CHAPTER 6

SUMMARY AND CONCLUSION

The present work has involved the formulation and development of computational procedures in order to obtain optimal order error estimates in the numerical solution of fourth order differential equations using mixed Galerkin finite element methods. Two ordinary differential equations and two time-dependent nonlinear partial differential equations in single space variable are considered for detailed investigation.

A quadrature based mixed Petrov-Galerkin finite element method is applied to a of fourth order linear ordinary differential equation in divergence form.

$$\frac{d^2}{dx^2}\left[a(x)\frac{d^2u}{dx^2}\right] + b(x)u = f(x), \;\; x \in I = (0,1);$$

subject to the boundary conditions

$$u(0) = 0,\; u(1) = 0;\; u_{xx}(0) = 0,\; u_{xx}(1) = 0,$$

where $a(x) \neq 0, \;\; x \in I$. Let $\alpha(x) = 1/a(x)$.
using a splitting technique, a cubic spline trial space and a piecewise linear test space are considered. The integrals are replaced by Gauss quadrature rule in the formulation itself. Optimal order *a priori* error estimates are obtained without any restriction on the mesh.

Further, a similar method is applied to the general fourth order problem

$$u_{xxxx} + a(x)u_{xxx} + b(x)u_{xx} + c(x)u_x + d(x)u = f(x),\; x \in I = (0,1);$$

subject to the boundary conditions

$$u(0) = 0,\ u(1) = 0;\ u_{xx}(0) = 0,\ u_{xx}(1) = 0.$$

The error estimates for both the above equations in general p-norms for $1 \leq p \leq \infty$ are as follows:

$$\|u - u_h\|_{i,p} \leq Ch^{4-i} \left[\|u\|_{6,p} + \|v\|_{6,p}\right]$$

and

$$\|v - v_h\|_{i,p} \leq Ch^{4-i} \left[\|u\|_{6,p} + \|v\|_{6,p}\right], \quad i = 0, 1, 2.$$

The result is validated with a numerical example.

An H^1-Galerkin mixed finite element method is employed to the extended Fisher-Kolmogorov equation.

$$u_t + \gamma u_{xxxx} - u_{xx} + f(u) = 0, \quad 0 < t < T,\ \gamma > 0,\ \ x \in I = (0, 1);$$

subject to the boundary and initial conditions

$$u(0,t) = 0,\ u(1,t) = 0,\ u_{xx}(0,t) = 0,\ u_{xx}(1,t) = 0;$$
$$u(x,0) = g(x);$$

where $f(u) = u^3 - u$. Using a splitting technique, optimal order error estimates are obtained without any restriction on the mesh. A semi descrete and a fully discrete schemes are developed and optimal order estimates are obtained for this non linear time dependent fourth order partial differential equation.

Further, a similar method is applied to the Kuramoto-Sivashinsky equation

$$u_t + \gamma u_{xxxx} + u_{xx} + uu_x = 0, \quad 0 < t < T,\ \gamma > 0,\ \ x \in I = (0, 1);$$

subject to the boundary and initial conditions

$$u(0,t) = 0,\ u(1,t) = 0,\ u_{xx}(0,t) = 0,\ u_{xx}(1,t) = 0;$$
$$u(x,0) = g(x).$$

The optimal order error estimates for extended Fisher-Kolmogorov equation and Kuramoto-Sivashinsky equations in the semidiscrete case are

$$
\begin{aligned}
\|v-V\|^2_{L_\infty(L_2)} &\leq Ch^8\left[\|u\|^2_{L_2(H^4)}+\|v\|^2_{L_2(H^4)}+\|v_t\|^2_{L_2(H^4)}+\|v\|^2_{L_\infty(H^4)}\right]\\
\|v-V\|^2_{L_2(H^2)} &\leq Ch^4\left[\|u\|^2_{L_2(H^4)}+\|v\|^2_{L_2(H^4)}+\|v_t\|^2_{L_2(H^4)}\right]\\
\|u_{xx}-U_{xx}\|^2_{L_\infty(L_2)} &\leq Ch^4\left[\|u\|^2_{L_2(H^4)}+\|v\|^2_{L_2(H^4)}+\|v_t\|^2_{L_2(H^4)}+\|v\|^2_{L_\infty(H^4)}\right]\\
\text{and}&\\
\|u-U\|^2_{L_\infty(L_2)} &\leq Ch^8\left[\|u\|^2_{L_2(H^4)}+\|v\|^2_{L_2(H^4)}+\|v_t\|^2_{L_2(H^4)}+\|v\|^2_{L_\infty(H^4)}\right].
\end{aligned}
$$

The optimal order error estimates for extended Fisher-Kolmogorov equation and Kuramoto-Sivashinsky equations in the fully discrete case are

$$
\begin{aligned}
\|u^{J+1}-Z^{J+1}\| &\leq C\left[k^{\frac{3}{2}}+h^4\right]\\
\|u^{J+1}_{xx}-Z^{J+1}_{xx}\| &\leq C\left[k^{\frac{3}{2}}+h^2\right] \text{ and}\\
\|v^{J+1}-W^{J+1}\| &\leq C\left[k^{\frac{3}{2}}+h^4\right] \text{ for } J=0,1,2,...,M-1,
\end{aligned}
$$

where C is a generic constant depending only on u and v. The results are validated with numerical examples in the case of both the above equations.

Following are the advantages of the method described in Chapter 4.

1. For the classical solutions of the extended Fisher-Kolmogorov equation, fourth order smoothness is required. The method described in this chapter requires sixth order regularity on the solution. But there are methods, for example, orthogonal cubic spline collocation method which demand eighth order regularity on the solution as given in the literature (Danumjaya *et al.* 2005).

2. The size of the combined linear system is $8n+4$ in (Danumjaya *et al.* 2005), where as the size of the decoupled system in the present method is $n+1$ each (*i.e.*,a total of $2n+2$). This is clearly explained in the next section, where the linear fully discrete scheme is described.

Following are the advantages of the method described in Chapter 5.

1. For the classical solutions of the Kuramoto -Sivashinsky equation, fourth order smoothness is required. The method described in this chapter requires sixth order regularity on the solution. But there are methods, for example, orthogonal cubic spline collocation method which demand eighth order regularity on the solution as given in the literature (Manickam *et al.* 1998).

2. The size of the combined linear system is $8n+4$ in (Manickam *et al.* 1998), where as the size of the decoupled system in the present method is $n+1$ each (*i.e.*,a total of $2n+2$). This is clearly explained in the next section, where the linear fully discrete scheme is described.

FUTURE WORK

The investigations of the present work can be continued to obtain optimal order *a priori* error estimates for:

(i) a quadrature based mixed H^1-Galerkin finite element method for the two time dependent non linear partial differential equations mentioned in the dissertation.

(ii) a quadrature based mixed Petrov-Galerkin finite element method for the two time dependent non linear partial differential equations mentioned in the dissertation.

(iii) $u-U$, $u^n - Z^n$ and $v^n - W^n$ in H^1 and H^2 norms in the case of the two time dependent non linear partial differential equations for the formulation mentioned in the dissertation.