CHAPTER 3

QUADRATURE BASED MIXED PETROV-GALERKIN METHOD FOR A GENERAL FOURTH ORDER BOUNDARY VALUE PROBLEM

3.1 INTRODUCTION

In this chapter, we discuss a quadrature based Petrov-Galerkin mixed finite element cubic spline approximation method for the following general fourth order two-point boundary value problem:

$$u_{xxxx} + a(x)u_{xxx} + b(x)u_{xx} + c(x)u_x + d(x)u = f(x),\ x \in I = (0,1); \tag{3.1}$$

subject to the boundary conditions

$$u(0) = 0,\ u(1) = 0;\ u_{xx}(0) = 0,\ u_{xx}(1) = 0. \tag{3.2}$$

We hereafter suppress the dependency of the independent variable x on these functions $a(x),\ b(x),\ c(x),\ d(x)$ and $f(x)$. We, therefore, write a, b, c, d and f respectively instead of these functions. Let us define the splitting of the above fourth order equation as follows:
Set

$$u_{xx} = v,\ \ x \in I.$$

Then the differential equation (3.1) and the boundary conditions (3.2) can be written as a coupled system of equations as follows:

$$\begin{aligned} u_{xx} &= v,\ \ x \in I \text{ with } u(0) = u(1) = 0, && (3.3)\\ v_{xx} + av_x + bv + cu_x + du &= f,\ \ x \in I \text{ with } v(0) = v(1) = 0. && (3.4) \end{aligned}$$

Weak Formulation: The weak formulation corresponding to the split equations (3.3) and (3.4) is defined as follows:
Find $u, v \in \ H^2(I) \bigcap \overset{0}{H^1}(I)$ such that

$$\begin{aligned} (u_{xx}, \phi) &= (v, \phi), \ \ \phi \in H^2(0,1), \qquad (3.5) \\ (v_{xx} + av_x + bv, \phi) + (cu_x + du, \phi) &= (f, \phi), \ \ \phi \in H^2(0,1). \qquad (3.6) \end{aligned}$$

Petrov-Galerkin Formulation: The Petrov-Galerkin formulation corresponding to the above weak formulation (3.5) and (3.6) is defined respectively as follows:
Find $u_h, v_h \in \overset{0}{S}_{h,3}$ such that

$$\begin{aligned} (u_{hxx}, \phi_h) &= (v_h, \phi_h), \ \ \phi_h \in S_{h,1}, \qquad (3.7) \\ (v_{hxx} + av_{hx} + bv_h, \phi_h) + (cu_{hx} + du_h, \phi_h) &= (f, \phi_h), \ \ \phi_h \in S_{h,1}. \qquad (3.8) \end{aligned}$$

Discrete Petrov-Galerkin Formulation: The discrete Petrov-Galerkin formulation corresponding to (3.5) and (3.6) is respectively defined as follows:
Find $u_h, v_h \in \overset{0}{S}_{h,3}$ such that

$$\langle u_{hxx}, \phi_h \rangle_h = \langle v_h, \phi_h \rangle_h, \ \ \phi_h \in S_{h,1}, \qquad (3.9)$$

$$\langle v_{hxx} + av_{hx} + bv_h, \phi_h \rangle_h + \langle cu_{hx} + du_h, \phi_h \rangle_h = \langle f, \phi_h \rangle_h, \phi_h \in S_{h,1}; \qquad (3.10)$$

with the corresponding boundary conditions

$$u_h(0) = 0, u_h(1) = 0, v_h(0) = 0, v_h(1) = 0. \qquad (3.11)$$

Using the cubic B-splines defined in Section 2.1, the mixed Petrov-Galerkin formulation leads to a coupled system of linear equations of size *(2n+2)* $\times$ *(2n+6)*. Combining the four equations related to the boundary conditions of u and v with the coupled system, we obtain a square system of size *(2n+6)* $\times$ *(2n+6)* as done in Chapter 2, which can be solved directly .
The presence of the term u_x in the general fourth order problem makes the estimation of error more difficult than in the case of the analysis of fourth order problem in divergence form. Here, we study the effect of quadrature rule in the

error analysis. Since we compute the approximations for the solution $u(x)$ as well as for its second derivative $v(x)$ with integrals replaced by Gauss quadrature rule in the formulation, this work may be considered as a quadrature based mixed Petrov-Galerkin method.

3.2 ASSUMPTIONS AND REGULARITY

We assume that the coefficients a, b, c and d are such that

$$a, b, c, d \in C^4(\bar{I}), \tag{3.12}$$

where $\bar{I}$ is the closure of $I = (0, 1)$. Let us use the following notations:

$$Lv := v_{xx} + av_x + bv \text{ and } Mu := u_{xx}. \tag{3.13}$$

Moreover, we assume that the operators L, M and the boundary conditions ensure the existence and uniqueness of the solutions u and v for a given sufficiently smooth function $f(x)$. We make a stronger assumption as in (Sloan *et al.* 1993) that for arbitrary $p \in [1, \infty]$, there exist a positive constant C_1 and C_2 such that

$$\|Lv\|_{0,p} \geq C_1 \|v\|_{2,q}\,, \tag{3.14}$$

$$\|Mu\|_{0,p} \geq C_2 \|u\|_{2,q}\,. \tag{3.15}$$

The adjoint operators L^* and M^* with corresponding adjoint boundary conditions are defined respectively as follows:

$$L^*\phi = \phi_{xx} - (a\phi)_x + b\phi, \tag{3.16}$$

$$\phi(0) = \phi(1) = 0 \tag{3.17}$$

and

$$M^*\phi = \phi_{xx}, \tag{3.18}$$

$$\phi(0) = \phi(1) = 0. \tag{3.19}$$

It can be proved as in (Sloan *et al.* 1993) that the above adjoint operators satisfy the following regularity:

Lemma 3.2.1 *Assume that (3.12), (3.14) and (3.15) hold. Then, there exists* $C > 0$ *such that*

$$\|L^*\phi\|_{0,q} \geq C\,\|\phi\|_{2,q}$$

for all ϕ *satisfying the adjoint boundary conditions (3.17) and*

$$\|M^*\phi\|_{0,q} \geq C\,\|\phi\|_{2,q}$$

for all ϕ *satisfying the adjoint boundary conditions (3.19).*

3.3 CONVERGENCE ANALYSIS

Let us denote the error between u and u_h by ε_h and that between v and v_h by e_h, *i.e.*, $\varepsilon_h = u - u_h$ and $e_h = v - v_h$. Using (3.10) and (3.4), we obtain the following error equations:

$$\begin{aligned}
\langle e_{hxx}, \phi_h \rangle_h &= \langle v_{xx}, \phi_h \rangle_h - \langle f - av_{hx} - bv_h - cu_{hx} - du_h, \phi_h \rangle_h \\
&= -\langle a(v - v_h)_x + b(v - v_h), \phi_h \rangle_h \\
&- \langle c(u - u_h)_x + d(u - u_h), \phi_h \rangle_h \\
&= -\langle ae_{hx} + be_h, \phi_h \rangle_h - \langle c\varepsilon_{hx} + d\varepsilon_h, \phi_h \rangle_h;
\end{aligned}$$

and therefore, using (3.13), we have

$$\langle Le_h, \phi_h \rangle_h = -\langle c\varepsilon_{hx} + d\varepsilon_h, \phi_h \rangle_h, \;\; \phi_h \in S_{h,1}. \tag{3.20}$$

Further, using (3.9) and (3.3), we obtain

$$\langle \varepsilon_{hxx}, \phi_h \rangle_h = \langle u_{xx} - u_{hxx}, \phi_h \rangle_h = \langle v - v_h, \phi_h \rangle_h = \langle e_h, \phi_h \rangle_h;$$

and therefore

$$\langle \varepsilon_{hxx}, \phi_h \rangle_h = \langle e_h, \phi_h \rangle_h, \;\; \phi_h \in S_{h,1}. \tag{3.21}$$

The following two lemmas give estimates for the error in the quadrature rule for the terms $(Le_h)\chi_h$ and $(M\varepsilon_h)\chi_h$ for $\chi_h \in S_{h,1}$. These estimates are required for our error analysis later. The proof of the lemma is similar to the proof of the Lemma 4.2 of (Sloan *et al.* 1993).

Lemma 3.3.1 *For all $\chi_h \in S_{h,1}$ and h sufficiently small*
(a) $E_h((Le_h)\chi_h) \leq C\left[h^4 \|v\|_{6,p} + h\|e_h\|_{0,p}\right] \|\chi_h\|_{1,q}$;
(b) $E_h((Le_h)\chi_h) \leq C\left[h^3 \|v\|_{6,p} + \|e_h\|_{0,p}\right] \|\chi_h\|_{0,q}$.

Proof: (a) From (1.12) we have that,

$$E_h((Le_h)\chi_h) \leq C\sum_{k=1}^{n} h_k^4 \left\|[(Le_h)\chi_h]^{(4)}\right\|_{0,1,k}.$$

Now, using Leibnitz's rule for derivative of product of functions, triangle inequality and the fact that χ_h belongs to $S_{h,1}$, we obtain

$$\begin{aligned}
\left\|[(Le_h)\chi_h]^{(4)}\right\|_{0,1,k} &= \left\|[(e_{hxx} + ae_{hx} + be_h)\chi_h]^{(4)}\right\|_{0,1,k} \\
&\leq C\left[\left\|(e_{hxx} + ae_{hx} + be_h)^{(4)}\right\|_{0,p,k} \|\chi_h\|_{0,q,k}\right. \\
&+ \left.\left\|(e_{hxx} + ae_{hx} + be_h)^{(3)}\right\|_{0,p,k} \left\|\chi_h^{(1)}\right\|_{0,q,k}\right] \\
&\leq C\left[\|v\|_{6,p,k} \|\chi_h\|_{0,q,k} + \|e_h\|_{3,p,k} \| \, \|\chi_h\|_{1,q,k}\right] \\
&\leq C\left[\|v\|_{6,p,k} \|\chi_h\|_{0,q,k} + (h_k^{-3} \|e_h\|_{0,p,k} + h_k^3 \|v\|_{6,p,k}) \|\chi_h\|_{1,q,k}\right] \\
&\leq C\left[\|v\|_{6,p,k} + h_k^{-3} \|e_h\|_{0,p,k} + h_k^3 \|v\|_{6,p,k}\right] \|\chi_h\|_{1,q,k};
\end{aligned}$$

where we have used (1.17) for $\|e_h\|_{3,p,k}$ with $i = 3, m = 6$.
Therefore we have,

$$\begin{aligned}
E_h((Le_h)\chi_h) &\leq C\sum_{k=1}^{n} h_k^4 \left[\|v\|_{6,p,k} + h_k^{-3} \|e_h\|_{0,p,k} + h_k^3 \|v\|_{6,p,k}\right] \|\chi_h\|_{1,q,k}, \\
E_h((Le_h)\chi_h) &\leq C\sum_{k=1}^{n} \left[h_k^4 \|v\|_{6,p,k} + h_k \|e_h\|_{0,p,k}\right] \|\chi_h\|_{1,q,k}. \qquad (3.22)
\end{aligned}$$

Using Hölder's inequality for sums, we get

$$E_h((Le_h)\chi_h) \leq C\left[h^4 \|v\|_{6,p} + h\|e_h\|_{0,p}\right] \|\chi_h\|_{1,q},$$

which completes the proof of (a).
For (b), using inverse inequality $\|\chi_h\|_{1,q,k} \leq Ch_k^{-1} \|\chi_h\|_{0,q,k}$ in (3.22), we obtain

$$E_h((Le_h)\chi_h) \leq C\sum_{k=1}^{n} \left[h_k^4 \|v\|_{6,p,k} + h_k \|e_h\|_{0,p,k}\right] h_k^{-1} \|\chi_h\|_{0,q,k}.$$

Using Hölder's inequality for sums in the above, we obtain

$$E_h((Le_h)\chi_h) \leq C\left[h^3\|v\|_{6,p} + \|e_h\|_{0,p}\right]\|\chi_h\|_{0,q}.$$

Hence the proof of (b) is completed. ■

The proof of the following lemma is similar to the proof of Lemma 2.3.1 in Chapter 2.

Lemma 3.3.2 *For all $\chi_h \in S_{h,1}$ and h sufficiently small*
(a) $E_h(\varepsilon_{hxx}\chi_h) \leq Ch^4\|u\|_{6,p}\|\chi_h\|_{1,q}$;
(b) $E_h(\varepsilon_{hxx}\chi_h) \leq Ch^3\|u\|_{6,p}\|\chi_h\|_{0,q}$.

The following three results give estimates for $\varepsilon_h(\overline{x})$, $\varepsilon_{hx}(\overline{x})$ and $e_h(\overline{x})$, where $\overline{x}$ is any arbitrary point in I. These estimates are crucial for our error analysis. Following result can be obtained as in Lemma 2.3.2 of Chapter 2.

Lemma 3.3.3 *Let u be the weak solution of* (3.3) *defined through* (3.5). *Further, let u_h be the corresponding discrete Petrov-Galerkin solution defined through* (3.9). *Then, the error $\varepsilon_h = u - u_h$ satisfies*

$$|\varepsilon_h(\overline{x})| \leq C\left[h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p} + h\|e_h\|_{1,p}\right],$$

where $\overline{x}$ is an arbitrary point in $[0,1]$.

We have a similar result for $|\varepsilon_{hx}(\overline{x})|$ given below, which is needed for the error analysis.

Lemma 3.3.4 *Let u be the weak solution of* (3.3) *defined through* (3.5). *Further, let u_h be the corresponding discrete Petrov-Galerkin solution defined through* (3.9). *Then the error $\varepsilon_h = u - u_h$ satisfies*

$$|\varepsilon_{hx}(\overline{x})| \leq C\left[h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p} + h\|e_h\|_{1,p}\right],$$

where $\overline{x}$ is an arbitrary point in $[0,1]$.

Proof: For a given $\overline{x} \in [0,1]$, let Φ be an element of $L_p(I) \bigcap C(I)$, $1 \leq p \leq \infty$ satisfying the following auxiliary problem:

$$\begin{aligned} \Phi_{xxx} &= 0,\ x \in I - \{\overline{x}\}, \\ \Phi_x(0) &= \Phi_x(1) = 0,\ \ \Phi_x^-(\overline{x}) - \Phi_x^+(\overline{x}) = 1, \\ \Phi_{xx}^-(\overline{x}) &- \ \Phi_{xx}^+(\overline{x}) = -1. \end{aligned}$$

The above problem has a solution. For example,

$$\Phi(x) = \begin{cases} (\overline{x}x^2)/2, & 0 \leq x \leq \overline{x}, \\ (\overline{x}+1)(\frac{x^2}{2} - x) + (2\overline{x} + \overline{x}^2)/2, & \overline{x} \leq x \leq 1. \end{cases}$$

satisfies the above differential equation and the boundary conditions.
Let us define Ψ as follows:

$$\Psi(x) = \begin{cases} \Phi_{xxx}, & x \in I - \{\overline{x}\}, \\ 0, & x = \overline{x}. \end{cases}$$

Then $\Psi = 0$ *a.e.* on I. We first multiply ε_h with Ψ and then integrate over the interval I. Proceeding as in the proof of Lemma 2.3.2, on applying integration by parts twice, using the fact that $\varepsilon_h(0) = \varepsilon_h(1) = 0$ and then using the given boundary condition and the jump conditions, we obtain

$$\begin{aligned} 0 &= (\varepsilon_h, \Psi) = \varepsilon_h(\overline{x}) \left[\Phi_{xx}^-(\overline{x}) - \Phi_{xx}^+(\overline{x})\right] - \int_0^{\overline{x}} \varepsilon_{hx}\Phi_{xx} - \int_{\overline{x}}^1 \varepsilon_{hx}\Phi_{xx} \\ &= -\varepsilon_h(\overline{x}) - \varepsilon_{hx}(\overline{x}) \left[\Phi_x^-(\overline{x}) - \Phi_x^+(\overline{x})\right] + (\varepsilon_{hxx}, \chi_x) \\ &= -\varepsilon_h(\overline{x}) - \varepsilon_{hx}(\overline{x}) + (\varepsilon_{hxx}, \chi_x)\,, \end{aligned}$$

where

$$\chi_x = \begin{cases} \Phi_x, & 0 \leq x < \overline{x}, \\ \Phi_x, & \overline{x} < x \leq 1, \\ x\overline{x}, & x = \overline{x}. \end{cases}$$

Since Φ_x is not defined at $\overline{x}$, a new function χ_x is introduced in the above manner in order to have a meaning for $(\varepsilon_{hxx}, \chi_x)$. Then, the above equation becomes

$$\begin{aligned} \varepsilon_{hx}(\overline{x}) &= -\varepsilon_h(\overline{x}) + (\varepsilon_{hxx}, \chi_x)\,, \\ i.e., |\varepsilon_{hx}(\overline{x})| &\leq |\varepsilon_h(\overline{x})| + |\,(\varepsilon_{hxx}, \chi_x)\,|. \end{aligned} \tag{3.23}$$

The estimate of the first term on the right hand side of (3.23) is given by Lemma 3.3.3. Now, we compute the estimate for the second term. Let χ_h be the linear interpolant of χ_x.

$$\begin{aligned}(\varepsilon_{hxx}, \chi_x) &= (\varepsilon_{hxx}, \chi_x - \chi_h) + (\varepsilon_{hxx}, \chi_h) - \langle \varepsilon_{hxx}, \chi_h \rangle_h + \langle \varepsilon_{hxx}, \chi_h \rangle_h . \\ |\,(\varepsilon_{hxx}, \chi_x)\,| &\leq |\,(\varepsilon_{hxx}, \chi_x - \chi_h)\,| + |E_h\,(\varepsilon_{hxx}\chi_h)\,| + |\,\langle \varepsilon_{hxx}, \chi_h \rangle_h\,| \\ &\leq T_{3,1} + T_{3,2} + T_{3,3}. \qquad (3.24)\end{aligned}$$

Using the fact that $\chi_x = \Phi_x$ *a.e.* on I and the definition of the auxiliary problem, it is easy to verify that $\|\chi_x\|_{2,q} \leq \|\chi\|_{2,q}$.
Therefore, we have that

$$\|\chi_h\|_{1,q} \leq \|\chi_x - \chi_h\|_{1,q} + \|\chi_x\|_{2,q} \leq C\,\|\chi_x\|_{2,q} \leq C\,\|\chi\|_{2,q} . \qquad (3.25)$$

We now compute estimates for the terms $T_{3,1}$, $T_{3,2}$ and $T_{3,3}$ as follows:

$$\begin{aligned}T_{3,1} = |(\varepsilon_{hxx}, \chi_x - \chi_h)| \leq \|\varepsilon_{hxx}\|_{0,p} \|\|\chi_x - \chi_h\|_{0,q} &\leq Ch^2\,\|\varepsilon_h\|_{2,p}\,\|\chi_x\|_{2,q} \\ &\leq Ch^2\,\|\varepsilon_h\|_{2,p}\,\|\chi\|_{2,q} .\end{aligned}$$

Using Lemma 3.3.2a and (3.25), we have

$$T_{3,2} = |E_h\,(\varepsilon_{hxx}\chi_h)\,| \leq Ch^4\,\|u\|_{6,p}\,\|\chi\|_{2,q} .$$

Using (3.21), (1.11) and Sobolev embedding theorem (1.13) locally on I_k for both $\|e_h\|_{0,\infty,k}$ and $\|\chi_h\|_{0,\infty,k}$, we have

$$\begin{aligned}T_{3,3} &= |\langle \varepsilon_{hxx}, \chi_h \rangle_h| = |\langle e_h, \chi_h \rangle_h| \leq C \sum_{k=1}^{n} \frac{h_k}{2}\,\|e_h\|_{0,\infty,k}\,\|\chi_h\|_{0,\infty,k} \\ &\leq C \sum_{k=0}^{n} \frac{h_k}{2}\,\|e_h\|_{1,p,k}\,\|\chi_h\|_{1,q,k} .\end{aligned}$$

Using Hölder's inequality for sums and (3.25), we have

$$T_{3,3} \leq Ch\,\|e_h\|_{1,p}\,\|\chi_h\|_{1,q} \leq Ch\,\|e_h\|_{1,p}\,\|\chi\|_{2,q} .$$

For Φ satisfying the auxiliary problem, it is easy to verify that $\|\chi\|_{2,q} = \|\Phi\|_{2,q} \leq K$, where K is a constant not depending on h. Using $T_{3,1}$, $T_{3,2}$ and $T_{3,3}$ in (3.24), we have that

$$|\,(\varepsilon_{hxx}, \chi_x)\,| \leq C\left[h^2\,\|\varepsilon_h\|_{2,p} + h^4\,\|u\|_{6,p} + h\,\|e_h\|_{1,p}\right] .$$

Using the above in (3.23) and using Lemma 3.3.3, we obtain

$$\text{Hence } |\varepsilon_{hx}(\overline{x}| \leq C\left[(h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p} + h\|e_h\|_{1,p})\right]$$

and this completes the proof. ■

In a similar manner, we present below the following third estimate $|e_h(\overline{x})|$ needed for the error analysis.

Lemma 3.3.5 *Let v be the weak solution of* (3.4) *defined through* (3.6). *Further, let v_h be the corresponding discrete Petrov-Galerkin solution defined through* (3.10). *Then the error $e_h = v - v_h$ satisfies*

$$|e_h(\overline{x})| \leq C\left[h\|e_h\|_{2,p} + h^4\|v\|_{5,p}\right],$$

where $\overline{x}$ is an arbitrary point in $[0,1]$.

Proof: For a given $\overline{x} \in [0,1]$, let Φ be an element of $L_p(I)\bigcap C(I)$, $1 \leq p \leq \infty$ satisfying the following auxiliary problem:

$$\begin{aligned}\Phi_{xx} &= 0,\ x \in I - \{\overline{x}\},\\ \Phi_x^-(\overline{x}) &- \Phi_x^+(\overline{x}) = -1.\end{aligned}$$

The example given in the proof of Lemma 2.3.2 is a solution of the above differential equation. Let us define Ψ as follows:

$$\Psi(x) = \begin{cases}\Phi_{xx}, & x \in I - \{\overline{x}\},\\ 0, & x = \overline{x}.\end{cases}$$

Then, $\Psi = 0$ *a.e.* on I. Proceeding as in the proof of Lemma 2.3.2, on applying integration by parts once, using the fact that $e_h(0) = e_h(1) = 0$ and then using the given jump condition,we obtain

$$\begin{aligned}0 &= (e_h,\Psi) = e_h(\overline{x})\left[\Phi_x^-(\overline{x}) - \Phi_x^+(\overline{x})\right] - \int_0^{\overline{x}} e_{hx}\Phi_x - \int_{\overline{x}}^1 e_{hx}\Phi_x\\ &= -e_h(\overline{x}) + (e_{hx},\Phi_x).\end{aligned}$$

Let χ_h be the linear interpolant of Φ_x.

$$\begin{aligned}e_h(\overline{x}) &= (e_{hx},\Phi_x) = (e_{hx},\Phi_x - \chi_h) + (e_{hx},\chi_h) - \langle e_{hx},\chi_h\rangle_h + \langle e_{hx},\chi_h\rangle_h\\ |e_h(\overline{x})| &\leq |(e_{hx},\Phi_x - \chi_h)| + |E_h(e_{hx}\chi_h)| + |\langle e_{hx},\chi_h\rangle_h|\\ &\leq T_{3,4} + T_{3,5} + T_{3,6}.\end{aligned} \tag{3.26}$$

We shall compute the estimates for the terms $T_{3,4}$, $T_{3,5}$ and $T_{3,6}$ as follows:

$$\begin{aligned} T_{3,4} = |(e_{hx}, \Phi_x - \chi_h)| \leq \|e_{hx}\|_{0,p} \|\|\Phi_x - \chi_h\|_{0,q} &\leq Ch^2 \|e_h\|_{1,p} \|\Phi_x\|_{2,q} \\ &\leq Ch^2 \|e_h\|_{1,p} \|\Phi\|_{2,q} \end{aligned}$$

as $\|\Phi_{xxx}\| = 0$.

We also have that

$$\|\chi_h\|_{1,q} \leq \|\Phi_x - \chi_h\|_{1,q} + \|\Phi_x\|_{2,q} \leq C \|\Phi_x\|_{2,q} \leq C \|\Phi\|_{2,q} \tag{3.27}$$

as $\|\Phi_{xxx}\| = 0$.

Using (1.12), Hölder's inequality for sums and (3.27), we obtain

$$\begin{aligned} T_{3,5} = |E_h (e_{hx}\chi_h)| &\leq C \sum_{k=1}^{n} h_k^4 \left\| [e_{hx}\chi_h]^{(4)} \right\|_{L^1(I_k)} \leq Ch^4 \|v\|_{5,p} \|\chi_h\|_{1,q} \\ &\leq Ch^4 \|v\|_{5,p} \|\Phi\|_{2,q} . \end{aligned}$$

Using (1.11) and Sobolev embedding theorem (1.13) locally on I_k for both $\|e_{hx}\|_{0,\infty,k}$ and $\|\chi_h\|_{0,\infty,k}$, we have

$$\begin{aligned} T_{3,6} &= |\langle e_{hx}, \chi_h \rangle_h| \leq C \sum_{k=1}^{n} \frac{h_k}{2} \|e_{hx}\|_{0,\infty,k} \|\chi_h\|_{0,\infty,k} \\ &\leq C \sum_{k=1}^{n} \frac{h_k}{2} \|e_{hx}\|_{1,p,k} \|\chi_h\|_{1,q,k} . \end{aligned}$$

Using Hölder's inequality for sums and (3.27), we have

$$T_{3,6} \leq Ch \|e_{hx}\|_{1,p} \|\chi_h\|_{1,q} \leq Ch \|e_h\|_{2,p} \|\Phi\|_{2,q} .$$

For Φ satisfying the auxiliary problem, it is easy to verify that $\|\Phi\|_{2,q} \leq K$, where K is a constant not depending on h. Using $T_{3,4}$, $T_{3,5}$ and $T_{3,6}$ in (3.26), we have

$$|e_h(\overline{x})| \leq C \left[h \|e_h\|_{2,p} + h^4 \|v\|_{5,p} \right] .$$

This completes the proof. ■

In the following lemma, we initially compute the error $(v - v_h)$ in terms of $(u - u_h)$ and then we establish an optimal estimate of error $(v - v_h)$ independent of $(u - u_h)$.

Lemma 3.3.6 *Let u and v be the weak solutions of the coupled equations* (3.3) *and* (3.4) *defined through* (3.5) *and* (3.6) *respectively. Further, let u_h and v_h be the corresponding discrete Petrov-Galerkin solutions defined through* (3.9) *and* (3.10) *respectively. Then the estimates of the errors $e_h = v - v_h$ in L_p, W_p^1 and W_p^2 norms, where $1 \le p \le \infty$ are given as follows:*

$$\begin{aligned}
\|e_h\|_{0,p} &\le C\left[h^4\|v\|_{6,p} + h^5\|u\|_{6,p} + h^3\|\varepsilon_h\|_{2,p}\right];\\
\|e_h\|_{1,p} &\le C\left[h^3\|v\|_{6,p} + h^4\|u\|_{6,p} + h^2\|\varepsilon_h\|_{2,p}\right];\\
\|e_h\|_{2,p} &\le C\left[h^2\|v\|_{6,p} + h^4\|u\|_{6,p} + h^2\|\varepsilon_h\|_{2,p}\right].
\end{aligned}$$

Proof: Let η be an arbitrary element of L_q and $\phi \in W_q^2$ be the solution of the auxiliary problem

$$\begin{aligned}
L^*\phi &= \eta,\\
\phi(0) &= \phi(1) = 0,
\end{aligned}$$

where $L^*\phi = \phi_{xx} - (a\phi)_x + b\phi$ as defined in (3.16).
We now have

$$\begin{aligned}
(e_h, \eta) &= (e_h, L^*\phi) = (Le_h, \phi) = (Le_h, \phi - \phi_h) + (Le_h, \phi_h)\\
&= (Le_h, \phi - \phi_h) + (Le_h, \phi_h) - \langle Le_h, \phi_h\rangle_h + \langle Le_h, \phi_h\rangle_h\\
|(e_h, \eta)| &\le |(Le_h, \phi - \phi_h)| + |E_h((Le_h)\phi_h)| + |\langle Le_h, \phi_h\rangle_h|\\
&\le T_{3,7} + T_{3,8} + T_{3,9}, \qquad (3.28)
\end{aligned}$$

where $\phi_h \in S_{h,1}$ is the linear interpolant of ϕ.
We know that

$$\|\phi_h\|_{1,q} \le \|\phi - \phi_h\|_{1,q} + \|\phi\|_{1,q} \le Ch\|\phi\|_{2,q} + \|\phi\|_{2,q} \le C\|\phi\|_{2,q}. \qquad (3.29)$$

We compute the estimates for the terms $T_{3,7}$, $T_{3,8}$ and $T_{3,9}$ as follows:

$$\begin{aligned}
T_{3,7} &= |(Le_h, \phi - \phi_h)| \le \|Le_h\|_{0,p}\|\phi - \phi_h\|_{0,q} \le Ch^2\|e_h\|_{2,p}\|\phi\|_{2,q}.\\
T_{3,8} &= |E_h((Le_h)\phi_h)| \le C\left[h^4\|v\|_{6,p} + h\|e_h\|_{0,p}\right]\|\phi_h\|_{1,q}\\
&\le C\left[h^4\|v\|_{6,p} + h\|e_h\|_{0,p}\right]\|\phi\|_{2,q}
\end{aligned}$$

by Lemma 3.3.1a and (3.29). Using (3.20), (1.11) and Sobolev embedding theorem (1.13) locally on I_k for $\|\phi_h\|_{0,\infty,k}$, we have

$$\begin{aligned} T_{3,9} &= |\langle Le_h, \phi_h\rangle_h| = |-\langle c\varepsilon_{hx} + d\varepsilon_h, \phi_h\rangle_h| \\ &\le C\left[\sum_{k=1}^{n} \frac{h_k}{2}\|\varepsilon_h\|_{0,\infty,k} + \sum_{k=0}^{n} \frac{h_k}{2}\|\varepsilon_{hx}\|_{0,\infty,k}\right]\|\phi_h\|_{0,\infty,k} \\ &\le C\left[\sum_{k=1}^{n} \frac{h_k}{2}\|\varepsilon_h\|_{0,\infty,k} + \sum_{k=1}^{n} \frac{h_k}{2}\|\varepsilon_{hx}\|_{0,\infty,k}\right]\|\phi_h\|_{1,q,k}. \end{aligned}$$

Using Hölder's inequality for sums, Lemma 3.3.3, Lemma 3.3.4 and (3.29), we obtain

$$\begin{aligned} T_{3,9} &\le Ch\left[\|\varepsilon_h\|_{0,\infty} + \|\varepsilon_{hx}\|_{0,\infty}\right]\|\phi_h\|_{1,q} \\ &\le Ch\left[h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p} + h\|e_h\|_{1,p}\right]\|\phi_h\|_{1,q} \\ &\le C\left[h^3\|\varepsilon_h\|_{2,p} + h^5\|u\|_{6,p} + h^2\|e_h\|_{1,p}\right]\|\phi\|_{2,q}. \end{aligned}$$

Substituting $T_{3,7}$, $T_{3,8}$ and $T_{3,9}$ in (3.28), we have

$$\begin{aligned} |(e_h, \eta)| &\le C\left[h^2\|e_h\|_{2,p} + h^4\|v\|_{6,p} + h\|e_h\|_{0,p} + h^3\|\varepsilon_h\|_{2,p}\right. \\ &+ \left. h^5\|u\|_{6,p} + h^2\|e_h\|_{1,p}\right]\|\phi\|_{2,q}. \end{aligned}$$

Using Lemma 3.2.1 and regularity of the auxiliary problem, we have $\|\phi\|_{2,q} \le \|\eta\|_{0,q}$. Since $\eta \in L_q$ is arbitrary, we have

$$\begin{aligned} \|e_h\|_{0,p} &\le C\left(h^2\|e_h\|_{2,p} + h^3\|\varepsilon_h\|_{2,p} + h^4\|v\|_{6,p} + h^5\|u\|_{6,p}\right. \\ &+ \left. h^2\|e_h\|_{1,p} + h\|e_h\|_{0,p}\right). \end{aligned}$$

For h sufficiently small, we obtain

$$\|e_h\|_{0,p} \le C\left(h^2\|e_h\|_{2,p} + h^4\|v\|_{6,p} + h^5\|u\|_{6,p} + h^3\|\varepsilon_h\|_{2,p}\right). \tag{3.30}$$

We now estimate $\|e_{hxx}\|_{0,p}$ *via* a projection argument. Let P_h be the orthogonal projection onto $S_{h,1}$ with respect to L_2 inner product defined by

$$(v_{xx} - P_h v_{xx}, \psi_h) = 0, \qquad \forall \psi_h \in S_{h,1}. \tag{3.31}$$

The domain of P_h may be taken as L_1. Referring to Crouzeix and Thomee (1987) and de Boor (1976), it is seen that the L_2 projection is stable in L_p, where $1 \leq p \leq \infty$. Thus

$$\|P_h v\|_{0,p} \leq C\|v\|_{0,p}. \tag{3.32}$$

Then the error e_{hxx} can be interpreted in terms of the error of the above projection as follows:

$$\|e_{hxx}\|_{0,p} = \|v_{xx} - v_{hxx}\|_{0,p} \leq \|v_{xx} - P_h v_{xx}\|_{0,p} + \|P_h v_{xx} - v_{hxx}\|_{0,p}. \tag{3.33}$$

From the stability property (3.32), the error in the projection follows as in (de Boor 1976),

$$i.e., \;\; \|v_{xx} - P_h v_{xx}\|_{0,p} \leq Ch^2 \|v_{xx}\|_{2,p} \leq Ch^2 \|v\|_{4,p}. \tag{3.34}$$

Then the remaining task is to compute the estimate of $\|P_h v_{xx} - v_{hxx}\|_{0,p}$.
For $\psi_h \in S_{h,1}$,

$$\begin{aligned}
(P_h v_{xx} - v_{hxx}, \psi_h) &= (P_h v_{xx} - v_{xx} + v_{xx} - v_{hxx}, \psi_h) \\
&= (P_h v_{xx} - v_{xx}, \psi_h) + (v_{xx} - v_{hxx}, \psi_h) \\
&= (v_{xx} - v_{hxx}, \psi_h) \text{ using (3.31)} \\
(P_h v_{xx} - v_{hxx}, \psi_h) &= (e_{hxx}, \psi_h) \\
&= (Le_h - ae_{hx} - be_h, \psi_h) \text{ using (3.13)} \\
&= (-ae_{hx} - be_h, \psi_h) + (Le_h, \psi_h) - \langle Le_h, \psi_h \rangle_h \\
&+ \langle Le_h, \psi_h \rangle_h \\
|(P_h v_{xx} - v_{hxx}, \psi_h)| &\leq |(-ae_{hx} - be_h, \psi_h)| + |E_h\left((Le_h)\psi_h\right)| + |\langle Le_h, \psi_h \rangle_h| \\
&\leq T_{3,10} + T_{3,11} + T_{3,12}.
\end{aligned} \tag{3.35}$$

We compute the estimates for the terms $T_{3,10}$, $T_{3,11}$ and $T_{3,12}$ as follows:

$$\begin{aligned}
T_{3,10} &= |(-ae_{hx} - be_h, \psi_h)| \leq C\|e_h\|_{1,p}\|\psi_h\|_{0,q}, \\
T_{3,11} &= |E_h\left((Le_h)\psi_h\right)| \leq C\left[h^3\|v\|_{6,p} + \|e_h\|_{0,p}\right]\|\psi_h\|_{0,q},
\end{aligned}$$

where we have used Lemma 3.3.1b.

Following the steps of computation involved in term $T_{3,9}$, we hence obtain the estimate of $T_{3,12}$ as

$$T_{3,12} \leq C\left[h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p} + h\|e_h\|_{1,p}\right]\|\psi_h\|_{0,q},$$

where we have used the inverse inequality $\|\phi_h\|_{1,q,k} \leq h_k^{-1}\|\phi_h\|_{0,q,k}$ locally.
Using $T_{3,10}$, $T_{3,11}$ and $T_{3,12}$ in (3.35), we get

$$\begin{aligned} |(P_h v_{xx} - v_{hxx}, \psi_h)| &\leq C\Big[\|e_h\|_{1,p} + h^3\|v\|_{6,p} \\ &+ h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p}\Big]\|\psi_h\|_{0,q}. \end{aligned} \tag{3.36}$$

We now show the above inequality for $\eta \in L_q$ to obtain $\|P_h v_{xx} - v_{hxx}\|_{0,p}$. Now let η be an arbitrary element of L_q. Then, since $v_{hxx} \in S_{h,1}$, it follows from the definition of $P_h\eta$, (3.36) and (3.32) with p replaced by q, that

$$\begin{aligned} 0 &= (P_h v_{xx} - v_{hxx}, \eta - P_h\eta), \\ i.e.\ & |(P_h v_{xx} - v_{hxx}, \eta)| = |(P_h v_{xx} - v_{hxx}, P_h\eta)| \\ &\leq C\left[\|e_h\|_{1,p} + h^3\|v\|_{6,p} + h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p}\right]\|P_h\eta\|_{0,q} \\ &\leq C\left[\|e_h\|_{1,p} + h^3\|v\|_{6,p} + h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p}\right]\|\eta\|_{0,q}. \end{aligned}$$

Since η is an arbitrary element of L_q, we have that

$$\|P_h v_{xx} - v_{hxx}\|_{0,p} \leq C\left[\|e_h\|_{1,p} + h^3\|v\|_{6,p} + h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p}\right]. \tag{3.37}$$

Using (3.34) and (3.37) in (3.33), we obtain that,

$$\|e_{hxx}\|_{0,p} \leq C\left[\|e_h\|_{1,p} + h^2\|v\|_{6,p} + h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p}\right].$$

Now, using the fact that $\|e_h\|_{2,p} \leq \|e_h\|_{1,p} + \|e_{hxx}\|_{0,p}$, we have

$$\|e_h\|_{2,p} \leq C\left[\|e_h\|_{1,p} + h^2\|v\|_{6,p} + h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p}\right].$$

Now, using (1.17) with $m = 2,\ i = 1$, we have

$$\|e_h\|_{1,p} \leq C(h^{-1}\|e_h\|_{0,p} + h\|e_h\|_{2,p}). \tag{3.38}$$

Then, we have

$$\|e_h\|_{2,p} \leq C\left[(h^{-1}\|e_h\|_{0,p} + h\|e_h\|_{2,p}) + h^2\|v\|_{6,p} + h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p}\right].$$

For sufficiently small h, we have

$$\|e_h\|_{2,p} \leq C\left[h^{-1}\|e_h\|_{0,p} + h^2\|v\|_{6,p} + h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p}\right]. \tag{3.39}$$

Using (3.39) in (3.30), we obtain

$$\begin{aligned}\|e_h\|_{0,p} &\leq C\Big[h^2(h^{-1}\|e_h\|_{0,p} + h^2\|v\|_{6,p} + h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p}) \\ &+ h^4\|v\|_{6,p} + h^5\|u\|_{6,p} + h^3\|\varepsilon_h\|_{2,p}\Big].\end{aligned}$$

Therefore for sufficiently small h, we have

$$\|e_h\|_{0,p} \leq C\left[h^4\|v\|_{6,p} + h^5\|u\|_{6,p} + h^3\|\varepsilon_h\|_{2,p}\right]. \tag{3.40}$$

Using (3.40) in (3.39), we have

$$\begin{aligned}\|e_h\|_{2,p} &\leq C\Big[h^{-1}(h^4\|v\|_{6,p} + h^5\|u\|_{6,p} + h^3\|\varepsilon_h\|_{2,p}) + h^2\|v\|_{6,p} \\ &+ h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p}\Big], \\ \|e_h\|_{2,p} &\leq C\left[h^2\|v\|_{6,p} + h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p}\right]. \end{aligned} \tag{3.41}$$

Using (3.40) and (3.41) in (3.38), we have

$$\begin{aligned}\|e_h\|_{1,p} &\leq C\Big[h^{-1}(h^4\|v\|_{6,p} + h^5\|u\|_{6,p} + h^3\|\varepsilon_h\|_{2,p}) \\ &+ h(h^2\|v\|_{6,p} + h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p})\Big], \\ \|e_h\|_{1,p} &\leq C\left(h^3\|v\|_{6,p} + h^4\|u\|_{6,p} + h^2\|\varepsilon_h\|_{2,p}\right). \end{aligned} \tag{3.42}$$

(3.40), (3.41) and (3.42) give the required result. ■

We now compute the error estimate of ε_h as has been done in the previous case.

Lemma 3.3.7 *Let u and v be the weak solutions of the coupled equations* (3.3) *and* (3.4) *defined through* (3.5) *and* (3.6) *respectively. Further, let u_h and v_h be the corresponding discrete Petrov-Galerkin solutions defined through* (3.9) *and* (3.10) *respectively. Then the estimates of the errors $\varepsilon_h = u - u_h$ in L_p, W_p^1 and W_p^2 norms, where $1 \leq p \leq \infty$ are given as follows:*

$$\|\varepsilon_h\|_{0,p} \leq C\left[h^4\|u\|_{6,p} + h^2\|e_h\|_{2,p} + h^5\|v\|_{5,p}\right];$$

$$\|\varepsilon_h\|_{1,p} \leq C\left[h^3\|u\|_{6,p} + h\|e_h\|_{2,p} + h^4\|v\|_{5,p}\right];$$

$$\|\varepsilon_h\|_{2,p} \leq C\left[h^2\|u\|_{6,p} + h\|e_h\|_{2,p} + h^4\|v\|_{5,p}\right].$$

Proof: Let ρ be an arbitrary element of L_q and $\phi \in W_q^2$ be the unique solution of the auxiliary problem

$$\begin{aligned} M^*\phi &= \rho, \\ \phi(0) &= \phi(1) = 0, \end{aligned}$$

where $M^*\phi = \phi_{xx}$ as defined in (3.18).
Then, we have

$$\begin{aligned} (\varepsilon_h, \rho) &= (\varepsilon_h, M^*\phi) = (M\varepsilon_h, \phi) = (\varepsilon_{hxx}, \phi) \\ &= (\varepsilon_{hxx}, \phi - \phi_h) + (\varepsilon_{hxx}, \phi_h) - \langle \varepsilon_{hxx}, \phi_h \rangle_h + \langle \varepsilon_{hxx}, \phi_h \rangle_h , \end{aligned}$$

where $\phi_h \in S_{h,1}$ is a linear interpolant of ϕ.

$$\begin{aligned} |(\varepsilon_h, \rho)| &\leq |(\varepsilon_{hxx}, \phi - \phi_h)| + |E_h\left(\varepsilon_{hxx}\phi_h\right)| + |\langle \varepsilon_{hxx}, \phi_h \rangle_h| \\ &\leq T_{3,13} + T_{3,14} + T_{3,15}. \end{aligned} \tag{3.43}$$

Now, we compute the estimates for the terms $T_{3,13}$, $T_{3,14}$ and $T_{3,15}$ as follows:

$$\begin{aligned} T_{3,13} &= |(\varepsilon_{hxx}, \phi - \phi_h)| \leq \|\varepsilon_{hxx}\|_{0,p} \|\phi - \phi_h\|_{0,q} \leq h^2 \|\varepsilon_h\|_{2,p} \|\phi\|_{2,q} , \\ T_{3,14} &= |E_h\left(\varepsilon_{hxx}\phi_h\right)| \leq Ch^4 \|u\|_{6,p} \|\phi_h\|_{1,q} \leq Ch^4 \|u\|_{6,p} \|\phi\|_{2,q} , \end{aligned}$$

where we have used Lemma 3.3.2a and (3.29) for estimating $T_{3,14}$.
Using (3.21) and (1.11) first, then Sobolev embedding theorem (1.13) locally on I_k for $\|\phi_h\|_{0,\infty,k}$ to estimate $T_{3,15}$, we have

$$\begin{aligned} T_{3,15} = |\langle \varepsilon_{hxx}, \phi_h \rangle_h| = |\langle e_h, \phi_h \rangle_h| &\leq C\sum_{k=1}^{n} \frac{h_k}{2} \|e_h\|_{0,\infty,k} \|\phi_h\|_{0,\infty,k} \\ &\leq C\sum_{k=1}^{n} \frac{h_k}{2} \|e_h\|_{0,\infty,k} \|\phi_h\|_{1,q,k} . \end{aligned}$$

Further, using Hölder's inequality for sums, Lemma 3.3.5 and (3.29), we obtain

$$T_{3,15} \leq Ch \|e_h\|_{0,\infty} \|\phi_h\|_{1,q} \leq C\left[h^2 \|e_h\|_{2,p} + h^5 \|v\|_{5,p}\right] \|\phi\|_{2,q} .$$

Substituting the estimates $T_{3,13}$, $T_{3,14}$ and $T_{3,15}$ in (3.43), we obtain

$$|(\varepsilon_h, \rho)| \leq C\left[h^2 \|\varepsilon_h\|_{2,p} + h^4 \|u\|_{6,p} + h^2 \|e_h\|_{2,p} + h^5 \|v\|_{5,p}\right] \|\phi\|_{2,q} .$$

Using Lemma 3.2.1 and regularity of the auxiliary problem, we have $\|\phi\|_{2,q} \leq \|\rho\|_{0,q}$. Since $\rho \in L_q$ is arbitrary, we have

$$\|\varepsilon_h\|_{0,p} \leq C\left[h^2 \|\varepsilon_h\|_{2,p} + h^4 \|u\|_{6,p} + h^2 \|e_h\|_{2,p} + h^5 \|v\|_{5,p}\right]. \tag{3.44}$$

We now estimate$\|\varepsilon_{hxx}\|_{0,p}$ *via* a projection argument as has been discussed in the earlier case.

Let P_h be the orthogonal projection onto $\mathrm{S}_{h,1}$ with respect to L_2 inner product defined by

$$(u_{xx} - P_h u_{xx}, \psi_h) = 0 \qquad \forall \psi_h \in S_{h,1}. \tag{3.45}$$

The domain of P_h may be taken to be L_1. From (Crouzeix and Thomee 1987) and (de Boor 1976), it is seen that the L_2 projection is stable L_p, where $1 \leq p \leq \infty$. Thus

$$\|P_h u\|_{0,p} \leq C\|u\|_{0,p}. \tag{3.46}$$

Then the error ε_{hxx} can be interpreted in terms of the error of the above projection.

$$\|\varepsilon_{hxx}\|_{0,p} = \|u_{xx} - u_{hxx}\|_{0,p} \leq \|u_{xx} - P_h u_{xx}\|_{0,p} + \|P_h u_{xx} - u_{hxx}\|_{0,p}. \tag{3.47}$$

From the stability property (3.46), the error in the projection follows as in (de Boor 1976),

$$i.e., \ \|u_{xx} - P_h u_{xx}\|_{0,p} \leq Ch^2 \|u_{xx}\|_{2,p} \leq Ch^2 \|u\|_{4,p}. \tag{3.48}$$

Then, for $\psi_h \in S_{h,1}$, using (3.45), we have

$$\begin{aligned}
(P_h u_{xx} - u_{hxx}, \chi_h) &= (P_h u_{xx} - u_{xx}, \chi_h) + (u_{xx} - u_{hxx}, \chi_h) \\
&= (u_{xx} - u_{hxx}, \chi_h) \\
&= (\varepsilon_{hxx}, \chi_h) - \langle \varepsilon_{hxx}, \chi_h \rangle_h + \langle \varepsilon_{hxx}, \chi_h \rangle_h \\
|(P_h u_{xx} - u_{hxx}, \chi_h)| &\leq |E_h(\varepsilon_{hxx}\phi_h)| + |\langle \varepsilon_{hxx}, \chi_h \rangle_h| \\
&\leq T_{3,16} + T_{3,17}.
\end{aligned} \tag{3.49}$$

We compute the estimates for the terms $T_{3,16}$ and $T_{3,17}$ as follows:

$$T_{3,16} \;=\; |E_h(\varepsilon_{hxx}\chi_h)| \leq Ch^3 \|u\|_{6,p} \|\chi_h\|_{0,q},$$

where we have used Lemma 3.3.2b.

Following the steps involved in the computation of the term $T_{3,15}$, we obtain the estimate of $T_{3,17}$ as

$$T_{3,17} = |\langle \varepsilon_{hxx}, \chi_h \rangle_h| \leq C \|e_h\|_{0,\infty} \|\chi_h\|_{0,q} \leq C \left[h \|e_h\|_{2,p} + h^4 \|v\|_{5,p} \right] \|\chi_h\|_{0,q},$$

where we have used inverse inequality $\|\chi_h\|_{1,q,k} \leq C h_k^{-1} \|\chi_h\|_{0,q,k}$ locally on I_k.

On substituting $T_{3,16}$ and $T_{3,17}$ in (3.49), we obtain

$$|(P_h u_{xx} - u_{hxx}, \chi_h)| \leq C \left[h^3 \|u\|_{6,p} + h \|e_h\|_{2,p} + h^4 \|v\|_{5,p} \right] \|\chi_h\|_{0,q}. \quad (3.50)$$

We now show the above inequality for $\rho \in L_q$ to obtain $\|P_h u_{xx} - u_{hxx}\|_{0,p}$.

Now let ρ be an arbitrary element of L_q. Then, since $v_{hxx} \in S_{h,1}$, it follows from the definition of $P_h\rho$, (3.50) and (3.46) with p replaced by q, that

$$\begin{aligned} 0 &= (P_h u_{xx} - u_{hxx}, \rho - P_h\rho) \\ |(P_h u_{xx} - u_{hxx}, \rho)| &= |(P_h u_{xx} - u_{hxx}, P_h\rho)| \\ &\leq C \left[h^3 \|u\|_{6,p} + h \|e_h\|_{2,p} + h^4 \|v\|_{5,p} \right] \|P_h\rho\|_{0,q} \\ &\leq C \left[h^3 \|u\|_{6,p} + h \|e_h\|_{2,p} + h^4 \|v\|_{5,p} \right] \|\rho\|_{0,q}. \end{aligned}$$

Hence,

$$\|P_h u_{xx} - u_{hxx}\|_{0,p} \leq C \left[h^3 \|u\|_{6,p} + h \|e_h\|_{2,p} + h^4 \|v\|_{5,p} \right]. \quad (3.51)$$

We now use (3.48) and (3.51) in (3.47) to obtain

$$\|\varepsilon_{hxx}\|_{0,p} \leq C \left[h^2 \|u\|_{6,p} + h \|e_h\|_{2,p} + h^4 \|v\|_{5,p} \right].$$

Following the same steps as in Lemma 3.3.6, we can compute the estimates for $\|\varepsilon_h\|_{0,p}$, $\|\varepsilon_h\|_{1,p}$ and $\|\varepsilon_h\|_{2,p}$ as

$$\begin{aligned} \|\varepsilon_h\|_{0,p} &\leq C \left[h^4 \|u\|_{6,p} + h^2 \|e_h\|_{2,p} + h^5 \|v\|_{5,p} \right]; && (3.52) \\ \|\varepsilon_h\|_{1,p} &\leq C \left[h^3 \|u\|_{6,p} + h \|e_h\|_{2,p} + h^4 \|v\|_{5,p}) \right]; && (3.53) \\ \|\varepsilon_h\|_{2,p} &\leq C \left[h^2 \|u\|_{6,p} + h \|e_h\|_{2,p} + h^4 \|v\|_{5,p} \right]. && (3.54) \end{aligned}$$

Hence the proof is completed. ■

Using all the estimates from Lemma 3.3.6 and Lemma 3.3.7, we have the following main error estimates.

Theorem 3.3.1 *Let u and v be the weak solutions of the coupled equations* (3.3) *and* (3.4) *defined through* (3.5) *and* (3.6) *respectively. Further, let u_h and v_h be the corresponding discrete Petrov-Galerkin solutions defined through* (3.9) *and* (3.10) *respectively. Further assume that* (3.12), (3.14) *and* (3.15) *hold. Assume also that $u \in W_p^6$ and $v \in W_p^6$, where $p \in [1, \infty]$. Then, for h sufficiently small, we have*

$$\begin{aligned} \|u - u_h\|_{i,p} &\leq Ch^{4-i} \left[\|u\|_{6,p} + \|v\|_{6,p}\right]; \\ \|v - v_h\|_{i,p} &\leq Ch^{4-i} \left[\|u\|_{6,p} + \|v\|_{6,p}\right], i = 0, 1, 2. \end{aligned}$$

Proof: Assume temporarily that solutions u_h and v_h of (3.9) and (3.10) respectively, exist. Using (3.54) in (3.41), we obtain

$$\begin{aligned} \|e_h\|_{2,p} &\leq C\left[h^2 \|v\|_{6,p} + h^2(h^2 \|u\|_{6,p} + h\|e_h\|_{2,p} + h^4 \|v\|_{5,p}) + h^4 \|u\|_{6,p}\right] \\ &\leq C\left(h^2 \|v\|_{6,p} + h^4 \|u\|_{6,p} + h^3 \|e_h\|_{2,p}\right). \end{aligned}$$

For sufficiently small h, we have

$$\|e_h\|_{2,p} \leq C\left(h^2 \|v\|_{6,p} + h^4 \|u\|_{6,p}\right). \tag{3.55}$$

An application of the above in (3.54), we get

$$\|\varepsilon_h\|_{2,p} \leq C\left[h^2 \|u\|_{6,p} + h^3 \|v\|_{6,p}\right]. \tag{3.56}$$

Apply (3.55) in (3.52) to have

$$\|\varepsilon_h\|_{0,p} \leq C\left[h^4 \|u\|_{6,p} + h^4 \|v\|_{6,p}\right]. \tag{3.57}$$

Use (3.56) in (3.40) to get

$$\|e_h\|_{0,p} \leq C\left[h^4 \|v\|_{6,p} + h^5 \|u\|_{6,p}\right]. \tag{3.58}$$

Using (3.58) and (3.55) in (3.38), we obtain

$$\|e_h\|_{1,p} \leq C\left[h^3 \|u\|_{6,p} + h^3 \|v\|_{6,p}\right]. \tag{3.59}$$

In a similar manner, using (3.57) and (3.56) in (3.38) with e_h replaced by ε_h, we have

$$\|\varepsilon_h\|_{1,p} \leq C\left[h^3 \|u\|_{6,p} + h^3 \|v\|_{6,p}\right]. \tag{3.60}$$

The required result can be obtained from estimates (3.55) to (3.60). ■

So far we have assumed temporarily that solutions u_h and v_h exist. We now discuss the existence and uniqueness of discrete Petrov-Galerkin approximation. Since the matrix corresponding to (3.9) and (3.10) with the boundary conditions (3.11) is square, existence of $u_h \in \overset{0}{S}_{h,3}$ and $v_h \in \overset{0}{S}_{h,3}$ for any $f \in C^0$ will follow from uniqueness, that is, from the property that the corresponding homogeneous equations have only trivial solutions.
Suppose that, u_h and v_h corresponding to u and v satisfy

$$\begin{aligned} \langle Mu_h - v_h, \chi_h \rangle &= 0 \\ \langle Lv_h + cu_{hx} + du_h, \chi_h \rangle &= 0, \quad \chi_h \in S_{h,1}. \end{aligned}$$

It follows from (3.57) and (3.58) (with u replaced by 0 and eventually $v \equiv 0$), that for sufficiently small h,

$$\|u_h\| \leq 0 \text{ and } \|v_h\| \leq 0$$

and hence $u_h \equiv 0$ and $v_h \equiv 0$. Thus uniqueness is proved and hence existence follows from uniqueness.

3.4 NUMERICAL EXPERIMENTS

In this section, we present a study on numerical implementation of the mixed discrete Petrov-Galerkin method (3.9) and (3.10), which is discussed in this chapter. We consider the following boundary value problem for our numerical experiment:

$$\begin{aligned} u_{xxxx} &+ \frac{4x}{1+x^2}u_{xxx} - \frac{24x^2}{(1+x^2)^2}u_{xx} - \frac{24x}{(1+x^2)^2}u_x + \frac{96x^2}{(1+x^2)^3}u \\ &= \frac{(24 - 8x - 192x^2 + 64x^3 + 40x^5)}{(1+x^2)^3}, x \in [0,1]. \end{aligned} \tag{3.61}$$

It can easily be checked that $u(x) = \dfrac{1}{1+x^2} - \dfrac{(12 + x - 12x^2 + 5x^3)}{12}$ is an exact solution of (3.61) satisfying $u(0) = 0,\ u(1) = 0,\ v(0) = 0$ and $v(1) = 0$. Though we do not impose any assumption on the partition of the interval, for the purpose of implementation, we consider a partition with uniform spacing. Using the cubic B-splines defined in Section 2.1, the approximate solutions $u_h \in S_{h,3}$ and $v_h \in S_{h,3}$ are expressed as follows:

$$u_h(x) = \sum_{j=-1}^{n+1} \gamma_j B_j(x) \text{ and } v_h(x) = \sum_{j=-1}^{n+1} \delta_j B_j(x).$$

The coefficient functions and right hand side function of (3.1) in the example given in this section are
$a(x) = \dfrac{4x}{1+x^2}$, $b(x) = -\dfrac{24x^2}{(1+x^2)^2}$, $c(x) = -\dfrac{24x}{(1+x^2)^2}$, $d(x) = \dfrac{96x^2}{(1+x^2)^3}$
and $f(x) = \dfrac{(24-8x-192x^2+64x^3+40x^5)}{(1+x^2)^3}$ respectively. Further, the operators L and M are as in (3.13). Then, the mixed discrete Petrov-Galerkin method for (3.61) is given as follows:

$$\langle Mu_h, \phi_i\rangle_h - \langle v_h, \phi_i\rangle_h = 0$$

$$\langle cu_{hx} + du_h, \phi_i\rangle_h + \langle Lv_h, \phi_i\rangle_h = \langle f, \phi_i\rangle_h, \text{ for } i = 0, 1, 2, \cdots, n.$$

subject to the boundary conditions
$u_h(0) = 0,\ u_h(1) = 0,\ v_h(0) = 0$ and $v_h(1) = 0$. The above scheme is equivalent to

$$\sum_{j=-1}^{n+1} \gamma_j \langle B_{jxx}, \phi_i\rangle_h - \sum_{j=-1}^{n+1} \delta_j \langle B_j, \phi_i\rangle_h = 0 \tag{3.62}$$

$$\sum_{j=-1}^{n+1} \gamma_j \langle cB_{jx} + dB_j, \phi_i\rangle_h + \sum_{j=-1}^{n+1} \delta_j \langle B_{jxx} + aB_{jx} + bB_j, \phi_i\rangle_h = \langle f, \phi_i\rangle_h, \quad i = 0, 1, 2, \ldots, n \tag{3.63}$$

subject to the boundary conditions

$$\sum_{j=-1}^{n+1} \gamma_j B_j(0) = 0; \quad \sum_{j=-1}^{n+1} \gamma_j B_j(1) = 0; \tag{3.64}$$

$$\sum_{j=-1}^{n+1} \delta_j B_j(0) = 0; \quad \sum_{j=-1}^{n+1} \delta_j B_j(1) = 0. \tag{3.65}$$

The values of $\langle B_{jxx}, \phi_i\rangle_h$, $\langle B_j, \phi_i\rangle_h$, $\langle cB_{jx} + dB_j, \phi_i\rangle_h$, $\langle B_{jxx} + aB_{jx} + bB_j, \phi_i\rangle_h$ and $\langle f, \phi_i\rangle_h$ are computed using the composite two-point Gauss rule.

The above set of equations (3.62) - (3.65) can be written as a set of *(2n + 6)* equations in *(2n + 6)* unknowns such that the coefficient matrix of the system is a block matrix of the form

$$\begin{bmatrix} P & Q \\ R & S \end{bmatrix},$$

where P, Q, R and S are banded square matrices of order *(n+3)* $\times$ *(n+3)* having bandwidth 5. The numerical solution is obtained by solving the above system using MATHEMATICA software. The result is tabulated in the following tables Table 3.1 - Table 3.6. The L_∞ norm of the errors are estimated by taking the maximum absolute value over the discrete set consisting of the multiples of 0.001. The L_1 and L_2 norms of the errors are estimated by using the two point composite Gaussian rule on a uniform partition of $[0,1]$ of length $h = 0.001$. The columns labeled *EOC* show the empirical orders of convergence computed from the errors with mesh spacing h and $2h$ by the formula $\log_2\left(\frac{e_{2h}}{e_h}\right)$.

The results in Table 3.1, Table 3.3 and Table 3.5 show that $O(h^4)$, $O(h^3)$ and $O(h^2)$ order convergence have been achieved for errors involved in the computation of u_h, u_{hx} and u_{hxx} in L_∞, L_1 and L_2 norms respectively. Table 3.2, Table 3.4 and Table 3.6 show similar results for v_h, v_{hx} and v_{hxx} in L_∞, L_1 and L_2 norms respectively.

Table 3.1: L_∞ errors in u_h, u_{hx}, u_{hxx}

n	$\|u-u_h\|_{0,\infty}$	EOC	$\|u_x-u_{hx}\|_{0,\infty}$	EOC	$\|(u-u_h)_{xx}\|_{0,\infty}$	EOC
10	$7.04471*10^{-6}$	-	$2.07632*10^{-4}$	-	$2.01442*10^{-2}$	-
20	$4.13383*10^{-7}$	4.09099	$2.48565*10^{-5}$	3.06233	$5.01010*10^{-3}$	2.00745
40	$2.56241*10^{-8}$	4.01191	$3.04779*10^{-6}$	3.02779	$1.25065*10^{-3}$	2.00216
80	$1.59464*10^{-9}$	4.00620	$3.76033*10^{-7}$	3.01883	$3.12541*10^{-4}$	2.00056
160	$9.96214*10^{-11}$	4.00630	$4.67143*10^{-8}$	3.00892	$7.81275*10^{-5}$	2.00014

Table 3.2: L_∞ errors in v_h, v_{hx} and v_{hxx}.

n	$\|\|v-v_h\|\|_{0,\infty}$	EOC	$\|\|v_x-v_{hx}\|\|_{0,\infty}$	EOC	$\|\|(v-v_h)_{xx}\|\|_{0,\infty}$	EOC
10	$1.95861*10^{-4}$	-	$5.90696*10^{-3}$		$6.12887*10^{-1}$	-
20	$1.17422*10^{-5}$	4.06006	$7.28718*10^{-4}$	3.01898	$1.50861*10^{-1}$	2.0224
40	$7.26878*10^{-7}$	4.01385	$9.05613*10^{-5}$	3.00839	$3.75545*10^{-2}$	2.00616
80	$4.54717*10^{-8}$	3.99867	$1.12856*10^{-5}$	3.00441	$9.37841*10^{-3}$	2.00157
160	$2.84602*10^{-9}$	3.99799	$1.40849*10^{-6}$	3.00226	$2.34396*10^{-3}$	2.00400

Table 3.3: L_1 **errors in** u_h, u_{hx}, u_{hxx}

n	$\|\|u - u_h\|\|_{0,1}$	EOC	$\|\|u_x - u_{hx}\|\|_{0,1}$	EOC	$\|\|(u - u_h)_{xx}\|\|_{0,1}$	EOC
10	$1.81460 * 10^{-6}$	-	$5.27478*10^{-5}$	-	$3.13037 * 10^{-3}$	-
20	$1.12305 * 10^{-7}$	4.01416	$6.21902*10^{-6}$	3.08435	$7.58217 * 10^{-4}$	2.04565
40	$7.00353 * 10^{-9}$	4.00320	$7.64925*10^{-7}$	3.02330	$1.87823 * 10^{-4}$	2.01324
80	$4.37582*10^{-10}$	4.00046	$9.52357*10^{-8}$	3.00574	$4.67457 * 10^{-5}$	2.00647
160	$2.77069*10^{-11}$	3.99795	$1.18758*10^{-8}$	3.00348	$1.17156 * 10^{-5}$	1.99640

Table 3.4: L_1 **errors in** v_h, v_{hx}, v_{hxx}

n	$\|\|v - v_h\|\|_{0,1}$	EOC	$\|\|v_x - v_{hx}\|\|_{0,1}$	EOC	$\|\|(v - v_h)_{xx}\|\|_{0,1}$	EOC
10	$3.18933 * 10^{-5}$	-	$1.38359*10^{-3}$	-	$8.01017 * 10^{-2}$	-
20	$1.79727 * 10^{-6}$	4.14937	$1.55574*10^{-4}$	3.15274	$1.88149 * 10^{-2}$	2.08996
40	$1.09468 * 10^{-7}$	4.03723	$1.88591*10^{-5}$	3.04427	$4.61979 * 10^{-3}$	2.02598
80	$6.80007 * 10^{-9}$	4.00882	$2.33717*10^{-6}$	3.01243	$1.14657 * 10^{-3}$	2.01050
160	$4.24372*10^{-10}$	4.00215	$2.91104*10^{-7}$	3.00516	$2.87126 * 10^{-4}$	1.99757

Table 3.5: L_2 **errors in** u_h, u_{hx}, u_{hxx}

n	$\|\|u - u_h\|\|$	EOC	$\|\|u_x - u_{hx}\|\|$	EOC	$\|\|u_{xx} - u_{hxx}\|\|$	EOC
10	$2.47388 * 10^{-6}$	-	$6.83956*10^{-5}$	-	$4.24257*10^{-3}$	-
20	$1.50063 * 10^{-7}$	4.04314	$8.10966*10^{-6}$	3.07619	$1.0396 * 10^{-3}$	2.02891
40	$9.31252 * 10^{-9}$	4.01025	$1.00033*10^{-6}$	3.01917	$2.58599*10^{-4}$	2.00724
80	$5.81097*10^{-10}$	4.00232	$1.24629*10^{-7}$	3.00476	$6.43293*10^{-5}$	2.00717
160	$3.66440*10^{-11}$	3.98713	$1.55651*10^{-8}$	3.00125	$1.61649*10^{-5}$	1.99261

Table 3.6: L_2 **errors in** v_h, v_{hx}, v_{hxx}

n	$\|\|v - v_h\|\|$	EOC	$\|\|v_x - v_{hx}\|\|$	EOC	$\|\|v_{xx} - v_{hxx}\|\|$	EOC
10	$5.03523 * 10^{-5}$	-	$1.95852*10^{-3}$	-	$1.18935*10^{-1}$	-
20	$2.82394 * 10^{-6}$	4.15628	$2.23966*10^{-4}$	3.12841	$2.85523*10^{-2}$	2.05849
40	$1.71744 * 10^{-7}$	4.03938	$2.73704*10^{-5}$	3.03259	$7.06574*10^{-3}$	2.01469
80	$1.06609 * 10^{-8}$	4.00986	$3.4023 * 10^{-6}$	3.00815	$1.75541*10^{-3}$	2.00903
160	$6.65126*10^{-10}$	4.00256	$4.24636*10^{-7}$	3.00210	$4.4096 * 10^{-4}$	1.99309