CHAPTER 5

H^1-GALERKIN MIXED FINITE ELEMENT METHOD FOR KURAMOTO-SIVASHINSKY EQUATION

5.1 INTRODUCTION

In this chapter, we discuss an H^1-Galerkin mixed finite element cubic spline approximation method for the following Kuramoto-Sivashinsky equation :

$$u_t + \gamma u_{xxxx} + u_{xx} + uu_x = 0, \quad 0 < t < T,\ \gamma > 0,\ x \in I = (0,1); \tag{5.1}$$

subject to the boundary and initial conditions

$$u(0,t) = 0,\ u(1,t) = 0,\ u_{xx}(0,t) = 0,\ u_{xx}(1,t) = 0; \tag{5.2}$$

$$u(x,0) = g(x). \tag{5.3}$$

The above equation is a nonlinear time dependent fourth order partial differential equation. Let us define the splitting of the above equation as follows:
Set

$$u_{xx} = v,\ x \in I. \tag{5.4}$$

Then (5.1) becomes

$$u_t + \gamma v_{xx} + v + uu_x = 0,\ x \in I. \tag{5.5}$$

In the present study, an H^1-Galerkin method with cubic splines as trial functions is formulated, analysed for the coupled equations (5.4) and (5.5) and error bounds are obtained for both semi discrete and fully discrete schemes.

5.2 SEMI DISCRETE H^1-GALERKIN FORMULATION

For $\varphi, \psi \in H^2(I) \bigcap \overset{0}{H^1}(I)$, let us define the following bilinear form

$$A(\lambda : \varphi, \psi) = (\gamma\varphi_{xx}, \psi_{xx}) + (\varphi, \psi_{xx}) + \lambda(\varphi, \psi) + \lambda(\varphi_x, \psi_x),$$

where $\lambda > 0$ is chosen appropriately later so that $A(\lambda : \varphi, \psi)$ is coercive. This can easily be seen from the following: For φ satisfying the the boundary conditions $\varphi(0) = 0$, $\varphi(1) = 0$, we have

$$\begin{aligned} A(\lambda : \varphi, \varphi) &= (\gamma\varphi_{xx}, \varphi_{xx}) + (\varphi, \varphi_{xx}) + \lambda(\varphi, \varphi) + \lambda(\varphi_x, \varphi_x) \\ &= (\gamma\varphi_{xx}, \varphi_{xx}) - (\varphi_x, \varphi_x) + \lambda(\varphi, \varphi) + \lambda(\varphi_x, \varphi_x) \\ &= \gamma\|\varphi_{xx}\|^2 + (\lambda - 1)\|\varphi_x\|^2 + \lambda\|\varphi\|^2 \\ &\geq \alpha_0\|\varphi\|_2^2, \end{aligned}$$

where we have used integration by parts. Further $\lambda > 0$ is chosen in such a way that $\alpha_0 = \min(\gamma, \lambda - 1)$.

It can also be shown that the bilinear form $A(\lambda : \varphi, \psi)$ is continuous,

$$i.e., \ |A(\lambda : \varphi, \psi)| \leq K\|\varphi\|_2\|\psi\|_2,$$

where K depends only on γ and λ.

We now see the weak formulation and H^1-Galerkin mixed finite element formulation for the split up equations (5.4) and (5.5) of the main equation (5.1). We obtain weak formulation for u by multiplying both sides of (5.4) with ϕ_{xx}, where $\phi \in H^2(I) \bigcap \overset{0}{H^1}(I)$ and then integrating the resulting expression with respect to x over the interval $[0, 1]$. In a similar manner, multiplying both sides of (5.5) by ϕ_{xx} with $\phi \in H^2(I) \bigcap \overset{0}{H^1}(I)$, integrating with respect to x over the interval $[0, 1]$ using integration by parts twice for the first term and then adding and subtracting $\lambda(v, \phi)$ and $\lambda(v_x, \phi_x)$ in the resulting equation, we obtain the following weak formulation for v. The weak formulation corresponding to the split up equations (5.4) and (5.5) is given below:

Weak Formulation: Find $u, v : [0, T] \to H^2(I) \bigcap \overset{0}{H^1}(I)$, such that

$$\begin{aligned} (u_{xx}, \phi_{xx}) &= (v, \phi_{xx}), \ \ \phi \in H^2(I) \bigcap \overset{0}{H^1}(I), \\ u(x, 0) &= g(x); \end{aligned} \tag{5.6}$$

$$\begin{aligned} (v_t, \phi) + A(\lambda : v, \phi) - \lambda(v, \phi) &- \lambda(v_x, \phi_x) + (uu_x, \phi_{xx}) = 0, \\ &\phi \in H^2(I) \bigcap \overset{0}{H^1}(I), \\ v(x, 0) &= g_{xx}(x). \end{aligned} \tag{5.7}$$

The Semi discrete H^1-Galerkin mixed finite element formulation corresponding to the above weak formulation (5.6) and (5.7) is defined respectively as follows:

<u>The Semi discrete H^1-Galerkin mixed finite element formulation:</u>

Find $U, V \in \overset{0}{S}_{h,3}$ such that

$$\begin{aligned} (U_{xx}, \phi_{hxx}) &= (V, \phi_{hxx}), \;\; \phi_h \in \overset{0}{S}_{h,3}, \\ U(x,0) &= g(x); \end{aligned} \tag{5.8}$$

$$\begin{aligned} (V_t, \phi_h) + A(\lambda : V, \phi_h) - \lambda(V, \phi_h) &- \lambda(V_x, \phi_{hx}) + (UU_x, \phi_{hxx}) = 0, \\ &\phi_h \in \overset{0}{S}_{h,3}, \\ V(x,0) &= g_{xx}(x). \end{aligned} \tag{5.9}$$

The above formulation leads to a system of coupled equations. This method may be regarded as a Petrov-Galerkin method with cubic spline space as trial space and piecewise linear space as test space, since the second derivative of a cubic spline is a piecewise linear spline.

5.3 AUXILIARY PROJECTION

As usual, the error analysis is generally carried over with the help of a comparison function. In this analysis, the comparison function is the auxiliary projection $\widehat{v} \in \overset{0}{S}_{h,3}$ of the weak solution v onto the finite dimensional subspace $\overset{0}{S}_{h,3}$ through the elliptic part appearing in the weak formulation.

Let $\widehat{v} : [0,T] \rightarrow \overset{0}{S}_{h,3}$ be the auxiliary projection of v defined by

$$A(\lambda : v - \widehat{v}, \phi_h) = 0, \;\; \phi_h \in \overset{0}{S}_{h,3} \tag{5.10}$$

and $\widehat{v}(x,0) = g_{xx}$.

We initially find the error involved in the auxiliary projection, *i.e.*, the error between the weak solution v and the intermediate comparison function $\widehat{v}$. In the next section, we compute the error between the comparison function $\widehat{v}$ and the mixed H^1-Galerkin approximation V. Let $v - \widehat{v} = \rho$. Error estimates for a similar auxiliary projection with complicated non linear terms are proved in (Jones and

Pani 1995) and (Pani and Das 1991). In the following lemma, we obtain the error estimates for the auxiliary projection.

Lemma 5.3.1 *There exists a constant* C *such that for sufficiently small* h *and* $i = 0, 1, 2$

$$\|\rho\|_i \leq Ch^{4-i}\|v\|_4;$$
$$\|\rho_t\|_i \leq Ch^{4-i}\|v_t\|_4.$$

Proof: We have from equation (5.10), that

$$A(\lambda : \rho, \phi_h) = 0, \ \ \phi_h \in \overset{0}{S}_{h,3}. \tag{5.11}$$

Choosing $\phi_h = \rho - (v - \chi)$ for $\chi \in \overset{0}{S}_{h,3}$, we have $A(\lambda : \rho, \rho) = A(\lambda : \rho, v - \chi)$. Using coercivity and continuity of $A(\lambda : \varphi, \psi)$, we obtain

$$\alpha_0\|\rho\|_2^2 \leq C\|\rho\|_2\|v - \chi\|_2.$$

Taking infimum over $\chi \in \overset{0}{S}_{h,3}$ first and then applying the approximation property, we obtain

$$\alpha_0\|\rho\|_2 \leq C \inf_{\chi \in \overset{0}{S}_{h,3}} \|v - \chi\|_2 \leq Ch^2\|v\|_4.$$

Hence, we obtain

$$\|\rho\|_2 \leq Ch^2\|v\|_4. \tag{5.12}$$

We now compute the error estimate of ρ in L_2 norm using the following duality argument.

Let $\psi \in H^4(I)$ be the solution of the auxiliary problem

$$L\psi := \gamma\psi_{xxxx} + (1 - \lambda)\psi_{xx} + \lambda\psi = \rho,$$

with boundary conditions

$$\psi_{xx}(0) = \psi_{xx}(1) = 0,$$
$$\psi(0) = \psi(1) = 0.$$

The above problem has the regularity property

$$\|\psi\|_4 \leq \|\rho\|.$$

It is easy to see that

$$\begin{aligned}
A(\lambda : \rho, \psi) &= \gamma(\rho_{xx}, \psi_{xx}) + (\rho, \psi_{xx}) + \lambda(\rho, \psi) + \lambda(\rho_x, \psi_x) \\
&= -\gamma(\rho_x, \psi_{xxx}) + \gamma\rho_x\psi_{xx}|_0^1 + (\rho, \psi_{xx}) + \lambda(\rho, \psi) - \lambda(\rho, \psi_{xx}) \\
&+ \lambda\rho\psi_x|_0^1 \\
&= \gamma(\rho, \psi_{xxxx}) - \gamma\psi_{xxx}\rho|_0^1 + \gamma\rho_x\psi_{xx}|_0^1 + (\rho, \psi_{xx}) + \lambda(\rho, \psi) \\
&- \lambda(\rho, \psi_{xx}) + \lambda\rho\psi_x|_0^1 \\
&= (\rho, \gamma\psi_{xxxx} + \psi_{xx}) + \lambda(\rho, \psi) - \lambda(\rho, \psi_{xx}) - \gamma\psi_{xxx}\rho|_0^1 \\
&+ \gamma\rho_x\psi_{xx}|_0^1 + \lambda\rho\psi_x|_0^1 \\
&= (\rho, L\psi),
\end{aligned}$$

where we have used integration by parts, boundary condition of ρ and the boundary conditions of the auxiliary problem.
Using the above and (5.11), we have that

$$(\rho, \rho) = (\rho, L\psi) = A(\lambda : \rho, \psi) = A(\lambda : \rho, \psi - \chi) \text{ for } \chi \in \overset{0}{S}_{h,3}.$$

Hence, applying the continuity of $A(\lambda : \varphi, \psi)$, the approximation property and regularity of ψ, we obtain

$$\|\rho\|^2 \leq C\|\rho\|_2\|\psi - \chi\|_2,$$

$$i.e., \; \|\rho\|^2 \leq C\|\rho\|_2 \inf_{\chi \in \overset{0}{S}_{h,3}} \|\psi - \chi\|_2 \leq C\|\rho\|_2 h^2\|\psi\|_4 \leq Ch^2\|\rho\|_2\|\|\rho\|.$$

Hence,

$$\|\rho\| \leq Ch^2\|\rho\|_2. \tag{5.13}$$

On an application of (5.12) in the above, we have

$$\|\rho\| \leq Ch^4\|v\|_4. \tag{5.14}$$

Using (5.12) and (5.14) in (1.17) with $m = 2$ and $i = 1$, we have

$$\begin{aligned} \|\rho\|_1 &\le C\left[h^{-1}\|\rho\| + h\|\rho\|_2\right], \\ i.e., \ \|\rho\|_1 &\le Ch^3\|v\|_4. \end{aligned} \tag{5.15}$$

To obtain similar error estimates for the temporal derivative of ρ, we differentiate the projection equation with respect to time variable t. Hence, we obtain

$$A(\lambda : \rho_t, \phi_h) = 0, \ \ \phi_h \in \overset{0}{S}_{h,3}.$$

It can be easily verified from the above analysis that

$$\begin{aligned} \|\rho_t\| &\le Ch^4\|v_t\|_4; \\ \|\rho_t\|_1 &\le Ch^3\|v_t\|_4; \\ \|\rho_t\|2 &\le Ch^2\|v_t\|_4. \end{aligned}$$

Thus, we obtain the error estimates for ρ and ρ_t in L_2, H^1 and H^2 norms. ■

5.4 ERROR ANALYSIS OF SEMI DICRETE SCHEME

In this section, we obtain *a priori* error estimate for the error between the comparison function $\widehat{v}$ and the H^1-Galerkin solution V. We also discuss the error analysis of the error between the weak solution u and its corresponding H^1-Galerkin approximation U. For this, we first write the error equation related to the H^1-Galerkin approximation. Subtracting (5.9) from (5.7), we obtain the following error equation:

$$\begin{aligned} (v_t - V_t, \phi_h) \ &+ \ A(\lambda : v - V, \phi_h) - \lambda(v - V, \phi_h) - \lambda(v_x - V_x, \phi_{hx}) \\ &+ \ (uu_x - UU_x, \phi_{hxx}) = 0, \ \ \phi_h \in \overset{0}{S}_{h,3}. \end{aligned} \tag{5.16}$$

In a similar manner, subtracting (5.8) from (5.6), we obtain

$$(u_{xx} - U_{xx}, \phi_{hxx}) = (v - V, \phi_{hxx}), \ \ \phi_h \in \overset{0}{S}_{h,3}. \tag{5.17}$$

Let e_1 be the error between u and U and e_2 be that between v and V. Then we have $e_1 = u - U$ and $e_2 = v - V = v - \widehat{v} + \widehat{v} - V = \rho + \zeta$, where $\zeta = \widehat{v} - V$.

Then, (5.16) can also be written as

$$\begin{aligned}(\rho_t + \zeta_t, \phi_h) + A(\lambda : \rho + \zeta, \phi_h) &- \lambda(\rho + \zeta, \phi_h) - \lambda(\rho_x + \zeta_x, \phi_{hx}) \\ &+ (uu_x - UU_x, \phi_{hxx}) = 0.\end{aligned}$$

Using projection (5.11) in the above equation, we obtain

$$\begin{aligned}(\zeta_t, \phi_h) &+ A(\lambda : \zeta, \phi_h) = -(\rho_t, \phi_h) + \lambda(\rho, \phi_h) + \lambda(\zeta, \phi_h) + \lambda(\rho_x, \phi_{hx}) \\ &+ \lambda(\zeta_x, \phi_{hx}) - (uu_x - UU_x, \phi_{hxx}) \text{ for } \phi_h \in \overset{0}{S}_{h,3}. \qquad (5.18)\end{aligned}$$

In a similar manner, (5.17) can also be written as

$$(u_{xx} - U_{xx}, \phi_{hxx}) = (\rho + \zeta, \phi_{hxx}) \text{ for } \phi_h \in \overset{0}{S}_{h,3}. \qquad (5.19)$$

The following lemma is similar to Lemma 4.4.1 of Chapter 4. In this, we compute $|e_1(\overline{x})|$, where $\overline{x}$ is an arbitrary point in $[0,1]$. This result is required for proving the main theorem in this section.

Lemma 5.4.1 *Let u and v be the weak solutions of* (5.4) *and* (5.5) *defined through* (5.6) *and* (5.7) *respectively. Further, let U and V be the corresponding semi discrete H^1-Galerkin mixed finite element formulation defined through* (5.8) *and* (5.9)*. Then the error $e_1 = u - U$ satisfies*

$$|e_1(\overline{x})| \leq C\left[h^2 \|e_{1xx}\| + \|e_2\|\right],$$

where $\overline{x}$ is an arbitrary point in $[0,1]$.

In the following lemma, we compute $|e_{1x}(\overline{x})|$, where $\overline{x}$ is an arbitrary point in $[0,1]$. This result is required for proving the main theorem in this section.

Lemma 5.4.2 *Let u and v be the weak solutions of* (5.4) *and* (5.5) *defined through* (5.6) *and* (5.7) *respectively. Further, let U and V be the corresponding semi discrete H^1-Galerkin mixed finite element formulation defined through* (5.8) *and* (5.9)*. Then the error $e_1 = u - U$ satisfies*

$$|e_{1x}(\overline{x})| \leq C\left[h^2 \|e_{1xx}\| + \|e_2\|\right],$$

where $\overline{x}$ is an arbitrary point in $[0,1]$.

Proof: For a given $\overline{x} \in [0,1]$, let Φ be an element of $L_2(I) \bigcap C(I)$ satisfying the following auxiliary problem:

$$\begin{aligned} \frac{d^3\Phi}{dx^3} &= 0, \; x \in I - \{\overline{x}\} \\ \Phi_x(0) &= \Phi_x(1) = 0, \; \Phi_x^-(\overline{x}) - \Phi_x^-(\overline{x}) = 1, \\ \Phi_{xx}^-(\overline{x}) &- \Phi_{xx}^+(\overline{x}) = -1. \end{aligned}$$

The above problem has a solution. For example,

$$\Phi(x) = \begin{cases} (\overline{x}x^2)/2, & 0 \le x \le \overline{x}, \\ (\overline{x}+1)(\frac{x^2}{2} - x) + (2\overline{x} + \overline{x}^2)/2, & \overline{x} \le x \le 1 \end{cases}$$

satisfies the above differential equation and the boundary conditions.
Let us define Ψ as follows.

$$\Psi(x) = \begin{cases} \Phi_{xxx}, & x \in I - \{\overline{x}\}, \\ 0, & x = \overline{x}. \end{cases}$$

Then $\Psi = 0$ *a.e.* on I. We first multiply e_1 with Ψ and then integrate over the interval I. Proceeding as in the proof of Lemma 5.4.1, on applying integration by parts twice, using the fact that $e_1(0) = e_1(1) = 0$ and then using the given boundary condition and the jump condition, we obtain

$$\begin{aligned} 0 &= (e_1, \Psi) = e_1(\overline{x}) \left[\Phi_{xx}^-(\overline{x}) - \Phi_{xx}^+(\overline{x}) \right] - \int_0^{\overline{x}} e_{1x}\Phi_{xx} - \int_{\overline{x}}^1 e_{1x}\Phi_{xx} \\ &= -e_1(\overline{x}) - e_{1x}(\overline{x}) \left[\Phi_x^-(\overline{x}) - \Phi_x^+(\overline{x}) \right] + (e_{1xx}, \chi_x) \\ &= -e_1(\overline{x}) - e_{1x}(\overline{x}) + (e_{1xx}, \chi_x), \end{aligned}$$

where

$$\chi_x = \begin{cases} \Phi_x, & 0 \le x < \overline{x}, \\ \Phi_x, & \overline{x} < x \le 1, \\ x\overline{x}, & x = \overline{x}. \end{cases}$$

Since Φ_x is not defined at $\overline{x}$, a new function χ_x is introduced in the above manner in order to have a meaning for (e_{1xx}, χ_x). Then the above becomes

$$\begin{aligned} e_{1x}(\overline{x}) &= -e_1(\overline{x}) + (e_{1xx}, \chi_x), \\ i.e., \; |e_{1x}(\overline{x})| &\le |e_1(\overline{x})| + |(e_{1xx}, \chi_x)|. \end{aligned} \tag{5.20}$$

The estimate of the first term on the right hand side of (5.20) is given by Lemma 5.4.1. Now we compute the estimate for the second term. Let χ_h be the linear interpolant of χ_x.

$$\begin{aligned} (e_{1xx}, \chi_x) &= (e_{1xx}, \chi_x - \chi_h) + (e_{1xx}, \chi_h) \\ |\,(e_{1xx}, \chi_x)\,| &\leq |\,(e_{1xx}, \chi_x - \chi_h)\,| + |\,(e_{1xx}, \chi_h)\,| \\ &\leq T_{5,A} + T_{5,B}. \end{aligned} \tag{5.21}$$

Using the fact that $\chi_x = \Phi_x$ $a.e$ on I and the definition of the auxiliary problem, it is easy to verify that $\|\chi_x\|_2 \leq \|\chi\|_2$.
Therefore we have that

$$\|\chi_h\|_1 \leq \|\chi_x - \chi_h\|_1 + \|\chi_x\|_2 \leq C\,\|\chi_x\|_2 \leq C\,\|\chi\|_2\,. \tag{5.22}$$

We now compute estimates for the terms $T_{5,A}$, and $T_{5,B}$ as follows:

$$\begin{aligned} T_{5,A} = |(e_{1xx}, \chi_x - \chi_h)| &\leq \|e_{1xx}\|\,\|\|\chi_x - \chi_h\| \leq Ch^2\,\|e_{1xx}\|\,\|\chi_x\|_2 \\ &\leq Ch^2\,\|e_{1xx}\|\,\|\chi\|_2\,. \end{aligned}$$

As in the case of the term $T_{4,B}$ in Chapter 4, using (5.22) and (5.17), we can compute $T_{5,B}$ as

$$T_{5,B} = |\,(e_{1xx}, \chi_h)\,| \leq C\,\|e_2\|\,\|\;\chi_h\| \leq C\,\|e_2\|\,\|\chi\|_2\,.$$

For Φ satisfying the auxiliary problem, it is easy to verify that $\|\chi\|_2 = \|\Phi\|_2 \leq \|\Phi\|_3 \leq K$, where K is a constant not depending on h. Using $T_{5,A}$ and $T_{5,B}$ in (5.21), we have that

$$|\,(e_{1xx}, \chi_x)\,| \leq C\left[h^2\,\|e_{1xx}\| + \|e_2\|\right].$$

Using the above in (5.20) and using Lemma 5.4.1, we obtain

$$\text{Hence }\; |e_{1x}(\overline{x})| \leq C\left[h^2\,\|e_{1xx}\| + \|e_2\|\right]$$

and this completes the proof. ■

The following lemma gives *a priori* error estimates for the semi discrete mixed H^1-Galerkin procedure. Initially we obtain the error $\|u - U\|$. The proof of this lemma is similar to the proof of Lemma 4.4.2 in Chapter 4.

Lemma 5.4.3 *Let u and v be the weak solutions of* (5.4) *and* (5.5) *defined through* (5.6) *and* (5.7) *respectively. Further, let U and V be the corresponding semi discrete H^1-Galerkin mixed finite element formulation defined through* (5.8) *and* (5.9)*. Also, let $\widehat{v}$ be the auxiliary projection of v as in (5.10). Then, we have*

$$\|u - U\|^2 \leq C\left[h^8\|u\|_4^2 + \|\rho\|^2 + \|\zeta\|^2\right], \ \ \textit{where} \ \ \rho = v - \widehat{v}, \ \zeta = \widehat{v} - V.$$

Proof: Choose $\phi_h = (u - U) - (u - \chi)$ in (5.17) for some $\chi \in \overset{0}{S}_{h,3}$. Then, it becomes

$$\begin{aligned}
(u_{xx} - U_{xx}, u_{xx} - U_{xx}) &= (u_{xx} - U_{xx}, u_{xx} - \chi_{xx}) + (v - V, u_{xx} - U_{xx}) \\
&- (v - V, u_{xx} - \chi_{xx}), \\
\|u_{xx} - U_{xx}\|^2 &\leq \|u_{xx} - U_{xx}\|\|u_{xx} - \chi_{xx}\| + \|v - V\|\|u_{xx} - U_{xx}\| \\
&+ \|v - V\|\|u_{xx} - \chi_{xx}\|.
\end{aligned}$$

Taking infimum over $\chi \in \overset{0}{S}_{h,3}$ first and then applying the approximation property, we obtain

$$\begin{aligned}
\|u_{xx} - U_{xx}\|^2 &\leq \|u_{xx} - U_{xx}\| \inf_{\chi \in \overset{0}{S}_{h,3}} \|u_{xx} - \chi_{xx}\| + \|v - V\|\|u_{xx} - U_{xx}\| \\
&+ \|v - V\| \inf_{\chi \in \overset{0}{S}_{h,3}} \|u_{xx} - \chi_{xx}\| \\
&\leq C\Big[\|u_{xx} - U_{xx}\| \ h^2 \ \|u\|_4 + \|v - V\|\|u_{xx} - U_{xx}\| \\
&+ \|v - V\| \ h^2 \ \|u\|_4\Big].
\end{aligned}$$

Using Young's inequality, we have

$$\begin{aligned}
\|u_{xx} - U_{xx}\|^2 &\leq C\left[\frac{\epsilon}{2}\|u_{xx} - U_{xx}\|^2 + \frac{1}{2\epsilon}h^4\|u\|_4^2 + \frac{\epsilon}{2}\|u_{xx} - U_{xx}\|^2\right. \\
&+ \left.\frac{1}{2\epsilon}\|v - V\|^2 + \frac{\epsilon}{2}\|v - V\|^2 + h^4\frac{1}{2\epsilon}\|u\|_4^2\right], \\
(1 - C\epsilon)\|u_{xx} - U_{xx}\|^2 &\leq C\left[\frac{1}{\epsilon}h^4\|u\|_4^2 + \left(\frac{1}{2\epsilon} + \frac{\epsilon}{2}\right)\|v - V\|^2\right].
\end{aligned}$$

Choosing $\epsilon > 0$ (for example $\epsilon = 1/2C$) properly so that inequality is maintained, we obtain that

$$\|e_{1xx}\|^2 = \|u_{xx} - U_{xx}\|^2 \leq C\left[h^4\|u\|_4^2 + \|v - V\|^2\right]. \tag{5.23}$$

Now we compute the estimate of e_1 in L_2 norm. For that, we apply the following duality argument:

Let $\Phi \in H^4(I)$ be the solution of the auxiliary problem

$$\Phi_{xxxx} = u - U = e_1, \;\; x \in I$$

satisfying the boundary conditions

$$\Phi_{xxx}(0) = \Phi_{xxx}(1) = 0, \; \Phi_{xx}(0) = \Phi_{xx}(1) = 0.$$

Then, using integration by parts and (5.17), we obtain for $\chi \in \overset{0}{S}_{h,3}$,

$$\begin{aligned} (e_1, e_1) &= (\Phi_{xxxx}, e_1) = (\Phi_{xx}, e_{1xx}) = (e_{1xx}, \Phi_{xx}) \\ &= (e_{1xx}, \Phi_{xx} - \chi_{xx}) + (e_{1xx}, \chi_{xx}) \\ &= (e_{1xx}, \Phi_{xx} - \chi_{xx}) + (\rho + \zeta, \chi_{xx}) \\ &= (e_{1xx}, \Phi_{xx} - \chi_{xx}) + (\rho + \zeta, \chi_{xx} - \Phi_{xx}) + (\rho + \zeta, \Phi_{xx}), \\ \|e_1\|^2 &\leq [\|e_{1xx}\| \|\Phi_{xx} - \chi_{xx}\| + \|\rho + \zeta\| \|\chi_{xx} - \Phi_{xx}\| + \|\rho + \zeta\| \|\Phi_{xx}\|] \, . \end{aligned}$$

Taking infimum over $\chi \in \overset{0}{S}_{h,3}$ first and then applying the approximation property, we obtain

$$\begin{aligned} \|e_1\|^2 &\leq C \left[h^2 \|e_{1xx}\| \|\Phi\|_4 + h^2 \|\rho + \zeta\| \|\Phi\|_4 + \|\rho + \zeta\| \|\Phi\|_4 \right] \\ &\leq C \left[h^2 \|e_{1xx}\| + h^2 \|\rho + \zeta\| + \|\rho + \zeta\| \right] \|\Phi\|_4. \end{aligned}$$

Using the regularity condition $\|\Phi\|_4 \leq \|e_1\|$ of the auxiliary problem, we have that

$$\begin{aligned} \|e_1\|^2 &\leq C \left[h^2 \|e_{1xx}\| + h^2 \|\rho + \zeta\| + \|\rho + \zeta\| \right] \|e_1\|, \\ i.e., \;\; \|e_1\| &\leq C \left[h^2 \|e_{1xx}\| + h^2 \|\rho + \zeta\| + \|\rho + \zeta\| \right], \\ i.e., \;\; \|e_1\| &\leq C \left[h^2 \|e_{1xx}\| + \|v - V\| \right]. \end{aligned}$$

Squaring both sides and using Hölder's inequality, we obtain

$$\|e_1\|^2 \leq C \left[h^4 \|e_{1xx}\|^2 + \|v - V\|^2 \right].$$

Using (5.23) in the above expression, we obtain

$$\begin{aligned} \|e_1\|^2 &\leq C \left[h^4 (h^4 \|u\|_4^2 + \|v - V\|^2) + \|v - V\|^2 \right], \\ i.e., \;\; \|u - U\|^2 = \|e_1\|^2 &\leq C \left[h^8 \|u\|_4^2 + \|v - V\|^2 \right], \qquad (5.24) \\ i.e., \;\; \|u - U\|^2 = \|e_1\|^2 &\leq C \left[h^8 \|u\|_4^2 + \|\rho\|^2 + \|\zeta\|^2 \right]. \end{aligned}$$

Hence the proof is completed. ■

We have now obtained the error bound for $\|u-U\|$ in terms of the estimates of ρ and ζ. Below, we shall obtain *a priori* bound for $\|\zeta\|$ in terms of ρ and ρ_t. This will help us in computing error bounds for $u-U$ and $v-V$ in terms of h, since the estimates of ρ and ρ_t have already been obtained.

Theorem 5.4.1 *Let u and v be the weak solutions of* (5.4) *and* (5.5) *defined through* (5.6) *and* (5.7) *respectively. Further, let U and V be the corresponding semi discrete H^1-Galerkin mixed finite element formulation defined through* (5.8) *and* (5.9). *Let $u,\ v,\ v_t \in L_2(H^4)$. Then, for a sufficiently small h, we have*

$$\begin{aligned}
\|v-V\|^2_{L_\infty(L_2)} &\le Ch^8\left[\|u\|^2_{L_2(H^4)}+\|v\|^2_{L_2(H^4)}+\|v_t\|^2_{L_2(H^4)}+\|v\|^2_{L_\infty(H^4)}\right];\\
\|v-V\|^2_{L_2(H^2)} &\le Ch^4\left[\|u\|^2_{L_2(H^4)}+\|v\|^2_{L_2(H^4)}+\|v_t\|^2_{L_2(H^4)}\right];\\
\|u_{xx}-U_{xx}\|^2_{L_\infty(L_2)} &\le Ch^4\left[\|u\|^2_{L_2(H^4)}+\|v\|^2_{L_2(H^4)}+\|v_t\|^2_{L_2(H^4)}+\|v\|^2_{L_\infty(H^4)}\right];\\
\|u-U\|^2_{L_\infty(L_2)} &\le Ch^8\left[\|u\|^2_{L_2(H^4)}+\|v\|^2_{L_2(H^4)}+\|v_t\|^2_{L_2(H^4)}+\|v\|^2_{L_\infty(H^4)}\right].
\end{aligned}$$

Proof: Set $\phi_h=\zeta$ in (5.18). Then, we have

$$\begin{aligned}
(\zeta_t,\zeta)+A(\lambda:\zeta,\zeta) &= -(\rho_t,\zeta)+\lambda(\rho,\zeta)+\lambda(\zeta,\zeta)+\lambda(\rho_x,\zeta_x)+\lambda(\zeta_x,\zeta_x)\\
&- (uu_x-UU_x,\zeta_{xx})\\
&\le |-(\rho_t,\zeta)|+|\lambda(\rho,\zeta)|+|\lambda(\zeta,\zeta)|+|\lambda(\rho_x,\zeta_x)|\\
&+ |\lambda(\zeta_x,\zeta_x)|+|(uu_x-UU_x,\zeta_{xx})|\\
&\le T_{5,1}+T_{5,2}+T_{5,3}+T_{5,4}+T_{5,5}+T_{5,6}.
\end{aligned}$$

Using coercivity of $A(\lambda:\varphi,\psi)$ and since $(\zeta_t,\zeta)=\dfrac{1}{2}\dfrac{d}{dt}(\zeta,\zeta)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\zeta\|^2+\alpha_0\|\zeta\|_2^2\le T_{5,1}+T_{5,2}+T_{5,3}+T_{5,4}+T_{5,5}+T_{5,6}. \tag{5.25}$$

We now estimate the terms on the right hand side of the above equation. Using Young's inequality for $T_{5,1}$, $T_{5,2}$, $T_{5,4}$ and $T_{5,5}$, we obtain

$$\begin{aligned}
T_{5,1} &= |-(\rho_t,\zeta)|\le C\left[\frac{1}{2\epsilon}\|\rho_t\|^2+\frac{\epsilon}{2}\|\zeta\|^2\right];\\
T_{5,2} &= |\lambda(\rho,\zeta)|\le C\left[\frac{1}{2\epsilon}\|\rho\|^2+\frac{\epsilon}{2}\|\zeta\|^2\right].
\end{aligned}$$

and

$$T_{5,3} = |\lambda(\zeta,\zeta)| \leq C\|\zeta\|^2.$$

Using integration by parts and the fact that $\rho(0,t)=0$, we obtain

$$T_{5,4} = |\lambda(\rho_x,\zeta_x)| = |\lambda(\rho,\zeta_{xx})| \leq C\left[\frac{1}{2\epsilon}\|\rho\|^2 + \frac{\epsilon}{2}\|\zeta_{xx}\|^2\right].$$

Using integration by parts and the fact that $\zeta(0,t)=0$, we obtain

$$T_{5,5} = |\lambda(\zeta_x,\zeta_x)| = |\lambda(\zeta_{xx},\zeta)| \leq C\left[\frac{1}{2\epsilon}\|\zeta\|^2 + \frac{\epsilon}{2}\|\zeta_{xx}\|^2\right].$$

$$T_{5,6} = |(uu_x - UU_x, \zeta_{xx})|$$

We now temporarily assume that $\|U\|_{0,\infty} \leq K^*$. After having obtained the relevant error estimate, we can prove that this assumption is no longer a strong condition. With this temporary assumption, we then have that

$$\begin{aligned}
\|uu_x - UU_x\| &= \|uu_x - Uu_x + Uu_x - UU_x\| = \|(u-U)u_x + U(u_x - U_x)\| \\
&\leq C\|u-U\| + C(K^*)\|u_x - U_x\| \\
&\leq C(K^*)\left[\|e_1\| + \|e_{1x}\|\right] \leq C(K^*)\|e_{1x}\|,
\end{aligned}$$

where we have used Poincarè inequality for $\|e_1\|$ and the assumption $\|u_x\| \leq C$ on the exact solution.

Using Young's inequality, we obtain an estimate for $T_{5,6}$ as follows:

$$\begin{aligned}
T_{5,6} = |(uu_x - UU_x, \zeta_{xx})| &\leq C\|uu_x - UU_x\|\|\zeta_{xx}\| \\
&\leq C\left[\frac{1}{2\epsilon}\|uu_x - UU_x\|^2 + \frac{\epsilon}{2}\|\zeta_{xx}\|^2\right] \\
&\leq C(K^*)\left[\frac{1}{2\epsilon}\|e_{1x}\|^2\right] + C\frac{\epsilon}{2}\|\zeta_{xx}\|^2. \qquad (5.26)
\end{aligned}$$

Using the estimates of $T_{5,1}$, $T_{5,2}$, $T_{5,3}$, $T_{5,4}$, $T_{5,5}$ and $T_{5,6}$ in (5.25), we obtain that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\zeta\|^2 &+ \alpha_0\|\zeta\|_2^2 \leq C\left[\frac{1}{2\epsilon}\left(\|\rho_t\|^2 + \|\rho\|^2 + \|\zeta\|^2\right)\right] \\
&+ \frac{1}{2\epsilon}C(K^*)\|e_{1x}\|^2 + C\left[\|\zeta\|^2 + \epsilon\|\zeta\|^2 + \frac{3\epsilon}{2}\|\zeta_{xx}\|^2\right].
\end{aligned}$$

Using the inequalities $\|\zeta\| \le \|\zeta\|_2$ and $\|\zeta_{xx}\| \le \|\zeta\|_2$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\zeta\|^2 + \left[\alpha_0 - C\frac{5\epsilon}{2}\right]\|\zeta\|_2^2 &\le C\left[\frac{1}{2\epsilon}\left(\|\rho_t\|^2 + \|\rho\|^2\right)\right.\\ &+ \left.(1+\frac{1}{2\epsilon})\|\zeta\|^2\right] + \frac{1}{2\epsilon}C(K^*)\|e_{1x}\|^2.\end{aligned}$$

Using the fact that $\|e_{1x}\| \le \|e_{1x}\|_{L_\infty}$ and also using the estimate for $\|e_{1x}\|_{L_\infty}$ from Lemma 5.4.2, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\zeta\|^2 + \left[\alpha_0 - C\frac{5\epsilon}{2}\right]\|\zeta\|_2^2 &\le C\left[\frac{1}{2\epsilon}\left(\|\rho_t\|^2 + \|\rho\|^2\right)\right.\\ &+ \left.(1+\frac{1}{2\epsilon})\|\zeta\|^2\right]\\ &+ \frac{1}{2\epsilon}C(K^*)\left(h^4\|e_{1xx}\|^2 + \|e_2\|^2\right)\end{aligned}$$

Using (5.23) in the above and also using the fact that $\|e_2\|^2 = \|v - V\|^2 \le \|\rho\|^2 + \|\zeta\|^2$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\zeta\|^2 + \left[\alpha_0 - C\frac{5\epsilon}{2}\right]\|\zeta\|_2^2 &\le C(K^*)\left[\frac{1}{2\epsilon}\left(\|\rho_t\|^2 + \|\rho\|^2\right.\right.\\ &+ \left.h^4(h^4\|u\|_4^2 + \|\rho\|^2 + \|\zeta\|^2) + \|\rho\|^2 + \|\zeta\|^2\right)\\ &+ \left.(1+\frac{1}{2\epsilon})\|\zeta\|^2\right],\end{aligned}$$

$$\begin{aligned}i.e., \frac{d}{dt}\|\zeta\|^2 + 2\left[\alpha_0 - C\frac{5\epsilon}{2}\right]\|\zeta\|_2^2 &\le C(K^*)\left[\frac{1}{2\epsilon}\left(\|\rho_t\|^2 + \|\rho\|^2\right.\right.\\ &+ \left.\left.h^8\|u\|_4^2\right) + (1+\frac{1}{\epsilon})\|\zeta\|^2\right].\end{aligned}$$

Now, $\epsilon > 0$ can be chosen appropriately (for example $\epsilon < \dfrac{2(\alpha_0 - 1)}{5C}$) in such a way that the above inequality is maintained. Then, we have

$$\begin{aligned}\frac{d}{dt}\|\zeta\|^2 + \|\zeta\|_2^2 &\le C(K^*)\left[\|\rho_t\|^2 + \|\rho\|^2 + h^8\|u\|_4^2\right]\\ &+ C(K^*)\|\zeta\|^2. \qquad (5.27)\end{aligned}$$

Integrate (5.27) with respect to time variable t from 0 to τ with $\tau \le T$, to obtain

$$\begin{aligned}&\|\zeta\|^2 + \int_0^\tau \|\zeta\|_2^2 dt\\ &\le C\|\zeta(x,0)\|^2 + C(K^*)\int_0^\tau (h^8\|u\|_4^2 + \|\rho_t\|^2 + \|\rho\|^2)dt\\ &+ C(K^*)\int_0^\tau \|\zeta\|^2 dt. \qquad (5.28)\end{aligned}$$

From the definition of auxiliary projection, we have $\zeta(x,0) = \widehat{v}(x,0) - V(x,0) = 0$. Then an application of Gronwall's lemma in (5.28) and usage of estimates for $\|\rho\|$ and $\|\rho_t\|$ from Lemma 5.3.1, we obtain

$$\|\zeta\|^2_{L_\infty(L_2)} + \|\zeta\|^2_{L_2(H^2)} \leq C(K^*)h^8 \left[\|u\|^2_{L_2(H^4)} + \|v\|^2_{L_2(H^4)} + \|v_t\|^2_{L_2(H^4)}\right]. \quad (5.29)$$

Using the triangle inequality, estimate for $\|\rho\|$ from Lemma 5.3.1 and the above expression, we obtain

$$\begin{aligned} \|v - V\|^2_{L_\infty(L_2)} &= \|e_2\|^2_{L_\infty(L_2)} \leq \left[\|\rho\|^2_{L_\infty(L_2)} + \|\zeta\|^2_{L_\infty(L_2)}\right] \\ &\leq C(K^*)h^8 \left[\|u\|^2_{L_2(H^4)} + \|v\|^2_{L_2(H^4)} + \|v_t\|^2_{L_2(H^4)} \right. \\ &+ \left. \|v\|^2_{L_\infty(H^4)}\right] \end{aligned} \quad (5.30)$$

and

$$\begin{aligned} \|v - V\|^2_{L_2(H^2)} &= \|e_2\|^2_{L_2(H^2)} \leq \left[\|\rho\|^2_{L_2(H^2)} + \|\zeta\|^2_{L_2(H^2)}\right] \\ &\leq C(K^*)h^4 \left[\|u\|^2_{L_2(H^4)} + \|v\|^2_{L_2(H^4)} + \|v_t\|^2_{L_2(H^4)}\right] \end{aligned} \quad (5.31)$$

Applying (5.30) in (5.23), we obtain

$$\begin{aligned} \|u_{xx} - U_{xx}\|^2 = \|e_{1xx}\|^2 &\leq C(K^*)h^4 \left[\|u\|^2_{L_2(H^4)} + \|v\|^2_{L_2(H^4)} + \|v_t\|^2_{L_2(H^4)} \right. \\ &+ \left. \|v\|^2_{L_\infty(H^4)}\right]. \end{aligned} \quad (5.32)$$

Applying (5.30) in (5.24), we obtain

$$\begin{aligned} \|u - U\|^2 = \|e_1\|^2 &\leq C(K^*)h^8 \left[\|u\|^2_{L_2(H^4)} + \|v\|^2_{L_2(H^4)} + \|v_t\|^2_{L_2(H^4)} \right. \\ &+ \left. \|v\|^2_{L_\infty(H^4)}\right]. \end{aligned} \quad (5.33)$$

From (5.30), (5.31), (5.32) and (5.33), we obtain the required result.

To complete our argument, we have to show that C can be chosen independent of K^* for sufficiently small h. For that, using Lemma 5.4.1, we have

$$\begin{aligned} \|U\|_{0,\infty} &\leq \|U - u\|_{0,\infty} + \|u\|_{0,\infty} \leq \|e_1\|_{0,\infty} + \|u\|_{0,\infty} \\ &\leq C[h^2\|e_{1xx}\| + \|e_2\|] + \|u\|_{0,\infty} \\ &\leq C[h^2\|u_{xx} - U_{xx}\| + \|v - V\|] + \|u\|_{0,\infty} \\ &\leq C(K^*)h^4\|u\|_4 + \|u\|_{0,\infty} \leq C(K^*)h^4 + C. \end{aligned}$$

Now, for sufficiently small h, the above expression can be written as

$$\|U\|_{0,\infty} \leq K^*.$$

Thus, once we obtain an error estimate for $\|u_{xx} - U_{xx}\|$ and $\|v - V\|$, the boundedness of $\|U\|_{0,\infty}$ can automatically be obtained. Therefore, the temporary assumption is not a strange or strong condition. Hence, C can be chosen independent of K^*. This completes the proof. ■

Remark 5.1: Using the interpolation inequality (1.17) with $m = 2$ and $i = 1$, and the estimates of $v - V$ in L_2 norm and in H^2 norm, we can easily obtain the following result.

$$\|v - V\|_1^2 \leq Ch^6 \left[\|u\|^2_{L_2(H^4)} + \|v\|^2_{L_2(H^4)} + \|v_t\|^2_{L_2(H^4)} + \|v\|^2_{L_\infty(H^4)}\right]$$

5.5 ERROR ANALYSIS OF FULLY DISCRETE EULER BACKWARD SCHEME

In this section, we see the fully discrete approximation for the split up equations (5.4) and (5.5). We retain the H^1-Galerkin mixed method in the spatial direction and replace the time derivative by backward finite difference in the time direction. For a time step $k = T/M$ with M a positive integer, let $t^n = nk$ be the time levels for $n = 0, 1, 2...M$. For a given continuous function ψ, let $d_t\psi^{n+1} = (\psi^{n+1} - \psi^n)/k$.

Linearised fully discrete scheme:

For the weak solutions v mentioned in (5.7) and u in (5.6), we consider a linearised fully discrete Euler backward approximations W and Z: $\{t^0, t^1, ...t^M\} \rightarrow \overset{0}{S}_{h,3}$ respectively defined as follows:

$$\begin{aligned}(d_tW^{n+1}, \phi_h) &+ A(\lambda : W^{n+1}, \phi_h) - \lambda(W^{n+1}, \phi_h) - \lambda(W_x^{n+1}, \phi_{hx}) \\ &+ (Z^n Z_x^n, \phi_{hxx}) = 0, \phi_h \in \overset{0}{S}_{h,3}, n = 0, 1, 2...M - 1 \quad (5.34)\end{aligned}$$

with $W^0 = g_{xx}$ and

$$(Z_{xx}^{n+1}, \phi_{hxx}) = (W^{n+1}, \phi_{hxx}), \quad \phi_h \in \overset{0}{S}_{h,3}, \quad n = 0, 1, 2...M - 1 \qquad (5.35)$$

with $Z^0 = g$.

Let us describe the numerical scheme as follows:

Express $W^{n+1} = \sum_{j=0}^{N} \alpha_j^{n+1}\phi_j$ and $Z^{n+1} = \sum_{j=0}^{N} \beta_j^{n+1}\phi_j$, where $\{\phi_j\}_{j=0}^{N}$ is a basis of $\overset{0}{S}_{h,3}$.

Step 1: Knowing Z^n and W^n, we compute W^{n+1} (*i.e.*, α_j^{n+1}, $j = 0, 1, ...N$) using (5.34) as follows:

$$\begin{aligned} & \sum_{j=0}^{N} \left[(\phi_j, \phi_i) + kA(\lambda, \phi_j, \phi_i) - k\lambda(\phi_j, \phi_i) - k\lambda(\phi_{jx}, \phi_{ix})\right] \alpha_j^{n+1} \\ = \; & (W^n, \phi_i) - k(Z^n Z_x^n, \phi_{ixx}), \;\; i = 0, 1, ...N. \end{aligned}$$

Step 2: With the recent value of W^{n+1}, we solve for Z^{n+1} (*i.e.*, β_j^{n+1}, $j = 0, 1, ...N$) using (5.35) as follows:

$$\sum_{j=0}^{N} (\phi_{jxx}, \phi_{ixx})\beta_j^{n+1} = (W^{n+1}, \phi_{ixx}), \;\; i = 0, 1, ...N.$$

We observe that the above linearised fully discrete scheme gives rise to a system of decoupled equations. Before discussing the error estimates we first obtain the error equations for the approximations of u and v. Evaluating (5.7) at t^{n+1} and then subtracting (5.34) from the resulting equation, we obtain

$$\begin{aligned} & (v_t^{n+1} - d_t W^{n+1}, \phi_h) + A(\lambda : v^{n+1} - W^{n+1}, \phi_h) - \lambda(v^{n+1} - W^{n+1}, \phi_h) \\ - \; & \lambda(v_x^{n+1} - W_x^{n+1}, \phi_{hx}) + (u^n u_x^n - Z^n Z_x^n), \phi_{hxx}) = 0, \;\; \phi_h \in \overset{0}{S}_{h,3}. \end{aligned} \tag{5.36}$$

In a similar manner, evaluating (5.6) at t^{n+1} and then subtracting (5.35) from the resulting equation, we obtain

$$(u_{xx}^{n+1} - Z_{xx}^{n+1}, \phi_{hxx}) \;=\; (v^{n+1} - W^{n+1}, \phi_{hxx}), \;\; \phi_h \in \overset{0}{S}_{h,3}. \tag{5.37}$$

Let us denote the error between u^n and Z^n by ε_1^n and that between v^n and W^n by ε_2^n, *i.e.*, $\varepsilon_1^n = u^n - Z^n$ and $\varepsilon_2^n = v^n - W^n$.

Let $v^n - W^n = v^n - \widehat{v}^n + \widehat{v}^n - W^n = \rho^n + \zeta^n$, where $\widehat{v}^n$ is defined earlier as in (5.10). Then (5.36) can be written as

$$\begin{aligned} & (v_t^{n+1} - d_t W^{n+1}, \phi_h) + A(\lambda : \rho^{n+1} + \zeta^{n+1}, \phi_h) - \lambda(\rho^{n+1} + \zeta^{n+1}, \phi_h) \\ - \; & \lambda(\rho_x^{n+1} + \zeta_x^{n+1}, \phi_{hx}) + (u^n u_x^n - Z^n Z_x^n, \phi_{hxx}) = 0, \;\; \phi_h \in \overset{0}{S}_{h,3}. \end{aligned} \tag{5.38}$$

But the term $v_t^{n+1} - d_t W^{n+1}$ can be written as

$$\begin{aligned} v_t^{n+1} - d_t W^{n+1} &= v_t^{n+1} - d_t v^{n+1} + d_t v^{n+1} - d_t W^{n+1} \\ &= \sigma_{n+1} + d_t(\rho^{n+1} + \zeta^{n+1}), \end{aligned}$$

where $\sigma_{n+1} = v_t^{n+1} - d_t v^{n+1}$.

Using the projection (5.11) in (5.38) and the above expression, we obtain the following error equation:

$$\begin{aligned} (d_t \zeta^{n+1}, \phi_h) &+ A(\lambda : \zeta^{n+1}, \phi_h) = -(d_t \rho^{n+1}, \phi_h) - (\sigma_{n+1}, \phi_h) \\ &+ \lambda(\rho^{n+1} + \zeta^{n+1}, \phi_h) + \lambda(\rho_x^{n+1} + \zeta_x^{n+1}, \phi_{hx}) \\ &- (u^n u_x^n - Z^n Z_x^n, \phi_{hxx}), \ \ \phi_h \in \overset{0}{S}_{h,3}. \end{aligned} \tag{5.39}$$

In a similar way, (5.37) can be written as

$$(u_{xx}^{n+1} - Z_{xx}^{n+1}, \phi_{hxx}) = (\rho^{n+1} + \zeta^{n+1}, \phi_{hxx}), \ \ \phi_h \in \overset{0}{S}_{h,3}. \tag{5.40}$$

In the following lemma, We compute $|\varepsilon_1^n(\overline{x}|$, where $\overline{x}$ is an arbitrary point in $[0,1]$, the proof of which is similar to the proof of Lemma 5.4.1. This result is used in the proof of the main theorem of this section.

Lemma 5.5.1 *Let u and v be the weak solutions of* (5.4) *and* (5.5) *defined through* (5.6) *and* (5.7) *respectively. Further, let Z^n and W^n be the corresponding semi discrete H^1-Galerkin mixed finite element formulation defined through* (5.35) *and* (5.34). *Also, let $\widehat{v}^n$ be the auxiliary projection of v^n as in (5.10). Then the error $\varepsilon_1^n = u^n - Z^n$ satisfies*

$$|\varepsilon_1^n(\overline{x})| \leq C\left[h^2 \, \|\varepsilon_{1xx}^n\| + \|\varepsilon_2\|\right],$$

where $\overline{x}$ is an arbitrary point in $[0,1]$.

In the following lemma, we compute $|\varepsilon_{1x}^n(\overline{x})|$, where $\overline{x}$ is an arbitrary point in $[0,1]$, the proof of which is similar to the proof of Lemma 5.4.2. This result is used in the proof of the main theorem of this section.

Lemma 5.5.2 *Let u and v be the weak solutions of* (5.4) *and* (5.5) *defined through* (5.6) *and* (5.7) *respectively. Further, let Z^n and W^n be the corresponding semi discrete H^1-Galerkin mixed finite element formulation defined through* (5.35) *and* (5.34). *Then the error $\varepsilon_1^n = u^n - Z^n$ satisfies*

$$|\varepsilon_{1x}^n(\overline{x})| \le C\left[h^2 \|\varepsilon_{1xx}^n\| + \|\varepsilon_2^n\|\right],$$

where $\overline{x}$ is an arbitrary point in $[0,1]$.

Below, we discuss the analysis of the error involved in the fully discrete scheme. We first compute the error estimate $\|u^n - Z^n\|$ in the following lemma, the proof of which is similar to the proof of Lemma 5.4.3.

Lemma 5.5.3 *Let u and v be the weak solutions of* (5.4) *and* (5.5) *defined through* (5.6) *and* (5.7) *respectively. Further, let Z^n and W^n be the corresponding semi discrete H^1-Galerkin mixed finite element formulation defined through* (5.35) *and* (5.34). *Also, let $\widehat{v}^n$ be the auxiliary projection of v^n as in (5.10). Then, we have*

$$\begin{aligned}\|u^n - Z^n\|^2 &\le C\left[h^8\|u^n\|_4^2 + \|\rho^n\|^2 + \|\zeta^n\|^2\right], \ \ \textit{where} \ \ \rho^n = v^n - \widehat{v}^n,\\ &\zeta^n = \widehat{v}^n - W^n, \ \ n = 0, 1, 2, ..., M.\end{aligned}$$

Proof: In (5.37), choose $\phi_h = (u^{n+1} - Z^{n+1}) - (u^{n+1} - \chi)$ for some $\chi \in \overset{0}{S}_{h,3}$. Then it becomes

$$\begin{aligned}(u_{xx}^{n+1} - Z_{xx}^{n+1}, u_{xx}^{n+1} - Z_{xx}^{n+1}) &= (u_{xx}^{n+1} - Z_{xx}^{n+1}, u_{xx}^{n+1} - \chi_{xx})\\ &+ (v^{n+1} - W^{n+1}, u_{xx}^{n+1} - Z_{xx}^{n+1})\\ &- (v^{n+1} - W^{n+1}, u_{xx}^{n+1} - \chi_{xx}).\end{aligned}$$

$$\begin{aligned}\|u_{xx}^{n+1} - Z_{xx}^{n+1}\|^2 &\le \|u_{xx}^{n+1} - Z_{xx}^{n+1}\|\|u_{xx}^{n+1} - \chi_{xx}\|\\ &+ \|v^{n+1} - W^{n+1}\|\|u_{xx}^{n+1} - Z_{xx}^{n+1}\|\\ &+ \|v^{n+1} - W^{n+1}\|\|u_{xx}^{n+1} - \chi_{xx}\|.\end{aligned}$$

Taking infimum over $\chi \in \overset{0}{S}_{h,3}$ first and then applying the approximation property, we obtain

$$\|u_{xx}^{n+1} - Z_{xx}^{n+1}\|^2 \le \|u_{xx}^{n+1} - Z_{xx}^{n+1}\| \inf_{\chi \in \overset{0}{S}_{h,3}} \|u_{xx}^{n+1} - \chi_{xx}\|$$

$$
\begin{aligned}
&+ \quad \|v^{n+1} - W^{n+1}\| \|u_{xx}^{n+1} - Z_{xx}^{n+1}\| \\
&+ \quad \|v^{n+1} - W^{n+1}\| \inf_{\chi \in \overset{0}{S}_{h,3}} \|u_{xx}^{n+1} - \chi_{xx}\| \\
&\leq \quad C \Big[\|u_{xx}^{n+1} - Z_{xx}^{n+1}\| \, h^2 \, \|u^{n+1}\|_4 \\
&+ \quad \|v^{n+1} - W^{n+1}\| \|u_{xx}^{n+1} - Z_{xx}^{n+1}\| \\
&+ \quad \|v^{n+1} - W^{n+1}\| \, h^2 \, \|u^{n+1}\|_4 \Big].
\end{aligned}
$$

Using Young's inequality, we have

$$
\begin{aligned}
\|u_{xx}^{n+1} - Z_{xx}^{n+1}\|^2 \quad &\leq \quad C \left[\frac{\epsilon}{2} \|u_{xx}^{n+1} - Z_{xx}^{n+1}\|^2 + \frac{1}{2\epsilon} h^4 \|u^{n+1}\|_4^2 \right. \\
&+ \quad \frac{\epsilon}{2} \|u_{xx}^{n+1} - Z_{xx}^{n+1}\|^2 + \frac{1}{2\epsilon} \|v^{n+1} - W^{n+1}\|^2 \\
&+ \quad \left. \frac{\epsilon}{2} \|v^{n+1} - W^{n+1}\|^2 + h^4 \frac{1}{2\epsilon} \|u\|_4^2 \right], \\
(1 - C\epsilon) \|u_{xx}^{n+1} - Z_{xx}^{n+1}\|^2 \quad &\leq \quad C \left[\frac{1}{\epsilon} h^4 \|u^{n+1}\|_4^2 + \left(\frac{1}{2\epsilon} + \frac{\epsilon}{2} \right) \|v^{n+1} - W^{n+1}\|^2 \right].
\end{aligned}
$$

Choosing $\epsilon > 0$ properly (for example, $\epsilon = 1/2C$) so that inequality is maintained, we obtain

$$
\|\varepsilon_{1xx}^{n+1}\|^2 = \|u_{xx}^{n+1} - Z_{xx}^{n+1}\|^2 \leq C \left[h^4 \|u^{n+1}\|_4^2 + \|v^{n+1} - W^{n+1}\|^2 \right]. \tag{5.41}
$$

We now apply the following duality argument to obtain the estimate of ε_1^{n+1} in L_2 norm.

Let Φ be the solution of the auxiliary problem

$$
\Phi_{xxxx} = u^{n+1} - Z^{n+1} = \varepsilon_1^{n+1}, \quad x \in I
$$

with the boundary conditions

$$
\Phi_{xxx}(0) = \Phi_{xxx}(1) = 0, \; \Phi_{xx}(0) = \Phi_{xx}(1) = 0.
$$

Then, using integration by parts, the boundary conditions and (5.40), we obtain for $\chi \in \overset{0}{S}_{h,3}$,

$$
\begin{aligned}
(\varepsilon_1^{n+1}, \varepsilon_1^{n+1}) \quad &= \quad (\Phi_{xxxx}, \varepsilon_1^{n+1}) = (\Phi_{xx}, \varepsilon_{1xx}^{n+1}) = (\varepsilon_{1xx}^{n+1}, \Phi_{xx}) \\
&= \quad (\varepsilon_{1xx}^{n+1}, \Phi_{xx} - \chi_{xx}) + (\varepsilon_{1xx}^{n+1}, \chi_{xx}) \\
&= \quad (\varepsilon_{1xx}^{n+1}, \Phi_{xx} - \chi_{xx}) + (\rho^{n+1} + \zeta^{n+1}, \chi_{xx})
\end{aligned}
$$

$$
\begin{aligned}
&= (\varepsilon_{1xx}^{n+1}, \Phi_{xx} - \chi_{xx}) + (\rho^{n+1} + \zeta^{n+1}, \chi_{xx} - \Phi_{xx}) \\
&+ (\rho^{n+1} + \zeta^{n+1}, \Phi_{xx}) \\
&\leq \|\varepsilon_{1xx}^{n+1}\| \|\Phi_{xx} - \chi_{xx}\| + \|\rho^{n+1} + \zeta^{n+1}\| \|\chi_{xx} - \Phi_{xx}\| \\
&+ \|\rho^{n+1} + \zeta^{n+1}\| \|\Phi_{xx}\|
\end{aligned}
$$

Taking infimum over $\chi \in \overset{0}{S}_{h,3}$ first and then applying the approximation property, we obtain

$$
\begin{aligned}
\|\varepsilon_1^{n+1}\|^2 &\leq C\left[h^2 \|\varepsilon_{1xx}^{n+1}\| \|\Phi\|_4 + h^2 \|\rho^{n+1} + \zeta^{n+1}\| \|\Phi\|_4\right. \\
&+ \left.\|\rho^{n+1} + \zeta^{n+1}\| \|\Phi\|_4\right] \\
&\leq C\left[h^2 \|\varepsilon_{1xx}^{n+1}\| + h^2 \|\rho^{n+1} + \zeta^{n+1}\| + \|\rho^{n+1} + \zeta^{n+1}\|\right] \|\Phi\|_4.
\end{aligned}
$$

Using the regularity of the auxiliary problem $\|\Phi\|_4 \leq \|\varepsilon_1^{n+1}\|$, we have

$$
\begin{aligned}
\|\varepsilon_1^{n+1}\|^2 &\leq C\left[h^2 \|\varepsilon_{1xx}^{n+1}\| + h^2 \|\rho^{n+1} + \zeta^{n+1}\| + \|\rho^{n+1} + \zeta^{n+1}\|\right] \|\varepsilon_1^{n+1}\|, \\
i.e.,\ \|\varepsilon_1^{n+1}\| &\leq C\left[h^2 \|\varepsilon_{1xx}^{n+1}\| + h^2 \|\rho^{n+1} + \zeta^{n+1}\| + \|\rho^{n+1} + \zeta^{n+1}\|\right], \\
i.e.,\ \|\varepsilon_1^{n+1}\| &\leq C\left[h^2 \|\varepsilon_{1xx}^{n+1}\| + \|v^{n+1} - W^{n+1}\|\right].
\end{aligned}
$$

Squaring both sides and using Hölder's inequality, we obtain

$$
\|\varepsilon_1^{n+1}\|^2 \leq C\left[h^4 \|\varepsilon_{1xx}^{n+1}\|^2 + \|v^{n+1} - W^{n+1}\|^2\right].
$$

Using (5.41) in the above expression, we obtain

$$
\|\varepsilon_1^{n+1}\|^2 \leq C\left[h^4 (h^4 \|u^{n+1}\|_4^2 + \|v^{n+1} - W^{n+1}\|^2) + \|v^{n+1} - W^{n+1}\|^2\right],
$$

$$
\begin{aligned}
i.e., \|u^{n+1} - Z^{n+1}\|^2 = \|\varepsilon_1^{n+1}\|^2 &\leq C\left[h^8 \|u^{n+1}\|_4^2 + \|v^{n+1} - W^{n+1}\|^2\right], \\
&n = 0,\ 1,\ 2,\ ...,\ M-1.
\end{aligned}
$$

Now, since $u^0 = Z^0$, we have that

$$
\begin{aligned}
\|u^n - Z^n\|^2 = \|\varepsilon_1^n\|^2 &\leq C\left[h^8 \|u^n\|_4^2 + \|\rho^n\|^2 + \|\zeta^n\|^2\right], \\
&n = 0,\ 1,\ 2,\ ...,\ M. \qquad (5.42)
\end{aligned}
$$

Hence the proof is completed. ■

We have now computed the error bound of $\|u^n - Z^n\|$ in terms of the estimates of ρ^n and ζ^n at time level n. Below, we shall obtain *a priori* bound for $\|\zeta^n\|$. This will help us in computing the error bounds of $(u^n - Z^n)$ and $(v^n - W^n)$ in terms of h and the regularity condition, since the estimates of ρ^n has already been obtained.

Theorem 5.5.1 *Let u and v be the weak solutions of* (5.4) *and* (5.5) *defined through* (5.6) *and* (5.7) *respectively. Further, let Z^n and W^n be the corresponding semi discrete H^1-Galerkin mixed finite element formulation defined through* (5.35) *and* (5.34)*. Then, for a sufficiently small h, the error in the fully discrete approximation of u and v by the backward Euler scheme is given by*

$$\begin{aligned}
\|u^{J+1} - Z^{J+1}\| &\leq C\left[k^{\frac{3}{2}} + h^4\right]; \\
\|u_{xx}^{J+1} - Z_{xx}^{J+1}\| &\leq C\left[k^{\frac{3}{2}} + h^2\right]; \\
\|v^{J+1} - W^{J+1}\| &\leq C\left[k^{\frac{3}{2}} + h^4\right] \text{ for } J = 0, 1, 2, ..., M-1,
\end{aligned}$$

where C is a generic constant depending only on u and v.

Proof: Substituting $\phi_h = \zeta^{n+1}$ in (5.39) and using coercivity of $A(\lambda : \varphi, \psi)$, we obtain

$$\begin{aligned}
(d_t\zeta^{n+1}, \zeta^{n+1}) &+ \alpha_0\|\zeta^{n+1}\|_2^2 \leq |(d_t\rho^{n+1}, \zeta^{n+1})| + |(\sigma_{n+1}, \zeta^{n+1})| \\
&+ |\lambda(\rho^{n+1}, \zeta^{n+1})| + |\lambda(\zeta^{n+1}, \zeta^{n+1})| + |\lambda(\rho_x^{n+1}, \zeta_x^{n+1})| \\
&+ |\lambda(\zeta_x^{n+1}, \zeta_x^{n+1})| + \left|\left(u^n u_x^n - Z^n Z_x^n, \zeta_{xx}^{n+1}\right)\right|. \qquad (5.43)
\end{aligned}$$

For the first term on the left hand side, using Young's inequality with $\epsilon = 1$, we have

$$(d_t\zeta^{n+1}, \zeta^{n+1}) = \frac{1}{k}(\zeta^{n+1} - \zeta^n, \zeta^{n+1}) = \frac{1}{k}\{\|\zeta^{n+1}\|^2 - (\zeta^{n+1}, \zeta^n)\} \geq \frac{1}{2}d_t\|\zeta^{n+1}\|^2.$$

Thus,

$$\frac{1}{2}d_t\|\zeta^{n+1}\|^2 \leq (d_t\zeta^{n+1}, \zeta^{n+1}).$$

Hence, (5.43) becomes

$$\begin{aligned}
\frac{\|\zeta^{n+1}\|^2 - \|\zeta^n\|^2}{2k} + \alpha_0\|\zeta^{n+1}\|_2^2 &\leq T_{5,1}^{n+1} + T_{5,2}^{n+1} + T_{5,3}^{n+1} + T_{5,4}^{n+1} \\
&+ T_{5,5}^{n+1} + T_{5,6}^{n+1} + T_{5,7}^{n+1}.
\end{aligned}$$

Summing the above from n=0, 1, 2, 3,...J after multiplying both sides by $2k$, we obtain

$$\begin{aligned}\|\zeta^{J+1}\|^2 - \|\zeta^0\|^2 &+ 2k\alpha_0 \sum_{n=0}^{J} \|\zeta^{n+1}\|_2^2 \\ &\leq 2k \sum_{n=0}^{J} \Big(T_{5,1}^{n+1} + T_{5,2}^{n+1} + T_{5,3}^{n+1} + T_{5,4}^{n+1} \\ &+ T_{5,5}^{n+1} + T_{5,6}^{n+1} + T_{5,7}^{n+1} \Big). \end{aligned} \tag{5.44}$$

We now estimate the terms on the right hand side of the above expression. For the term $T_{5,1}^{n+1} = |(d_t\rho^{n+1}, \zeta^{n+1})|$, we use Young's inequality to obtain

$$kT_{5,1}^{n+1} \leq C\left[\frac{1}{2\epsilon}k\|d_t\rho^{n+1}\|^2 + \frac{\epsilon}{2}k\|\zeta^{n+1}\|^2\right]. \tag{5.45}$$

We compute the error bound for $T_{5,2}^{n+1} = |(\sigma_{n+1}, \zeta^{n+1})|$ as follows. Recall that

$$\sigma_{n+1} = v_t^{n+1} - d_t v^{n+1}.$$

From the Taylor series expansion

$$v^{n+1} = v^n + \frac{k}{1!}v_t^n + \frac{k^2}{2!}v_{tt}^n(x, \theta_1) \text{ for } t_n < \theta_1 < t_{n+1}.$$

we have that

$$d_t v^{n+1} = \frac{v^{n+1} - v^n}{k} = v_t^n + \frac{k}{2!}v_{tt}^n(x, \theta_1) \text{ for } t_n < \theta_1 < t_{n+1}.$$

Using Taylor series expansion $v_t^{n+1} = v_t^n + \frac{k}{1!}v_{tt}^n(x, \theta_2)$ for $t_n < \theta_2 < t_{n+1}$ for the first term, we obtain

$$\sigma_{n+1} = kv_{tt}^n(x, \theta_2) - \frac{k}{2}v_{tt}^n(x, \theta_1) \text{ for } t_n < \theta_1 < t_{n+1},\ t_n < \theta_2 < t_{n+1}.$$

Therefore,

$$\|\sigma_{n+1}\| \leq k\|v_{tt}^n(x, \theta_2)\| + \frac{k}{2}\|v_{tt}^n(x, \theta_1)\| \text{ for } t_n < \theta_1 < t_{n+1},\ t_n < \theta_2 < t_{n+1}.$$

Hence, we have that

$$\|\sigma_{n+1}\|^2 \leq Ck^2\|v_{tt}^n\|_{L_\infty(L_2)}^2. \tag{5.46}$$

For the term $|T_{5,2}^{n+1}|$, using Young's inequality, we obtain

$$T_{5,2}^{n+1} \leq C\left[\frac{1}{2\epsilon}\|\sigma_{n+1}\|^2 + \frac{\epsilon}{2}\|\zeta^{n+1}\|^2\right].$$

On substituting (5.46) in the above, we have that

$$kT_{5,2}^{n+1} \leq C\left[\frac{1}{2\epsilon}k^3\|v_{tt}^n\|_{L_\infty(L_2)}^2 + k\frac{\epsilon}{2}\|\zeta^{n+1}\|^2\right]. \tag{5.47}$$

Similarly, an application of Young's inequality gives the estimates of the term $T_{5,3}^{n+1} = |\lambda(\rho^{n+1}, \zeta^{n+1})|$ as follows:

$$kT_{5,3}^{n+1} \leq C\left[\frac{1}{2\epsilon}k\|\rho^{n+1}\|^2 + k\frac{\epsilon}{2}\|\zeta^{n+1}\|^2\right]. \tag{5.48}$$

Further, for $T_{5,4}^{n+1} = |\lambda(\zeta^{n+1}, \zeta^{n+1})|$, we obtain

$$kT_{5,4}^{n+1} = Ck\|\zeta^{n+1}\|^2. \tag{5.49}$$

Using integration by parts for the term $T_{5,5}^{n+1} = |\lambda(\rho_x^{n+1}, \zeta_x^{n+1})|$ and the fact that $\rho^{n+1}(0) = 0$, we obtain

$$T_{5,5}^{n+1} = |\lambda(\rho^{n+1}, \zeta_{xx}^{n+1})|.$$

An application of Young's inequality gives

$$kT_{5,5}^{n+1} \leq C\left[\frac{1}{2\epsilon}k\|\rho^{n+1}\|^2 + k\frac{\epsilon}{2}\|\zeta_{xx}^{n+1}\|^2\right]. \tag{5.50}$$

Further, using integration by parts for the term $T_{5,6}^{n+1} = |\lambda(\zeta_x^{n+1}, \zeta_x^{n+1})|$ and the fact that $\zeta^{n+1}(0) = 0$, we obtain

$$kT_{5,6}^{n+1} \leq Ck|(\zeta^{n+1}, \zeta_{xx}^{n+1})|.$$

Hence using Young's inequality, we obtain

$$kT_{5,6}^{n+1} \leq Ck\left[\frac{1}{2\epsilon}\|\zeta^{n+1}\|^2 + \frac{\epsilon}{2}\zeta_{xx}^{n+1}\|^2\right]. \tag{5.51}$$

For the estimate of $T_{5,7}^{n+1} = |(u^n u_x^n - Z^n Z_x^n, \zeta_{xx}^{n+1})|$, we observe that,

$$T_{5,7}^{n+1} = \left|\left(u^n u_x^n - Z^n Z_x^n, \zeta_{xx}^{n+1}\right)\right|. \tag{5.52}$$

As done in semidiscrete case, we now temporarily assume that $\|Z^n\| \leq K^*$. After having obtained the relevant error estimate, we prove that this assumption is no longer a strong condition. With this assumption we then have that,

$$\begin{aligned}
\|u^n u_x^n - ZZ_x^n\| &= \|u^n u_x^n - Z^n u_x^n + Z^n u_x^n - Z^n Z_x^n\| \\
&= \|(u^n - Z^n)u_x^n + Z^n(u_x^n - Z_x^n)\| \\
&\leq C\|u^n - Z^n\| + C(K^*)\|u_x^n - Z_x^n\| \\
&\leq C(K^*)\left[\|\varepsilon_1^{n+1}\| + \|\varepsilon_{1x}^{n+1}\|\right] \leq C(K^*)\|\varepsilon_{1x}^{n+1}\|,
\end{aligned}$$

where we have used Poincarè inequality for $\|\varepsilon_1^{n+1}\|$ and the assumption $\|u_x^n\| \leq C$ on the exact solution. Using Young's inequality, we obtain an estimate for $|T_{5,7}^{n+1}|$ as follows:

$$\begin{aligned}
T_{5,7}^{n+1} = |(u^n u_x^n - Z^n Z_x^n, \zeta_{xx}^{n+1})| &\leq C\|u^n u_x^n - Z^n Z_x^n\|\|\zeta_{xx}^{n+1}\| \\
&\leq C\left[\frac{1}{2\epsilon}\|u^n u_x^n - Z^n Z_x^n\|^2 + \frac{\epsilon}{2}\|\zeta_{xx}\|^2\right] \\
&\leq C(K^*)\left[\frac{1}{2\epsilon}\|\varepsilon_{1x}^{n+1}\|^2\right] + C\frac{\epsilon}{2}\|\zeta_{xx}^{n+1}\|^2.
\end{aligned}$$

Hence,

$$kT_{5,7}^{n+1} \leq C(K^*)k\left[\frac{1}{2\epsilon}\|\varepsilon_{1x}^{n+1}\|^2\right] + Ck\frac{\epsilon}{2}\|\zeta_{xx}^{n+1}\|^2. \tag{5.53}$$

Using the estimates of $T_{5,1}^{n+1}$, $T_{5,2}^{n+1}$, $T_{5,3}^{n+1}$, $T_{5,4}^{n+1}$, $T_{5,5}^{n+1}$, $T_{5,6}^{n+1}$ and $T_{5,7}^{n+1}$ in (5.44), we obtain

$$\begin{aligned}
\|\zeta^{J+1}\|^2 &- \|\zeta^0\|^2 + 2k\sum_{n=0}^{J}\alpha_0\|\zeta^{n+1}\|_2^2 \leq C\left[\frac{1}{2\epsilon}\left(k\sum_{n=0}^{J}\|d_t\rho^{n+1}\|^2\right.\right. \\
&+ \left.\left. k^3\sum_{n=0}^{J}\|v_{tt}^n\|_{L_\infty(L_2)}^2 + k\sum_{n=0}^{J}\|\rho^{n+1}\|^2 + k\sum_{n=0}^{J}\|\zeta^{n+1}\|^2\right)\right] \\
&+ C(K^*)\frac{1}{2\epsilon}k\sum_{n=0}^{J}\|\varepsilon_{1x}^{n+1}\|^2 \\
&+ C\left[\frac{3\epsilon}{2}k\sum_{n=0}^{J}\|\zeta^{n+1}\|^2 + k\sum_{n=0}^{J}\|\zeta^{n+1}\|^2 + k\frac{3\epsilon}{2}\sum_{n=0}^{J}\|\zeta_{xx}^{n+1}\|^2\right].
\end{aligned}$$

We now use the inequalities $\|\zeta^{n+1}\| \leq \|\zeta^{n+1}\|_2$ and $\|\zeta_{xx}^{n+1}\| \leq \|\zeta^{n+1}\|_2$. Using the result of Lemma 5.5.2, we obtain $\|\varepsilon_{1x}^{n+1}\| \leq \|\varepsilon_{1x}^{n+1}\|_{L_\infty} \leq C\left[h^2\|\varepsilon_{1xx}^{n+1}\| + \|\varepsilon_2^{n+1}\|\right]$.

Substituting all these in the above, then using (5.41) and from the fact that $\|\varepsilon_2^{n+1}\|^2 \leq \|\rho^{n+1}\|^2 + \|\zeta^{n+1}\|^2$, we obtain

$$\begin{aligned} & \|\zeta^{J+1}\|^2 - \|\zeta^0\|^2 + k(2\alpha_0 - 3C\epsilon)\sum_{n=0}^{J}\|\zeta^{n+1}\|_2^2 \\ \leq\ & C(K^*)\left[\frac{1}{2\epsilon}\left(k\sum_{n=0}^{J}\|d_t\rho^{n+1}\|^2 + k^3\sum_{n=0}^{J}\|v_{tt}^n\|_{L^\infty(L^2)}^2 + k\sum_{n=0}^{J}\|\rho^{n+1}\|^2\right.\right. \\ +\ & \left.\left. k\sum_{n=0}^{J}\left(h^8\|u^{n+1}\|_4^2 + \|\rho^{n+1}\|^2\right)\right) + k(1+\frac{1}{\epsilon})\sum_{n=0}^{J}\|\zeta^{n+1}\|^2\right]. \end{aligned}$$

Bringing $C(K^*)k(1+\frac{1}{\epsilon})\|\zeta^{J+1}\|^2$ term to the left hand side, we obtain

$$\begin{aligned} & \left(1 - C(K^*)k(1+\frac{1}{\epsilon})\right)\|\zeta^{J+1}\|^2 - \|\zeta^0\|^2 + k(2\alpha_0 - 3C\epsilon)\sum_{n=0}^{J}\|\zeta^{n+1}\|_2^2 \\ \leq\ & C(K^*)\left[\frac{1}{2\epsilon}\left(k\sum_{n=0}^{J}\|d_t\rho^{n+1}\|^2 + k^3\sum_{n=0}^{J}\|v_{tt}^n\|_{L_\infty(L_2)}^2\right.\right. \\ +\ & \left.\left. k\sum_{n=0}^{J}\|\rho^{n+1}\|^2 + k\sum_{n=0}^{J}\left(h^8\|u^{n+1}\|_4^2 + \|\rho^{n+1}\|^2\right)\right) + k(1+\frac{1}{\epsilon})\sum_{n=0}^{J}\|\zeta^{n}\|^2\right]. \end{aligned}$$

Choose $\epsilon > 0$ properly (for example, $\epsilon < \dfrac{(2\alpha_0 - 1)}{3C}$) so that the inequality is maintained. After having chosen such an ϵ, select k sufficiently small so that

$$\left(1 - C(K^*)(1+\frac{1}{\epsilon})k\right) > 0.$$

Then, using discrete version of Gronwall's inequality and the fact that $\zeta^0 = 0$, we obtain

$$\begin{aligned} & \|\zeta^{J+1}\|^2 + k\sum_{n=0}^{J}\|\zeta^{n+1}\|_2^2 \leq C(K^*)\left[k\sum_{n=0}^{J}\|d_t\rho^{n+1}\|^2 + k^3\sum_{n=0}^{J}\|v_{tt}^n\|_{L_\infty(L_2)}^2\right. \\ +\ & \left. k\sum_{n=0}^{J}\|\rho^{n+1}\|^2 + k\sum_{n=0}^{J}\left(h^8\|u^{n+1}\|_4^2 + \|\rho^{n+1}\|^2\right)\right]. \end{aligned} \tag{5.54}$$

The estimates of $\|\rho^{n+1}\|$ and $\|kd_t\rho^{n+1}\|$ can be obtained from the results of Lemma 5.3.1 since $kd_t\rho^{n+1} = \rho^{n+1} - \rho^n$. Thus

$$\|\zeta^{J+1}\|^2 \leq O\left[k^3 + h^8\right];$$

and therefore, we have that

$$\|\zeta^{J+1}\| \leq O\left[k^{\frac{3}{2}} + h^4\right]. \tag{5.55}$$

Using the estimate for $\|\rho^{n+1}\|$ from Lemma 5.3.1 and triangle inequality, we obtain the required result for $\|\varepsilon_2\|$ as follows:

$$\|v^{J+1} - W^{J+1}\| = \|\varepsilon_2^{J+1}\| \leq O\left[k^{\frac{3}{2}} + h^4\right]. \tag{5.56}$$

Using (5.56) in (5.42), we get

$$\|u_{xx}^{J+1} - Z_{xx}^{J+1}\| \leq O\left[k^{\frac{3}{2}} + h^2\right]. \tag{5.57}$$

Further, using (5.55) in (5.42), we obtain that

$$\|u^{J+1} - Z^{J+1}\| \leq O\left[k^{\frac{3}{2}} + h^4\right]. \tag{5.58}$$

We obtain the required result from (5.56) to (5.58) .

To complete our argument, we have to show that C can be chosen independent of K^* for sufficiently small h. For that, using Lemma 5.5.1, we have

$$\begin{aligned}
\|Z^n\|_{0,\infty} &\leq \|Z^n - u^n\|_{0,\infty} + \|u^n\|_{0,\infty} \leq \|\varepsilon_1^n\|_{0,\infty} + \|u^n\|_{0,\infty} \\
&\leq C[h^2\|\varepsilon_{1xx}^n\| + \|\varepsilon_2^n\|] + \|u^n\|_{0,\infty} \\
&\leq C[h^2\|u_{xx}^n - Z_{xx}^n\| + \|v^n - W^n\|] + \|u^n\|_{0,\infty} \\
&\leq C(K^*)h^4\|u^n\|_4 + \|u^n\|_{0,\infty} \leq C(K^*)h^4 + C.
\end{aligned}$$

Now, for sufficiently small h, the above expression can be written as

$$\|Z^n\|_{0,\infty} \leq K^*.$$

Once we obtain an error estimate for $\|u_{xx}^n - Z_{xx}^n\|$ and $\|v^n - W^n\|$, the boundedness of $\|Z^n\|_{0,\infty}$ can automatically be obtained. Therefore, the temporary assumption is not a strange or strong condition Hence, C can be chosen independent of K^*. This completes the proof. ■

Remark 5.2:

We have assumed that $\zeta^0 = 0$.

If $\zeta^0 \neq 0$, then

$$\|\zeta^0\| = \|\hat{v}^0 - W^0\| = \|g_{xx} - W^0\| \leq h^4\|g_{xx}\|_4.$$

Then for an optimal order error estimate, it is demanded that $g \in H^6(I)$.

Remark 5.3:

From (4.51), we also have that

$$k \sum_{n=0}^{J} \|\zeta^{n+1}\|_2^2 \leq O\left[k^3 + h^8\right].$$

Using the estimate for $\|\rho^{n+1}\|_2^2$ from Lemma 5.3.1 and triangle inequality, we obtain

$$k \sum_{n=0}^{J} \|v^{n+1} - W^{n+1}\|_2^2 \leq O\left[k^3 + h^4\right].$$

Remark 5.4:

Following are the advantages of the method described in this chapter.

1. For the classical solutions of the Kuramoto -Sivashinsky equation, fourth order smoothness is required. The method described in this chapter requires sixth order regularity on the solution. But there are methods, for example, orthogonal cubic spline collocation method which demand eighth order regularity on the solution as given in the literature (Manickam *et al.* 1998).

2. The size of the combined linear system is $8n + 4$ in (Manickam *et al.* 1998), where as the size of the decoupled system in the present method is $n + 1$ each ($i.e.$,a total of $2n + 2$). This is clearly explained in the next section, where the linear fully discrete scheme is described.

5.6 NUMERICAL EXPERIMENTS

Though we do not impose any assumption on the partition of the interval, for the purpose of implementation we consider a partition with uniform spacing. Using the modified cubic splines as defined in Section 4.6 of Chapter 4, let us now describe the numerical scheme as follows:

Express $W^{n+1} = \sum_{j=0}^{N} \alpha_j^{n+1} \widehat{B}_j$ and $Z^{n+1} = \sum_{j=0}^{N} \beta_j^{n+1} \widehat{B}_j$, where $\{\widehat{B}_j\}_{j=0}^{N}$ is a basis of $\overset{0}{S}_{h,3}$.

Step 1: Knowing Z^n and W^n, we compute W^{n+1} ($i.e.$, α_j^{n+1}, $j = 0, 1, ...N$) using

(5.34) as follows:

$$\sum_{j=0}^{N}\left[(\widehat{B}_j,\widehat{B}_i)+kA(\lambda,\widehat{B}_j,\widehat{B}_i)-k\lambda(\widehat{B}_j,\widehat{B}_i)-k\lambda(\widehat{B}_{jx},\widehat{B}_{ix})\right]\alpha_j^{n+1}$$
$$= (W^n,\widehat{B}_i)-k(Z^nZ_x^n),\widehat{B}_{ixx}),\ \ i=0,\,1,\,...N.$$

Step 2: With the recent value of W^{n+1}, we solve for Z^{n+1} ($i.e.,\ \ \beta_j^{n+1},\ j=0,\,1,\,...N$) using (5.35) as follows:

$$\sum_{j=0}^{N}(\widehat{B}_{jxx},\widehat{B}_{ixx})\beta_j^{n+1}=(W^{n+1},\widehat{B}_{ixx}),\ \ i=0,\,1,\,...N.$$

The above two steps can equivalently be posed as solving the following systems of linear equations:

We first solve the system

$$B\overline{\alpha}=\overline{C}$$

for $\overline{\alpha}=(\alpha_0^{n+1},\alpha_1^{n+1},...\alpha_N^{n+1})^T$, where $B=\{b_{ij}\}_{i,j=0}^N$ with

$$b_{ij}=(\widehat{B}_j,\widehat{B}_i)+kA(\lambda,\widehat{B}_j,\widehat{B}_i)-k\lambda(\widehat{B}_j,\widehat{B}_i)-k\lambda(\widehat{B}_{jx},\widehat{B}_{ix})$$

and $\overline{C}=(c_0,c_1...c_N)^T$ with

$$c_i=(W^n,\widehat{B}_i)-k(Z^nZ_x^n,\widehat{B}_{ixx}).$$

We then solve the second system

$$D\overline{\beta}=\overline{\delta}$$

for $\overline{\beta}=(\beta_0^{n+1},\beta_1^{n+1},...\beta_N^{n+1})^T$, where $D=\{d_{ij}\}_{i,j=0}^N$ with $d_{ij}=(\widehat{B}_{jxx},\widehat{B}_{ixx})$ and $\overline{\delta}=(\delta_0,\delta_1...\delta_N)^T$ with $\delta_i=(W^{n+1},\phi_{ixx})$.

The size of the combined linear system is $8n+4$ in (Manickam *et al.* 1998), where as the size of the decoupled system in the present method is $n+1$ each (*i.e.*, $2n+2$).

Example 5.1:

We consider the following problem for the Kuramoto-Sivashinsky equation

$$u_t+\gamma u_{xxxx}+u_{xx}+uu_x=0,\quad 0<t<T,\ \ x\in I=(0,1); \tag{5.59}$$

subject to the boundary and initial conditions

$$u(0,t) = 0,\ u(1,t) = 0,\ \ u_{xx}(0,t) = 0,\ u_{xx}(1,t) = 0;$$
$$u(x,0) = \sin(\pi x).$$

The profile of the approximate solution of (5.59) for $\gamma = 1$ is given in the following figures.

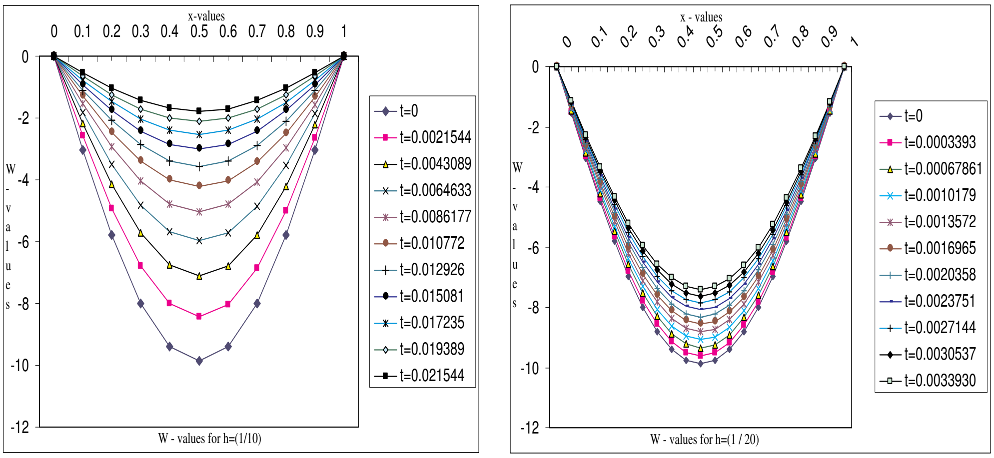

Figure 5.1: Approximate solution of $v(x) = u_{xx}(x)$, (*i.e.*, $W(x)$) at different time levels taking $h = 1/10$ and $h = 1/20$ respectively.

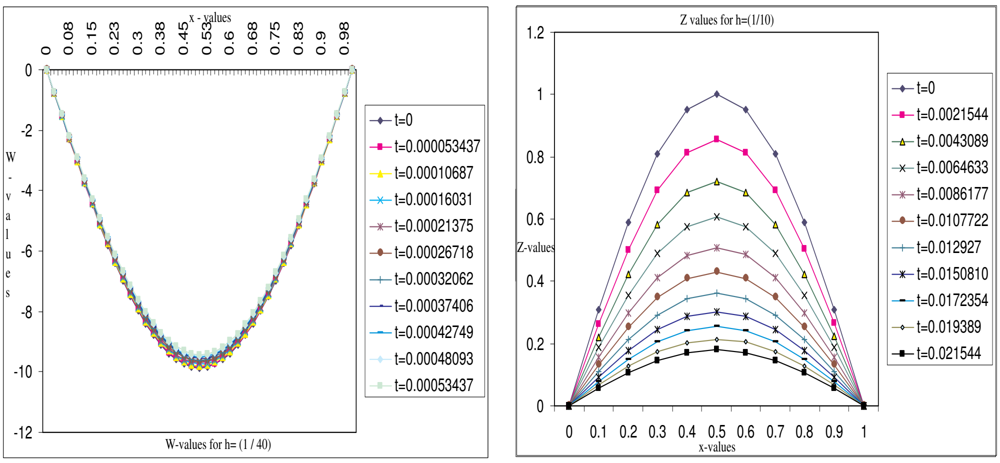

Figure 5.2: Approximate solution of $v(x) = u_{xx}(x)$, (*i.e.*, $W(x)$) at different time levels taking $h = 1/40$ and approximate solution of $u(x)$, (*i.e.*, $Z(x)$) at different time levels taking $h = 1/10$ respectively.

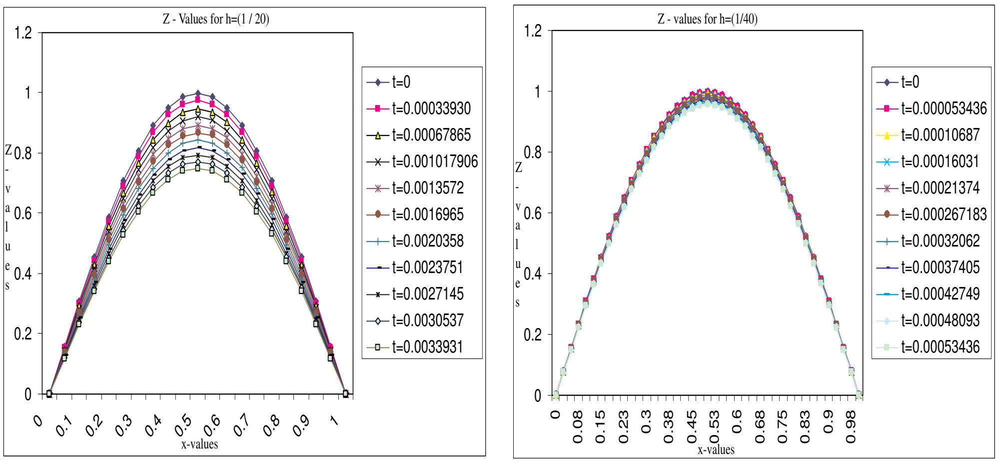

Figure 5.3: Approximate solution of $u(x)$, (*i.e.*, $Z(x)$) at different time levels taking $h = 1/20$ and $h = 1/40$ respectively.

The profile of the approximate solution of (5.59) for $\gamma = 0.00001$ for two time levels is given in the following figures. This shows the chaotic behaviour of the approximate solutions.

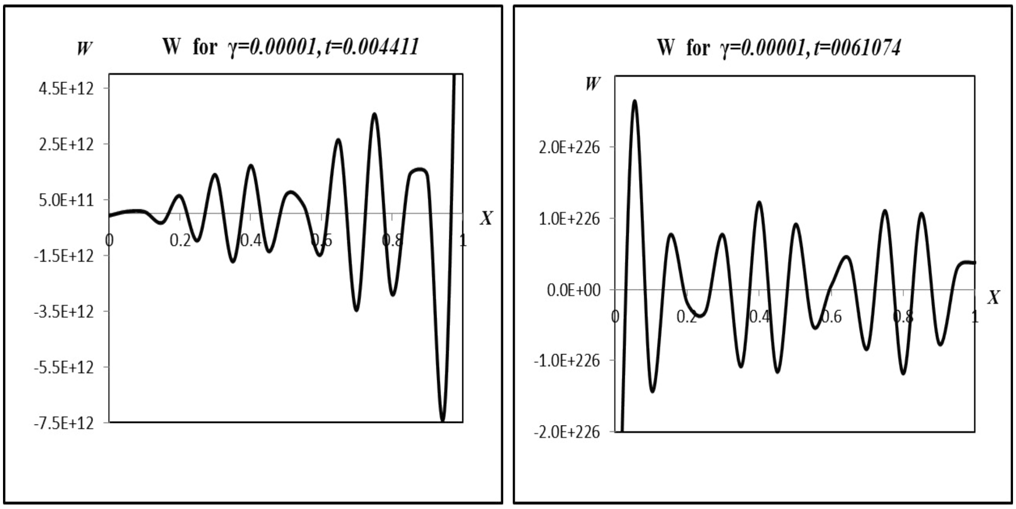

Figure 5.4: Approximate solution of $v(x) = u_{xx}(x)$, (*i.e.*, $W(x)$) when $\gamma = 0.00001$ for $h = 1/20$ at different time levels $t = 0.004411$ and $t = 0.0061074$ respectively.

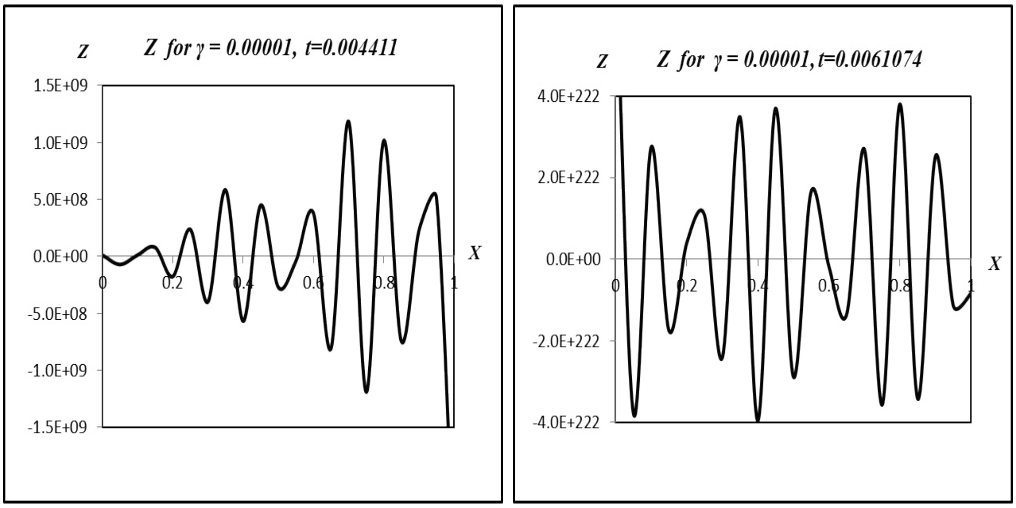

Figure 5.5: Approximate solution of $u(x)$, (*i.e.*, $Z(x)$) when $\gamma = 0.00001$ for $h = 1/20$ at different time levels $t = 0.004411$ and $t = 0.0061074$ respectively.

Example 5.2:

We now consider the following non homogenous Kuramoto-Sivashinsky equation

$$u_t + \gamma u_{xxxx} - u_{xx} + uu_x = \phi(x,t), \quad 0 < t < T, \;\; x \in I = (0,1);$$

with initial condition

$$u(x,0) = \sin(2\pi x)$$

and boundary conditions

$$u(0,t) = 0, \; u(1,t) = 0, \;\; u_{xx}(0,t) = 0, \; u_{xx}(1,t) = 0;$$

where

$$\phi(x,t) = e^{-t}\sin(2\pi x)\left[2\pi e^{-t}\cos(2\pi x) - 1 + 16\pi^4 - 4\pi^2\right].$$

The exact solution of the above problem for $\gamma = 1$ is

$$u(x,t) = e^{-t}\sin(2\pi x).$$

It has been theoretically proved that the error of this approximation is of order $\left(k^{\frac{3}{2}} + h^4\right)$,
i.e.,

$$\|u^{J+1} - Z^{J+1}\| \;\; \leq \;\; C\left[k^{\frac{3}{2}} + h^4\right]$$

and

$$\|v^{J+1} - W^{J+1}\| \leq C\left[k^{\frac{3}{2}} + h^4\right] \text{ for } J = 0, 1, 2, ..., M-1,$$

The numerical computations with number of sub intervals 10, 20 and 40 are considered separately, *i.e.*, with $h = \frac{1}{10}$, $h = \frac{1}{20}$ and $h = \frac{1}{40}$ respectively. For each of these spacial mesh length, the corresponding time step lengths k's are taken satisfying $k^{\frac{3}{2}} = h^4$. With these time step length choice, the error of convergence becomes $O(h^4)$ instead of $O\left(k^{\frac{3}{2}} + h^4\right)$.

We denote $h_1 = \frac{1}{10}$, $h_2 = \frac{1}{20}$ and $h_3 = \frac{1}{40}$. Let $Z^n_{h_i}$ and $W^n_{h_i}$ be the approximate solution in the space $\overset{0}{S}_{h,3}$ of the exact solution $u(x,t)$ and $v(x,t)$ respectively at time $t = t^n$, taking the spatial mesh length h_i for i=1, 2 and 3.

The order of convergence for this method for u is calculated by the formula

$$Order = \left(\log \frac{\|u^n - Z^n_{h_i}\|_{L_p}}{\|u^n - Z^n_{h_{i+1}}\|_{L_p}}\right) / (\log 2)$$

at the n^{th} time level t^n, where $1 \leq p \leq \infty$. Similarly the order of convergence of this method for v is calculated by the formula

$$Order = \left(\log \frac{\|v^n - W^n_{h_i}\|_{L_p}}{\|v^n - W^n_{h_{i+1}}\|_{L_p}}\right) / (\log 2)$$

at the n^{th} time level t^n, where $1 \leq p \leq \infty$. Order of errors in L_2 and L_∞ norms are computed and tabulated as follows.

Table 5.1: L_2 errors in Z and W at time $t = 0.00215443469$

n	h	$\|u - Z\|$	order	$\|v - W\|$	order
10	0.1	$1.21086 * 10^{-4}$	-	$2.96842 * 10^{-3}$	-
20	0.05	$6.62302 * 10^{-6}$	4.1924	$1.39837 * 10^{-4}$	4.40788
40	0.025	$3.90733 * 10^{-7}$	4.08323	$7.7812 * 10^{-6}$	4.16761

Table 5.2: L_2 errors in Z and W at time $t = 0.01077217300$

n	h	$\|u - Z\|$	order	$\|v - W\|$	order
10	0.1	$1.32437 * 10^-4$	-	$3.31288 * 10^-3$	-
20	0.05	$6.8287 * 10^-6$	4.27755	$1.48026 * 10^-4$	4.48416
40	0.025	$3.95784 * 10^{-7}$	4.10883	$8.0288 * 10^{-6}$	4.20452

Table 5.3: L_∞ errors in Z and W at time $t = 0.00215443469$

n	h	$\|u-Z\|_{0,\infty}$	order	$\|v-W\|_{0,\infty}$	order
10	0.1	$3.9652 * 10^{-4}$	-	$1.23219 * 10^{-2}$	-
20	0.05	$2.32104 * 10^{-5}$	4.09455	$7.36052 * 10^{-4}$	4.065273
40	0.025	$1.42503 * 10^{-6}$	4.02571	$4.53213 * 10^{-5}$	4.02155

Table 5.4: L_∞ errors in Z and W at time $t = 0.01077217300$

n	h	$\|u-Z\|_{0,\infty}$	order	$\|v-W\|_{0,\infty}$	order
10	0.1	$4.12933 * 10^-4$	-	$1.29999 * 10^-2$	-
20	0.05	$2.33883 * 10^-5$	4.14205	$7.44624 * 10^-4$	4.12584
40	0.025	$1.42472 * 10^{-6}$	4.03704	$4.54009 * 10^{-5}$	4.03572