}^{n+1}$, using Young's inequality, we obtain

$$T_{4,2}^{n+1} \leq C\left[\frac{1}{2\epsilon}\|\sigma_{n+1}\|^2 + \frac{\epsilon}{2}\|\zeta^{n+1}\|^2\right].$$

On substituting (4.45) in the above, we have that

$$kT_{4,2}^{n+1} \leq C\left[\frac{1}{2\epsilon}k^3\|v_{tt}^n\|_{L_\infty(L_2)}^2 + k\frac{\epsilon}{2}\|\zeta^{n+1}\|^2\right]. \tag{4.46}$$

Similarly, an application of Young's inequality gives the estimates of the term $T_{4,3}^{n+1} = |\lambda(\rho^{n+1}, \zeta^{n+1})|$ as follows:

$$kT_{4,3}^{n+1} \leq C\left[\frac{1}{2\epsilon}k\|\rho^{n+1}\|^2 + k\frac{\epsilon}{2}\|\zeta^{n+1}\|^2\right]. \tag{4.47}$$

Further, for $T_{4,4}^{n+1} = |\lambda(\zeta^{n+1}, \zeta^{n+1})|$, we obtain

$$kT_{4,4}^{n+1} = Ck\|\zeta^{n+1}\|^2. \tag{4.48}$$

For the estimate of $T_{4,5}^{n+1} = |(f(u^{n+1}) - f(u^n), \zeta_{xx}^{n+1})|$, we observe that,

$$T_{4,5}^{n+1} = \left|\left(f(u^{n+1}) - f(u^n), \zeta_{xx}^{n+1}\right)\right| \leq C\left\|f(u^{n+1}) - f(u^n)\right\| \left\|\zeta_{xx}^{n+1}\right\|.$$

We now estimate the first expression appearing on the right hand side Taylor series expansion for $u^{n+1}$ and then Young's inequality, as follows:

$$\begin{aligned}
\left\|f(u^{n+1}) - f(u^n)\right\| &= \|[(u^{n+1})^3 - u^{n+1}] - [(u^n)^3 - u^n]\| \\
&= \|[(u^{n+1})^3 - (u^n)^3] - (u^{n+1} - u^n)\| \\
&\leq \|u^{n+1} - u^n\|\|[(u^{n+1})^2 + u^{n+1}u^n + (u^n)^2 - 1]\| \\
&\leq Ck\|u_t^n(x, \theta_3)\| \text{ for } t_n < \theta_3 < t_{n+1}.
\end{aligned}$$

We therefore have that

$$\left\|f(u^{n+1}) - f(u^n)\right\| \leq Ck\|u_t^n\|_{L_\infty(L_2)}.$$

Hence the estimate of $T_{4,5}^{n+1}$ can be written as

$$T_{4,5}^{n+1} \leq C\left[\frac{1}{2\epsilon}k^2\|u_t^n\|_{L_\infty(L_2)}^2 + \frac{\epsilon}{2}\|\zeta_{xx}^{n+1}\|^2\right],$$

$$i.e.,\ kT_{4,5}^{n+1} \leq C\left[\frac{1}{2\epsilon}k^3\|u_t^n\|_{L_\infty(L_2)}^2 + \frac{\epsilon}{2}k\|\zeta_{xx}^{n+1}\|^2\right], \tag{4.49}$$

We now temporarily assume that $\|Z^n\|_{0,\infty} \leq K^*$ . After having obtained the relevant error estimate, we can prove that this assumption is no longer a strong condition. With this temporary assumption and using Young's inequality, we now estimate $T_{4,6}^{n+1}$ as follows:

$$\begin{aligned}
T_{4,6}^{n+1} &= |(f(u^n) - f(Z^n), \zeta_{xx}^{n+1})| \leq \|f(u^n) - f(Z^n)\|\|\zeta_{xx}^{n+1}\| \\
&\leq C\left[\frac{1}{2\epsilon}\|f(u^n) - f(Z^n)\|^2 + \frac{\epsilon}{2}\|\zeta_{xx}^{n+1}\|^2\right] \\
&\leq C\left[\frac{1}{2\epsilon}\|[(u^n)^3 - u] - [(Z^n)^3 - Z^n]\|^2 + \frac{\epsilon}{2}\|\zeta_{xx}^{n+1}\|^2\right] \\
&\leq C\left[\frac{1}{2\epsilon}\|(u^n - Z^n)[u^n)^2 + u^nZ^n + (Z^n)^2 - 1]\|^2 + \frac{\epsilon}{2}\|\zeta_{xx}^{n+1}\|^2\right] \\
&\leq C(K^*)\frac{1}{2\epsilon}\|(u^n - Z^n)\|^2 + C\frac{\epsilon}{2}\|\zeta_{xx}^{n+1}\|^2 \\
&\leq C(K^*)\frac{1}{2\epsilon}\|\varepsilon_1^n\|^2 + C\frac{\epsilon}{2}\|\zeta_{xx}^{n+1}\|^2
\end{aligned} \tag{4.50}$$

Substituting (4.44) and (4.46) to (4.50) in (4.43), we obtain that

$$
\begin{aligned}
\|\zeta^{J+1}\|^2 &- \|\zeta^0\|^2 + 2k\sum_{n=0}^{J}\alpha_0\|\zeta^{n+1}\|_2^2 \le C\left[\frac{1}{2\epsilon}\left(k\sum_{n=0}^{J}\|d_t\rho^{n+1}\|^2\right.\right.\\
&+ \left.\left. k^3\sum_{n=0}^{J}\|v_{tt}^n\|^2_{L_\infty(L_2)} + k\sum_{n=0}^{J}\|\rho^{n+1}\|^2 + k^3\sum_{n=0}^{J}\|u_t^n\|^2_{L_\infty(L_2)}\right)\right]\\
&+ kC(K^*)\frac{1}{2\epsilon}\sum_{n=0}^{J}\|\varepsilon_1^n\|^2 + C\left[\frac{3\epsilon}{2}k\sum_{n=0}^{J}\|\zeta^{n+1}\|^2\right.\\
&+ \left. k\sum_{n=0}^{J}\|\zeta^{n+1}\|^2 + \epsilon k\sum_{n=0}^{J}\|\zeta_{xx}^{n+1}\|^2\right].
\end{aligned}
$$

Using the inequalities $\|\zeta^{n+1}\| \le \|\zeta^{n+1}\|_2$, $\|\zeta_{xx}^{n+1}\| \le \|\zeta^{n+1}\|_2$ and rearranging the terms, we obtain

$$
\begin{aligned}
\|\zeta^{J+1}\|^2 &- \|\zeta^0\|^2 + k(2\alpha_0 - \frac{5C}{2}\epsilon)\sum_{n=0}^{J}\|\zeta^{n+1}\|_2^2 \le C(K^*)\left[\frac{1}{2\epsilon}\left(k\sum_{n=0}^{J}\|d_t\rho^{n+1}\|^2\right.\right.\\
&+ \left. k^3\sum_{n=0}^{J}\|v_{tt}^n\|^2_{L_\infty(L_2)} + k\sum_{n=0}^{J}\|\rho^{n+1}\|^2 + k^3\sum_{n=0}^{J}\|u_t^n\|^2_{L_\infty(L_2)}\right)\\
&+ \left. k\frac{1}{2\epsilon}\sum_{n=0}^{J}\|\varepsilon_1^n\|^2 + k\sum_{n=0}^{J}\|\zeta^{n+1}\|^2\right].
\end{aligned}
$$

Using the result of Lemma 4.5.2 and bringing $kC(K^*)\|\zeta^{J+1}\|^2$ term to the left hand side, we obtain

$$
\begin{aligned}
(1-kC(K^*))\|\zeta^{J+1}\|^2 &- \|\zeta^0\|^2 + k(2\alpha_0 - \frac{5C}{2}\epsilon)\sum_{n=0}^{J}\|\zeta^{n+1}\|_2^2\\
&\le C(K^*)\left[\frac{1}{2\epsilon}\left(k\sum_{n=0}^{J}\|d_t\rho^{n+1}\|^2 + k^3\sum_{n=0}^{J}\|v_{tt}^n\|^2_{L^\infty(L^2)}\right.\right.\\
&+ \left. k\sum_{n=0}^{J}\|\rho^{n+1}\|^2 + k^3\sum_{n=0}^{J}\|u_t^n\|^2_{L_\infty(L_2)} + kh^8\sum_{n=0}^{J}\|u^n\|_4^2\right)\\
&+ \left. k(1+\frac{1}{2\epsilon})\sum_{n=0}^{J}\|\zeta^n\|^2\right].
\end{aligned}
$$

Choose $\epsilon > 0$ properly (for example, $\epsilon < \dfrac{4\alpha_0 - 2}{5C}$) so that the inequality is maintained. After having chosen such an $\epsilon$, select $k$ so that $(1 - C(K^*)k) > 0$. Then, using discrete version of Gronwall's inequality and the fact that $\zeta^0 = 0$, we obtain

$$
\|\zeta^{J+1}\|^2 + k\sum_{n=0}^{J}\|\zeta^{n+1}\|_2^2 \le C(K^*)\left[k\sum_{n=0}^{J}\|d_t\rho^{n+1}\|^2 + k^3\sum_{n=0}^{J}\|v_{tt}^n\|^2_{L^\infty(L^2)}\right.
$$

$$+ \quad k\sum_{n=0}^{J}\|\rho^{n+1}\|^2 + k^3\sum_{n=0}^{J}\|u_t^n\|^2_{L^\infty(L^2)} + kh^8\sum_{n=0}^{J}\|u^n\|_4^2\Bigg]. \tag{4.51}$$

The estimates of $\|\rho^{n+1}\|$ and $\|kd_t\rho^{n+1}\|$ can be obtained from the results of Lemma 4.3.1 since $kd_t\rho^{n+1} = \rho^{n+1} - \rho^n$. Thus

$$\|\zeta^{J+1}\|^2 \le O\left[k^3 + h^8\right];$$

and therefore, we have that

$$\|\zeta^{J+1}\| \le O\left[k^{\frac{3}{2}} + h^4\right]. \tag{4.52}$$

Using the estimate for $\|\rho^{n+1}\|$ and triangle inequality, we obtain the required result for $\|\varepsilon_2\|$ as follows:

$$\|v^{J+1} - W^{J+1}\| = \|\varepsilon_2^{J+1}\| \le O\left[k^{\frac{3}{2}} + h^4\right]. \tag{4.53}$$

Using (4.53) in (4.40), we get

$$\|u_{xx}^{J+1} - Z_{xx}^{J+1}\| \le O\left[k^{\frac{3}{2}} + h^2\right]. \tag{4.54}$$

Further, using (4.52) in (4.41), we obtain that

$$\|u^{J+1} - Z^{J+1}\| \le O\left[k^{\frac{3}{2}} + h^4\right]. \tag{4.55}$$

We obtain the required result from (4.53) to (4.55).

To complete our argument, we have to show that $C$ can be chosen independent of $K^*$ for sufficiently small $h$. For that, using Lemma 4.5.1, we have

$$\begin{aligned}
\|Z^n\|_{0,\infty} &\le \|Z^n - u^n\|_{0,\infty} + \|u^n\|_{0,\infty} \le \|\varepsilon_1^n\|_{0,\infty} + \|u^n\|_{0,\infty} \\
&\le C[h^2\|\varepsilon_{1xx}^n\| + \|\varepsilon_2^n\|] + \|u^n\|_{0,\infty} \\
&\le C[h^2\|u_{xx}^n - Z_{xx}^n\| + \|v^n - W^n\|] + \|u^n\|_{0,\infty} \\
&\le C(K^*)h^4\|u^n\|_4 + \|u^n\|_{0,\infty} \le C(K^*)h^4 + C.
\end{aligned}$$

Now, for sufficiently small $h$, the above expression can be written as

$$\|Z^n\|_{0,\infty} \le K^*.$$

Once we obtain an error estimate for $\|u_{xx}^n - Z_{xx}^n\|$ and $\|v^n - W^n\|$, the boundedness of $\|Z^n\|_{0,\infty}$ can automatically be obtained. Therefore, the temporary assumption is not a strange or strong condition. Hence, $C$ can be chosen independent of $K^*$. This completes the proof. ■

**Remark 4.2**:

We have assumed that $\zeta^0 = 0$.

If $\zeta^0 \neq 0$, then

$$\|\zeta^0\| = \|\widehat{v}^0 - W^0\| = \|g_{xx} - W^0\| \leq h^4 \|g_{xx}\|_4.$$

Then for an optimal order error estimate, it is demanded that $g \in H^6(I)$.

**Remark 4.3**:

From (4.51), we also have that

$$k \sum_{n=0}^{J} \|\zeta^{n+1}\|_2^2 \leq O\left[k^3 + h^8\right].$$

Using the estimate for $\|\rho^{n+1}\|_2^2$ and triangle inequality, we obtain

$$k \sum_{n=0}^{J} \|v^{n+1} - W^{n+1}\|_2^2 \leq O\left[k^3 + h^4\right].$$

**Remark 4.4**:

Following are the advantages of the method described in this chapter.

1. For the classical solutions of the extended Fisher-Kolmogorov equation, fourth order smoothness is required. The method described in this chapter requires sixth order regularity on the solution. But there are methods, for example, orthogonal cubic spline collocation method which demand eighth order regularity on the solution as given in the literature (Danumjaya *et al.* 2005).

2. The size of the combined linear system is $8n+4$ in (Danumjaya *et al.* 2005), where as the size of the decoupled system in the present method is $n+1$ each (*i.e.*,a total of $2n+2$). This is clearly explained in the next section, where the linear fully discrete scheme is described.

### 4.6 NUMERICAL EXPERIMENTS

In this section, we present a study on numerical implementation of the $H^1$-Galerkin mixed finite element cubic spline approximation method, which is discussed in this chapter. Though we do not impose any assumption on the partition of the interval, for the purpose of implementation we consider a partition with uniform spacing. Replacing $n$ by $N$ in the definition of cubic B-Splines mentioned in Chapter 2, the trial space and test space with these cubic B-Splines as basis functions are of dimension $N+3$. In order to include the zero Dirichlet boundary condition in the trial and test spaces, we modify the above cubic B-splines space as follows. The $j^{th}$ basis $\widehat{B}_j(x)$ of this modified cubic splines $\overset{0}{S}_{h,3}$ for $j = 0, 1, 2, \cdots, N$ is given by

$$\begin{aligned}
\widehat{B}_0(x) &= B_0(x) - \frac{B_0(x_0)}{B_{-1}(x_0)} B_{-1}(x) \\
\widehat{B}_1(x) &= B_1(x) - \frac{B_1(x_0)}{B_{-1}(x_0)} B_{-1}(x) \\
\widehat{B}_j(x) &= B_j(x) \text{ if } j = 2, 3, ...N-2 \\
\widehat{B}_{N-1}(x) &= B_{N-1}(x) - \frac{B_{N-1}(x_N)}{B_{N+1}(x_N)} B_{N+1}(x) \\
\widehat{B}_N(x) &= B_N(x) - \frac{B_N(x_N)}{B_{N+1}(x_N)} B_{N+1}(x).
\end{aligned}$$

Let us now describe the numerical scheme as follows:
Express $W^{n+1} = \sum_{j=0}^{N} \alpha_j^{n+1} \widehat{B}_j$ and $Z^{n+1} = \sum_{j=0}^{N} \beta_j^{n+1} \widehat{B}_j$, where $\{\widehat{B}_j\}_{j=0}^{N}$ is a basis of $\overset{0}{S}_{h,3}$.

Step 1: Knowing $Z^n$ and $W^n$, we compute $W^{n+1}$ (*i.e.*, $\alpha_j^{n+1}$, $j = 0, 1, ...N$) using (4.33) as follows:

$$\begin{aligned}
&\sum_{j=0}^{N} \left[ (\widehat{B}_j, \widehat{B}_i) + kA(\lambda, \widehat{B}_j, \widehat{B}_i) - k\lambda(\widehat{B}_j, \widehat{B}_i) \right] \alpha_j^{n+1} \\
&= (W^n, \widehat{B}_i) - k(f(Z^n), \widehat{B}_{ixx}), \;\; i = 0, 1, ...N.
\end{aligned}$$

Step 2: With the recent value of $W^{n+1}$, we solve for $Z^{n+1}$ (*i.e.*, $\beta_j^{n+1}$, $j = 0, 1, ...N$) using (4.34) as follows:

$$\sum_{j=0}^{N} (\widehat{B}_{jxx}, \widehat{B}_{ixx}) \beta_j^{n+1} = (W^{n+1}, \widehat{B}_{ixx}), \;\; i = 0, 1, ...N.$$

The above two steps can equivalently be posed as solving the following systems of linear equations:

We first solve the system

$$B\overline{\alpha} = \overline{C}$$

for $\overline{\alpha} = (\alpha_0^{n+1}, \alpha_1^{n+1}, ...\alpha_N^{n+1})^T$, where $B = \{b_{ij}\}_{i,j=0}^N$ with

$$b_{ij} = (\widehat{B}_j, \widehat{B}_i) + kA(\lambda, \widehat{B}_j, \widehat{B}_i) - k\lambda(\widehat{B}_j, \widehat{B}_i)$$

and $\overline{C} = (c_0, c_1...c_N)^T$ with

$$c_i = (W^n, \widehat{B}_i) - k(f(Z^n), \widehat{B}_{ixx}).$$

We then solve the second system

$$D\overline{\beta} = \overline{\delta}$$

for $\overline{\beta} = (\beta_0^{n+1}, \beta_1^{n+1}, ...\beta_N^{n+1})^T$, where $D = \{d_{ij}\}_{i,j=0}^N$ with $d_{ij} = (\widehat{B}_{jxx}, \widehat{B}_{ixx})$ and $\overline{\delta} = (\delta_0, \delta_1...\delta_N)^T$ with $\delta_i = (W^{n+1}, \phi_{ixx})$.

The size of the combined linear system is $8n + 4$ in (Danumjaya *et al.* 2005), where as the size of the decoupled system in the present method is $n + 1$ each (*i.e.*, $2n + 2$).

**Example 4.1**:

We consider the following problem for the extended Fisher- Kolmogorov equation

$$u_t + \gamma u_{xxxx} - u_{xx} + f(u) = 0, \quad 0 < t < T, \;\; x \in I = (0, 1); \tag{4.56}$$

subject to the boundary and initial conditions

$$u(0, t) = 0, \; u(1, t) = 0, \;\; u_{xx}(0, t) = 0, \; u_{xx}(1, t) = 0;$$
$$u(x, 0) = \sin(\pi x);$$

where $f(u) = u^3 - u$. The profile of the approximate solution of (4.56) for $\gamma = 1$ is given in figure 4.1 and figure 4.2, for different values of $h$.

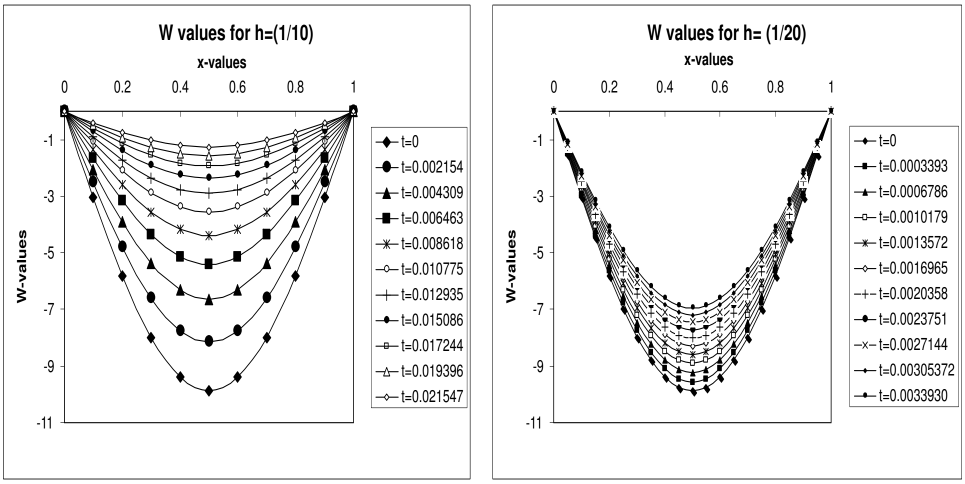


Figure 4.1: Approximate solution of $v(x) = u_{xx}$, (*i.e.*, $W(x)$) at different time levels taking $h = 1/10$ and $h = 1/20$ respectively.

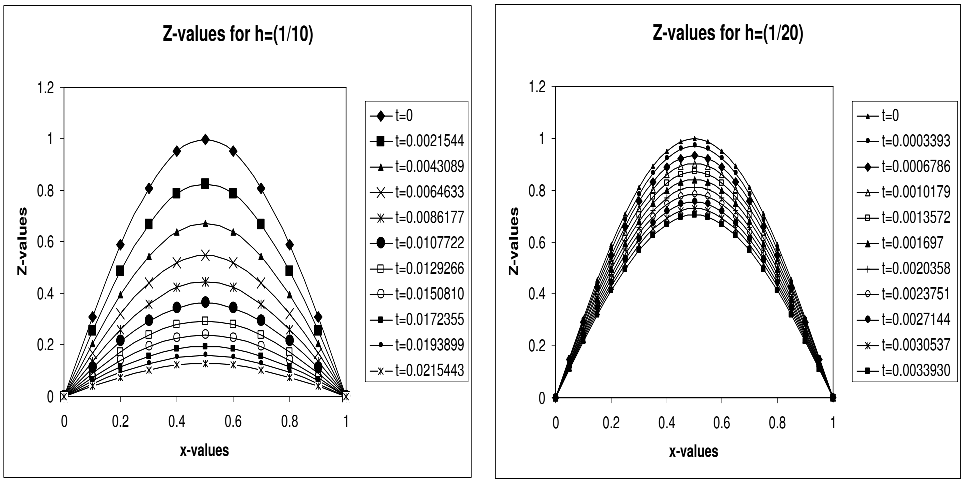


Figure 4.2: Approximate solution of $u(x)$, (*i.e.*, $Z(x)$) at different time levels taking $h = 1/10$ and $h = 1/20$ respectively.

**Example 4.2**:

We now consider the following non homogenous extended Fisher- Kolmogorov equation

$$u_t + \gamma u_{xxxx} - u_{xx} + f(u) = \phi(x,t), \quad 0 < t < T, \;\; x \in I = (0,1);$$

with initial condition

$$u(x,0) = \sin(2\pi x)$$

and boundary conditions

$$u(0,t)=0,\ u(1,t)=0,\ \ u_{xx}(0,t)=0,\ u_{xx}(1,t)=0;$$

where

$$\phi(x,t)=e^{-t}\sin(2\pi x)\left[e^{-2t}\sin^2(2\pi x)-2+16\pi^4+4\pi^2\right]$$

and $f(u)=u^3-u$ (Noomen and Khaled 2011). The exact solution of the above problem for $\gamma=1$ is

$$u(x,t)=e^{-t}\sin(2\pi x).$$

It has been theoretically proved that the error of this approximation is of order $\left(k^{\frac{3}{2}}+h^4\right)$,

*i.e.*,

$$\|u^{J+1}-Z^{J+1}\| \ \leq \ C\left[k^{\frac{3}{2}}+h^4\right]$$

and

$$\|v^{J+1}-W^{J+1}\| \ \leq \ C\left[k^{\frac{3}{2}}+h^4\right] \text{ for } J=0,1,2,...,M-1,$$

The numerical computations with number of sub intervals 10, 20 and 40 are considered separately, *i.e.*, with $h=\frac{1}{10},\ h=\frac{1}{20}$ and $h=\frac{1}{40}$ respectively. For each of these spacial mesh length, the corresponding time step lengths $k$'s are taken satisfying $k^{\frac{3}{2}}=h^4$. With these time step length choice, the error of convergence becomes $O(h^4)$ instead of $O\left(k^{\frac{3}{2}}+h^4\right)$. We denote $h_1=\frac{1}{10},\ \ h_2=\frac{1}{20}$ and $h_3=\frac{1}{40}$.

Let $Z^n_{h_i}$ and $W^n_{h_i}$ be the approximate solution in the space $\overset{0}{S}_{h,3}$ of the exact solution $u(x,t)$ and $v(x,t)$ respectively at time $t=t^n$, taking the spacial mesh length $h_i$ for i=1, 2 and 3.

The order of convergence for this method for $u$ is calculated by the formula

$$Order=\left(\log\frac{\|u^n-Z^n_{h_i}\|_{L_p}}{\|u^n-Z^n_{h_{i+1}}\|_{L_p}}\right)/(\log 2)$$

at the $n^{th}$ time level $t^n$, where $1\leq p\leq\infty$. Similarly the order of convergence of this method for $v$ is calculated by the formula

$$Order=\left(\log\frac{\|v^n-W^n_{h_i}\|_{L_p}}{\|v^n-W^n_{h_{i+1}}\|_{L_p}}\right)/(\log 2)$$

at the $n^{th}$ time level $t^n$, where $1 \leq p \leq \infty$. Order of errors in $L_2$ and $L_\infty$ norms are computed and tabulated as follows.

**Table 4.1:** $L_2$ **errors in** $Z$ **and** $W$ **at time** $t = 0.00215443469$

| n | h | $\|u - Z\|$ | order | $\|v - W\|$ | order |
|---|---|---|---|---|---|
| 10 | 0.1 | $1.20361 * 10^{-4}$ | - | $2.94993 * 10^{-3}$ | - |
| 20 | 0.05 | $6.6649 * 10^{-6}$ | 4.17464 | $1.41083 * 10^{-4}$ | 4.386064 |
| 40 | 0.025 | $4.10368 * 10^{-7}$ | 4.02159 | $8.39797 * 10^{-6}$ | 4.07036 |

**Table 4.2:** $L_2$ **errors in** $Z$ **and** $W$ **at time** $t = 0.01077217300$

| n | h | $\|u - Z\|$ | order | $\|v - W\|$ | order |
|---|---|---|---|---|---|
| 10 | 0.1 | $1.30764 * 10^{-4}$ | - | $3.26184 * 10^{-3}$ | - |
| 20 | 0.05 | $6.83905 * 10^{-6}$ | 4.25703 | $1.48182 * 10^{-4}$ | 4.46024 |
| 40 | 0.025 | $4.1387 * 10^{-7}$ | 4.04655 | $8.5968 * 10^{-6}$ | 4.10743 |

**Table 4.3:** $L_\infty$ **errors in** $Z$ **and** $W$ **at time** $t = 0.00215443469$

| n | h | $\|u - Z\|_{0,\infty}$ | order | $\|v - W\|_{0,\infty}$ | order |
|---|---|---|---|---|---|
| 10 | 0.1 | $3.95331 * 10^{-4}$ | - | $1.22774 * 10^{-2}$ | - |
| 20 | 0.05 | $2.32643 * 10^{-5}$ | 4.08687 | $7.35631 * 10^{-4}$ | 4.060878 |
| 40 | 0.025 | $1.45127 * 10^{-6}$ | 4.00273 | $4.6196 * 10^{-5}$ | 3.99314 |

**Table 4.4:** $L_\infty$ **errors in** $Z$ **and** $W$ **at time** $t = 0.01077217300$

| n | h | $\|u - Z\|_{0,\infty}$ | order | $\|v - W\|_{0,\infty}$ | order |
|---|---|---|---|---|---|
| 10 | 0.1 | $4.10278 * 10^{-4}$ | - | $1.28968 * 10^{-2}$ | - |
| 20 | 0.05 | $2.33967 * 10^{-5}$ | 4.13222 | $7.42453 * 10^{-4}$ | 4.11857 |
| 40 | 0.025 | $1.44877 * 10^{-6}$ | 4.0134 | $4.61911 * 10^{-5}$ | 4.00661 |