CHAPTER 5

SCALING GROUP TRANSFORMATION FOR MHD BOUNDARY-LAYER FLOW OF A NANOFLUID PAST A VERTICAL STRETCHING SURFACE IN THE PRESENCE OF SUCTION/INJECTION

5.1 INTRODUCTION

Nanofluid is a new class of heat transfer fluids that contain a base fluid and nanoparticles. The use of additives is a technique applied to enhance the heat transfer performance of base fluids. The thermal conductivity of the ordinary heat transfer fluids is not adequate to meet today’s cooling rate requirements. Nanofluids have been shown to increase the thermal conductivity and convective heat transfer performance of the base liquids. One of the possible mechanisms for anomalous increase in the thermal conductivity of nanofluids is the Brownian motion of the nanoparticles inside the base fluids. Variety of nuclear reactor designs featured by enhanced safety and improved economics are being proposed by the nuclear power industry around the world to more realistically solve the future energy supply shortfall.

Nanofluid coolants showing an improved thermal performance are being considered as a new key technology to secure nuclear safety and economics. Nanofluids are suspensions of nanoparticles in fluids that show significant enhancement of their properties at modest nanoparticle concentrations. Many of the publications on nanofluids about understand their behavior so that they can be utilized where straight heat transfer enhancement
is paramount as in many industrial applications, nuclear reactors, transportation, electronics as well as biomedicine and food.

In nuclear reactors, the heat is removed from fuel elements via forced convection, making this a much more important heat transfer process. Although nanofluids exhibit better heat transfer properties than pure substances, they also have a higher viscosity, which corresponds to an increase in pumping power. Nanofluids are engineered colloidal suspensions of nanoparticles in water and exhibit a very significant enhancement of the boiling critical heat flux at modest nanoparticle concentrations. They are also used in other electronic applications which use microfluidic applications.

Magnetic nanofluid is a unique material that has both the liquid and magnetic properties. Many of the physical properties of these fluids can be tuned by varying magnetic field. In addition, they have been wonderful model system for fundamental studies. As the magnetic nanofluids are easy to manipulate with an external magnetic field, they have been used for a variety of studies. Particle transport and deposition in flowing suspensions onto collector surfaces is of importance in a broad field of applications. Brownian diffusion and thermophoresis are important slip mechanisms in nanofluids.

Nanofluids are suspensions of submicronic solid particles (nanoparticles) in common fluids. The term “nanofluid” refers to a liquid containing a suspension of submicronic solid particles (nanoparticles). The term was coined by Choi (1995). The characteristic feature of nanofluids is thermal conductivity enhancement, a phenomenon observed by Masuda et al (1993). This phenomenon suggests the possibility of using nanofluids in advanced nuclear systems (Buongiorno and Hu (2005)). A comprehensive survey of convective transport in nanofluids was made by Buongiorno (2006), who says that a satisfactory explanation for the abnormal increase of the
thermal conductivity and viscosity is yet to be found and focused on the further heat transfer enhancement observed in convective situations.

Kuznetsov and Nield (2009) have examined the influence of nanoparticles on natural convection boundary-layer flow past a vertical plate, using a model in which Brownian motion and thermophoresis are accounted for. The authors have assumed the simplest possible boundary conditions, namely those in which both the temperature and the nanoparticle fraction are constant along the wall. Further, Nield and Kuznetsov (2009) have studied the Cheng and Minkowycz (1977) problem of natural convection past a vertical plate in a porous medium saturated by a nanofluid. The model used for the nanofluid incorporates the effects of Brownian motion and thermophoresis. For the porous medium the Darcy model has been employed.

In the present study, the combined effects of Brownian motion and thermophoresis to get the gradient of nanoparticles volume fraction are considered. Using Lie group analysis, three-dimensional, unsteady, laminar boundary layer equations of non-Newtonian fluids were studied by Yurusoy and Pakdemirli (1997, 1999). Using Lie group analysis, they obtained two different reductions to ordinary differential equations. They studied the effects of a moving surface with vertical suction/injection through the porous surface and also analyzed the exact solution of boundary layer equations of a special non-Newtonian fluid over a stretching sheet by the method of Lie group analysis. Yurusoy et al (2001) investigated the Lie group analysis of creeping flow of a second grade fluid. They constructed an exponential type of exact solution using the translation symmetry and a series type of approximate solution using the scaling symmetry and also discussed some boundary value problems.
5.2 OBJECTIVE AND SCOPE

The problem of laminar flow of the fluid which results from the stretching of a vertical surface with variable stream conditions in a nanofluid has been investigated numerically. The model used for the nanofluid incorporates the effects of Brownian motion and thermophoresis in the presence of magnetic field. The symmetry groups admitted by the corresponding boundary value problem are obtained by using a special form of Lie group transformations viz. scaling group of transformations. An exact solution is obtained for translation symmetry and numerical solutions for scaling symmetry. This solution depends on a Lewis number, magnetic field, Brownian motion parameter and thermophoretic parameter. The conclusion is drawn that the flow field and temperature and nanoparticle volume fraction profiles are significantly influenced by these parameters.

Nanofluids are suspensions of nanoparticles in fluids that show significant enhancement of their heat transfer properties at modest nanoparticle concentrations. So, it can be used in many industrial applications, nuclear reactors, transportation, electronics as well as biomedicine and food. Nanofluid as a smart fluid, where heat transfer can be reduced or enhanced at will, has also been reported. Owing to their enhanced properties as thermal transfer fluids for instance, nanofluids can be used in a plethora of engineering applications ranging from use in the automotive industry to the medical arena to use in power plant cooling systems as well as computers.
5.3 FORMULATION OF THE PROBLEM

Consider a two-dimensional problem and select a coordinate frame in which the $x$-axis is aligned vertically upwards. Also, consider a vertical plate at $y = 0$ as shown in Figure 5.1. A uniform transverse magnetic field of strength $\vec{B}_0$ is applied parallel to the $y$-axis. It is assumed that the induced magnetic field, the external electric field and the electric field due to the polarization of charges are negligible. At this boundary the temperature $T$ and the nanoparticle fraction $C$ take constant values $T_w$ and $C_w$, respectively. When $y$ tends to infinity, the ambient values of $T$ and $C$ are denoted by $T_\infty$ and $C_\infty$, respectively.

![Figure 5.1](image)

**Figure 5.1 Physical model of boundary layer flow over a vertical stretching surface**

The Oberbeck–Boussinesq approximation is employed. The following four field equations embody the conservation of total mass, momentum, thermal energy, and nanoparticles, respectively. The field
variables are the velocity \((v)\), the temperature \((T)\) and the nanoparticle volume fraction \((C)\).

\[
\nabla \bar{v} = 0
\]

\[
\rho_f \left( \frac{\partial \bar{v}}{\partial t} + \bar{v} \cdot \nabla \bar{v} \right) = -\nabla p - \sigma \bar{B}_0^2 \bar{v} + \mu \nabla^2 \bar{v} + \left[ C \rho_p + (1 - C) \left( \rho_f (1 - \beta (T - T_w)) \right) \right] \bar{g}
\]

\[
(\rho c)_f \left( \frac{\partial T}{\partial t} + \bar{v} \cdot \nabla T \right) = k \nabla^2 T + (\rho c)_p \left[ D_B \nabla C \cdot \nabla T + \left( \frac{\nabla T}{T_w} \right) \nabla T \cdot \nabla T \right]
\]

\[
\frac{\partial C}{\partial t} + \bar{v} \cdot \nabla C = D_B \nabla^2 C + \left( \frac{\nabla T}{T_w} \right) \nabla^2 T
\]

Consider \(\bar{v} = (u, v)\). Here \(\rho_f\) is the density of the base fluid, \(\bar{B}_0\) is a constant magnetic field of strength, \(\sigma\) is electric conductivity, \(\mu, k, \beta\) are the viscosity, thermal conductivity and volumetric volume expansion coefficient of the nanofluid, while \(\rho_p\) is the density of the particles. The gravitational acceleration is denoted by \(\bar{g}\). The coefficients that appear in Equations (5.3) and (5.4) are the Brownian diffusion coefficient \((D_B)\) and the thermophoretic diffusion coefficient \((D_T)\). Details of the derivation of Equations (5.3) and (5.4) are given in the paper by Buongiorno (2006) and Nield and Kuznetsov (2009). The boundary conditions are taken to be

\[
u = U(x), v = V(x), C = C_w, T = T_w \quad \text{at} \quad y = 0;
\]

\[
u = 0, C = C_w, T = T_w \quad \text{as} \quad y \rightarrow \infty
\]

A steady state flow is considered. In keeping with the Oberbeck–Boussinesq approximation and an assumption that the nanoparticle
concentration is dilute, and with a suitable choice for the reference pressure, the momentum equation can be linearized and Equation (5.2) is written as

$$\rho_t \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p - \sigma \mathbf{B}_0^2 \mathbf{v} + \mu \nabla^2 \mathbf{v}$$

$$+ \left[ (\rho_p - \rho_{t\infty})(C - C_{\infty}) + (1 - C_{\infty})\rho_{t\infty} \beta (T - T_{\infty}) \right] \mathbf{g} \quad (5.6)$$

Let us make the standard boundary-layer approximation, based on a scale analysis, and write the governing equations

$$\frac{\partial \mathbf{u}}{\partial x} + \frac{\partial \mathbf{v}}{\partial y} = 0 \quad (5.7)$$

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 \mathbf{u}}{\partial y^2} - \sigma \mathbf{B}_0^2 \mathbf{u} - \rho_t \left( \frac{\partial \mathbf{u}}{\partial x} + \frac{\partial \mathbf{u}}{\partial y} \right)$$

$$+ \left[ (1 - C_{\infty})\rho_{t\infty} \beta g(T - T_{\infty}) - (\rho_p - \rho_{t\infty}) g(C - C_{\infty}) \right] \quad (5.8)$$

$$\frac{\partial p}{\partial y} = 0 \quad (5.9)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \tau \left[ D_B \frac{\partial C}{\partial y} \frac{\partial T}{\partial y} + D_T \left( \frac{\partial T}{\partial y} \right)^2 \right] \quad (5.10)$$

$$u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D_B \frac{\partial^2 C}{\partial y^2} + D_T \frac{\partial^2 T}{T_{\infty} \frac{\partial T}{\partial y}^2} \quad (5.11)$$

where \( u \) and \( v \) are the velocity components along the x and y axes, respectively, \( \alpha = \frac{k}{(\rho c)_t} \) is the thermal diffusivity of the fluid, \( v \) is the kinematic viscosity coefficient and \( \tau = \frac{(\rho c)_p}{(\rho c)_t} \) is the ratio of nanoparticle heat capacity and the base fluid heat capacity.
The stream wise velocity and the suction/injection velocity are taken as

\[ U(x) = c x^m, \ V(x) = V_0 \ x^{\frac{m-1}{2}} \]  

(5.12)

Here \( c > 0 \) is constant, \( T_w \) is the wall temperature, the power-law exponent \( m \) is also constant. In this study consider \( c = 1 \).

One can eliminate \( p \) from Equations (5.8) and (5.9) by cross-differentiation. At the same time one can introduce a stream function \( \psi \) defined by

\[ u = \frac{\partial \psi}{\partial y}, \ v = -\frac{\partial \psi}{\partial x}, \ \theta = \frac{T - T_w}{T_w - T_\infty}, \ \text{and} \ \phi = \frac{C - C_w}{C_w - C_\infty} \]  

(5.13)

So that Equation (5.7) is satisfied identically and left with the following three equations.

\[ \frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} - \nu \frac{\partial^3 \psi}{\partial y^3} = -\frac{\sigma B_0^2}{\rho_\ell} \frac{\partial \psi}{\partial y} \]  
\[ + (1 - \phi_\infty) \rho \beta g \Delta \theta - (\rho_p - \rho_{\infty}) g \phi \Delta \phi \]  

(5.14)  

\[ \frac{\partial \psi}{\partial y} \frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \theta}{\partial y} = \alpha \frac{\partial^2 \theta}{\partial y^2} + \tau \left[ D_\theta \frac{\partial \phi}{\partial y} \frac{\partial \theta}{\partial y} + \frac{D_\phi}{T_\infty} \left( \frac{\partial \theta}{\partial y} \right)^2 \right] \]  

(5.15)  

\[ \frac{\partial \psi}{\partial y} \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \phi}{\partial y} = D_\psi \frac{\partial^2 \phi}{\partial y^2} + \frac{D_\psi}{T_\infty} \frac{\partial^2 \theta}{\partial y^2} \]  

(5.16)
\[ \frac{\partial \psi}{\partial y} = x^m, \quad \frac{\partial \psi}{\partial x} = -V_0 x^{m-1}, \quad \theta = \phi = 1 \text{ at } y = 0; \]

\[ \frac{\partial \psi}{\partial y} \rightarrow 0, \theta \rightarrow 0, \phi \rightarrow 0 \text{ as } y \rightarrow \infty \]

where \( \Delta \theta = T_w - T_\infty \) and \( \Delta \phi = C_w - C_\infty \).

Introduce the simplified form of Lie-group transformations namely, the scaling group of transformations (Mukhopadhyay et al, 2005),

\[ \Gamma: x^* = x e^{\varepsilon \alpha_1}, \quad y^* = y e^{\varepsilon \alpha_2}, \quad \psi^* = \psi e^{\varepsilon \alpha_3}, \]

\[ u^* = u e^{\varepsilon \alpha_4}, \quad v^* = v e^{\varepsilon \alpha_5}, \quad \theta^* = \theta e^{\varepsilon \alpha_6}, \quad \phi^* = \phi e^{\varepsilon \alpha_7} \]

Equation (5.18) may be considered as a point-transformation which transforms the co-ordinates \((x, y, \psi, u, v, \theta, \phi)\) to the coordinates \((x^*, y^*, \psi^*, u^*, v^*, \theta^*, \phi^*)\).

Substituting equation (5.18) in (5.14), (5.15) and (5.16) the following equations are obtained

\[ e^{\varepsilon(\alpha_1 + 2\alpha_2 - 3\alpha_3)} \left( \frac{\partial \psi^*}{\partial y^*} \frac{\partial^2 \psi^*}{\partial x^* \partial y^*} - \frac{\partial \psi^*}{\partial x^*} \frac{\partial^2 \psi^*}{\partial y^*} \right) = \nu e^{\varepsilon(3\alpha_2 - \alpha_3)} \frac{\partial^3 \psi^*}{\partial y^3} \]
\[ + e^{-\varepsilon \alpha_6} (1 - \phi_\infty) \rho_f \beta \frac{\partial}{\partial y} \Delta \theta - e^{\varepsilon(\alpha_2 - \alpha_3)} \frac{\partial \psi}{\partial y} - e^{-\varepsilon \alpha_7} (\rho_p - \rho_f) \frac{\partial \phi}{\partial y} \Delta \phi \]

\[ e^{\varepsilon(\alpha_1 + 2\alpha_2 - 3\alpha_3 - \alpha_6)} \left( \frac{\partial \theta^*}{\partial y^*} \frac{\partial^2 \theta^*}{\partial x^* \partial y^*} - \frac{\partial \psi^*}{\partial x^*} \frac{\partial \theta^*}{\partial y^*} \right) = \alpha e^{\varepsilon(2\alpha_2 - \alpha_6)} \frac{\partial^2 \theta^*}{\partial y^2} \]
\[ + \tau \left[ D_B e^{\varepsilon(2\alpha_2 - \alpha_7)} \frac{\partial \phi^*}{\partial y^*} \frac{\partial^2 \theta^*}{\partial y^2} + e^{\varepsilon(2\alpha_2 - \alpha_6)} \frac{D_f}{T_\infty} \left( \frac{\partial \theta^*}{\partial y^*} \right)^2 \right] \]
\[ e^{(\alpha_1 + \alpha_2 - \alpha_3 - \alpha_7)} \left( \frac{\partial \psi^*}{\partial y^*} \frac{\partial \phi^*}{\partial x^*} - \frac{\partial \psi^*}{\partial x^*} \frac{\partial \phi^*}{\partial y^*} \right) = D_B e^{(2\alpha_2 - \alpha_7)} \frac{\partial^2 \phi^*}{\partial y^{*2}} + \frac{D_T}{T_\infty} e^{(2\alpha_2 - \alpha_6)} \frac{\partial^2 \theta^*}{\partial y^{*2}} \] (5.21)

The system will remain invariant under the group of transformations \( \Gamma \) and would have the following relations among the parameters, namely

\[
\begin{align*}
\alpha_1 + 2\alpha_2 - 2\alpha_3 &= 3\alpha_2 - \alpha_3 = \alpha_2 - \alpha_3 = -\alpha_6 = -\alpha_7; \\
\alpha_1 + \alpha_2 - \alpha_3 - \alpha_6 &= 2\alpha_2 - \alpha_6 = 2\alpha_2 - \alpha_6 - \alpha_7 = 2\alpha_2 - 2\alpha_6 \quad \text{and} \\
\alpha_1 + \alpha_2 - \alpha_3 - \alpha_7 &= 2\alpha_2 - \alpha_7 = 2\alpha_2 - \alpha_6
\end{align*}
\]

These relations \( 3\alpha_2 - \alpha_3 = \alpha_2 - \alpha_3 \) give the value \( \alpha_2 = 0 \)

Hence, \( \alpha_1 + 2\alpha_2 - 2\alpha_3 = 3\alpha_2 - \alpha_3 \) gives \( \alpha_6 = \alpha_7 = 0, \alpha_2 = \frac{1}{4} \alpha_1 = \frac{1}{3} \alpha_3. \)

The boundary conditions yield

\[ \alpha_4 = m \alpha_1 = \frac{1}{2} \alpha_1, \alpha_5 = \frac{m-1}{2} \alpha_1 = -\frac{1}{4} \alpha_1 \text{ (as } m = \frac{1}{2}) \]

In view of these, the boundary conditions become

\[ \frac{\partial \psi^*}{\partial y^*} = x^* \frac{\alpha_1}{2}, \quad \frac{\partial \psi^*}{\partial x^*} = -V_0 x^* \left( \frac{1}{4} \right), \quad \theta^* = \phi^* = 1 \text{ at } y^* = 0 \text{ and } \]
\[ \frac{\partial \psi^*}{\partial y^*} \to 0, \theta^* \to 0, \phi^* \to 0 \text{ as } y^* \to \infty \] (5.22)

The set of transformations \( \Gamma \) reduces to

\[ x^* = x e^{\frac{\alpha_1}{4}}, \quad y^* = y e^{\frac{\alpha_1}{4}}, \quad \psi^* = \psi e^{\frac{3\alpha_1}{4}}, \quad u^* = u e^{\frac{-\alpha_1}{2}}, \quad v^* = v e^{\frac{-\alpha_1}{4}}, \quad \theta^* = \theta, \phi^* = \phi \]
Expanding by Taylor’s method in powers of $\varepsilon$ and keeping terms up to the order $\varepsilon$, Equation (5.23) is obtained.

\[
x^* - x = x \varepsilon \alpha_1, \quad y^* - y = y \varepsilon \frac{\alpha_1}{4}, \quad \psi^* - \psi = \psi \varepsilon \frac{3\alpha_1}{4},
\]
\[
u^* - \nu = \nu \varepsilon \frac{\alpha_1}{2}, \quad \theta^* - \theta = \theta \varepsilon 
\]

(5.23)

From the above equations, Equation (5.24) is acquired.

\[
y^* x^* = \eta, \quad \psi^* = x^* \frac{3}{4} f(\eta), \quad \theta^* = \theta(\eta), \quad \phi^* = \phi(\eta)
\]

(5.24)

With the help of these relations, the Equations (5.19), (5.20) and (5.21) become

\[
f'' + \frac{1}{Pr} [0.75 f' f'' - 0.5 f'^2] + Ra_x [\theta - N\phi] + M f' = 0
\]

(5.25)

\[
\theta'' + 0.75 f' \theta' + Nb \theta' \phi' + Nt \theta'^2 = 0
\]

(5.26)

\[
\phi'' + 0.75 Le f' \phi' + \frac{Nt}{Nb} \theta'' = 0
\]

(5.27)

The boundary conditions take the following form

\[
f' = 1, \quad f = -\frac{4N_0}{3}, \quad \theta = \phi = 1 \text{ at } \eta = 0 \quad \text{and} \quad f' \to 0, \theta \to 0, \phi \to 0 \text{ as } \eta \to \infty
\]

(5.28)

where $Pr = \frac{v^*}{\alpha}$ is the Prandtl number, $Ra_x = \frac{(1-\phi_{\infty}) \beta g \Delta \theta}{\nu \alpha \phi_{\infty}}$ is the local Rayleigh number, $N = \frac{(\rho_p - \rho_{\infty}) \Delta \phi}{\rho_{\infty} \beta \Delta \theta (1-\phi_{\infty})}$ is the buoyancy ratio, $Nb = \frac{(\rho c)_p D_B \Delta \phi}{\alpha (\rho c)_i}$.
is the Brownian motion parameter, \( N_t = \frac{(\rho \, c)_p \, D_T \, \Delta \theta}{\alpha (\rho \, c)_T \, T_\infty} \) is the thermophoresis parameter, \( M = \frac{\sigma \, B^2 \, U}{\rho_i} \) is the magnetic parameter and \( Le = \frac{V^*}{D} \) is the Lewis number.

The boundary conditions take the following forms.

\[
f' = 1, f = S, \theta = \phi = 1, \text{ at } \eta^* = 0 \text{ and } f' \to 0, \theta \to 0, \phi \to 0 \text{ as } \eta^* \to \infty \tag{5.29}
\]

Where \( S = -\frac{4}{3} V_0 \), \( S > 0 \) corresponds to suction and \( S < 0 \) corresponds to injection whereas \( V_0 \) is the velocity of suction if \( V_0 < 0 \) and injection if \( V_0 > 0 \).

### 5.4 NUMERICAL SOLUTION

The set of non-linear ordinary differential Equations (5.25) – (5.27) with boundary conditions (5.29) have been solved by using the Runge-Kutta Gill algorithm, (Gill, 1951) with a systematic guessing of \( f''(0), \theta'(0) \) and \( \phi'(0) \) by the Shooting Technique with until the boundary conditions at infinity \( f''(\infty), \theta'(\infty) \) and \( \phi'(\infty) \) decay exponentially to zero. The step size \( \Delta \eta = 0.001 \) is used while obtaining the numerical solution with \( \eta_{\max} \), and accuracy to the fifth decimal place is sufficient for convergence. The computations were done in Matlab. A step size of \( \Delta \eta = 0.001 \) was selected to be satisfactory for a convergence criterion of \( 10^{-7} \) in nearly all cases. The value of \( \eta_\infty \) was found to each iteration loop by assignment statement \( \eta_i = \eta_i + \Delta \eta \). The maximum value of \( \eta_\infty \), to each group of parameters \( Pr, Le, M, N, Nb \) and \( Nt \) determined when the values of unknown boundary conditions at \( \eta = 0 \) not change to successful loop with error less
than $10^{-7}$. Effects of development of the steady boundary layer flow, heat transfer and nanoparticle volume fraction over a stretching surface in a nanofluid are studied for different values of Brownian motion parameter, thermophoresis parameter, magnetic parameter and Lewis number.

5.5 RESULTS AND DISCUSSION

Numerical analysis is carried out for $0.5 \leq \text{Nb} \leq 2.5$, $0.5 \leq \text{Nt} \leq 2.0$, $0.5 \leq \text{M} \leq 2.0$ and $-2.0 \leq \text{S} \leq 2.0$. Equations (5.25), (5.26) and (5.27) subject to the boundary conditions (5.29) have been solved numerically for some values of the governing parameters $\text{Pr, Le, M, N, Nb and Nt}$ using Runge- Kutta Gill algorithm with shooting technique. Neglecting the effects of $\text{Nb}$ and $\text{Nt}$ numbers, the results for the reduced Nusselt number $-\theta'(0)$ are compared with those obtained by Khan and Pop (2010), Wang (1989), and Gorla and Sidawi (1994) for different values of $\text{Pr}$ in Table 5.1. This shows that the comparison shows good agreement for each value of $\text{Pr}$.

Table 5.1 Comparison of results for $-\theta'(0)$ with previous studies

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In the absence of Local Rayleigh number $Ra_x$, in order to ascertain the accuracy of our numerical results, the present study is compared with the available exact solution in the literature. The nanoparticle fraction profiles for Lewis number $Le$ are compared with the available exact solution of Khan and Pop (2010), are shown in Figures 5.2 (a) and 5.2(b). It is observed that the agreements with the theoretical solution of nanoparticle fraction profiles are excellent. For a given $Nb$ and $Nt$, it is clear that there is a fall in nanoparticle fraction with increasing the Lewis number. This is due to the fact that there would be a decrease of nanoparticle volume fraction boundary layer thickness with the increase of Lewis number as one can see from Figures 5.2(a) and 5.2(b) by comparing the curves with $Le=10, 20$ and 30.

**Figures 5.2 (a)-(b)  Comparison of the Nanoparticle volume fraction profiles**
Figure 5.3  Temperature profiles for various values of Nb when \( \text{Pr} = 2.0, \text{Le} = 3.0, \text{Nt} = 1.0, \text{S} = 2.0, \text{N} = 0.5 \) and \( M = 1.0 \)

Figure 5.4  Effect of Nb over the Nanoparticle volume fraction profiles with \( \text{Pr} = 2.0, \text{Le} = 3.0, \text{Nt} = 1.0, \text{S} = 2.0, \text{N} = 0.5, M = 1.0 \)
Volume fraction of nanoparticles is a key parameter for studying the effect of nanoparticles on flow fields and temperature distributions. Thus Figures 5.3 – 5.4 presents the effect of Brownian motion on temperature distribution and volume fraction of nanoparticle. It also illustrates the typical temperature and nanoparticle volume fraction profiles for various values of Brownian motion parameter (Nb). Temperature of the fluid increases and the nanoparticle volume fraction decreases with increase of Nb. It is noted that Brownian motion of nanoparticles at the molecular and nanoscale levels is a key nanoscale mechanism governing their thermal behavior. In nanofluid systems, due to the size of the nanoparticles Brownian motion takes place which can affect the heat transfer properties. As the particle size scale approaches to the nano-meter scale, the particle Brownian motion and its effect on the surrounding liquids play an important role in heat transfer.

Figure 5.5  Temperature profiles for various values of Nt with Pr=2.0, Le = 3.0, Nb = 1.0, S = 2.0, N = 0.5, M = 1.0
Figures 5.5 and 5.6 display the effect of thermophoretic parameter \( Nt \) on temperature and nanoparticle volume fraction profiles. It is to note that the temperature of the fluid increases whereas the nanoparticle volume fraction decreases with increase of \( Nt \). For hot surfaces, thermophoresis tends to blow the nanoparticle volume fraction boundary layer away from the surface since a hot surface repels the sub-micron sized particles from it, thereby forming a relatively particle-free layer near the surface. As a consequence, the nanoparticle distribution is formed just outside. In particular, the effect of increasing the thermophoretic parameter \( Nt \) is limited to increasing slightly the wall slope of the nanoparticle volume fraction profiles but decreasing the nanoparticle volume fraction. This is true only for small values of Lewis number for which the Brownian diffusion effect is large compared to the convection effect. However, for large values of Lewis number, the diffusion effect is minimal compared to the convection effect.
and, therefore, the thermophoretic parameter $N_t$ is expected to alter the nanoparticle volume fraction boundary layer significantly.

Although thermophoresis effect is important in natural convection of nanofluids, there are other parameters that may have effect, and should be considered. These effects include increase in effective viscosity of nanofluids due to the presence of nanoparticles and density variation due to variable volume fraction. More volume fraction of nanoparticles makes nanofluid much viscous and the mixture’s convection takes place weaker, thus natural convective Nusselt number decreases due to high viscosity. On the other hand it is showed that the separation factor for common nanofluids is positive and density variation due to variable volume fraction of nanoparticles, called particulate buoyancy force, helps nanofluid to have strong convection heat transfer.

![Figure 5.7](image)

**Figure 5.7** Effect of suction/injection over the velocity profiles with $Pr = 2.0, Le = 3.0, Nb = Nt = 1.0, N = 0.5, M = 1.0$
Figure 5.8 Effect of suction/injection over the temperature profiles with $Pr = 2.0$, $Le = 3.0$, $Nb = Nt = 1.0$, $N = 0.5$, $M = 1.0$

Figure 5.9 Effect of suction/injection over the Nanoparticle volume fraction profiles with $Pr=2.0$, $Le=3.0$, $Nb = Nt =1.0$, $N = 0.5$, $M = 1.0$
Figures 5.7 – 5.9 depict the influence of the suction/injection parameter $S$ on the velocity, temperature and nanoparticle volume fraction profiles in the boundary layer when the thermophoretic particle deposition is uniform, i.e. $N_t = 1.0$. With the increasing value of the suction ($S > 0$), the velocity is found to decrease (Figure 5.7), i.e. suction causes to decrease the velocity of the fluid in the boundary layer region. In case of suction, the heated fluid is pushed towards the wall where the buoyancy forces can act to retard the fluid due to high influence of the Brownian motion effects. This effect acts to decrease the wall shear stress.

Figures 5.8 and 5.9 exhibit that the temperature $\theta(\eta)$ and nanoparticle volume fraction $\phi(\eta)$ in boundary layer also decrease with the increasing suction parameter ($S > 0$) (the fluid is of uniform thermophoretic particle deposition, i.e. $N_t = 1.0$) ($Pr = 2.0$ and $Nb = 1.0$). The explanation for such behavior is that the fluid is brought closer to the surface and reduces the thermal and nanoparticle volume boundary layer thickness in case of suction. As such then the presence of wall suction decreases velocity boundary layer thicknesses and also decreases the thermal and nanoparticle volume fraction boundary layers thickness, i.e. thins out the thermal and nanoparticle volume fraction boundary layers. However, the exact opposite behavior is produced by imposition of wall fluid blowing or injection. These behaviors are also clear from Figures 5.7 – 5.9. The samples of velocity, temperature and the nanoparticle volume fraction profiles are given in Figures 5.7 – 5.9, respectively. These profiles satisfy the far field boundary equation (5.28) asymptotically, which support the numerical results obtained.
Figure 5.10 Magnetic effect over the velocity profiles with $Pr=2.0$, $Le=3.0$, $Nb = Nt=1.0$, $S=2.0$, $N=0.5$.

Figure 5.11 Magnetic effect over the temperature profiles with $Pr=2.0$, $Le=3.0$, $Nb = Nt = 1.0$, $S=2.0$, $N=0.5$.
Table 5.2  Analysis for skin friction, rate of heat, mass transfer for \( \text{Nb} \)

\( ( \text{Pr} = 2.0, \text{Le} = 3.0, \text{M} = 1.0, \text{Nt} = 1.0, \text{S} = 2.0 \text{ and } N = 0.5 ) \)

<table>
<thead>
<tr>
<th>Nb ( (\times 10^{-2}) )</th>
<th>( f'(0) )</th>
<th>( \theta'(0) )</th>
<th>( \phi'(0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>2.40860</td>
<td>-1.7183</td>
<td>-1.7322</td>
</tr>
<tr>
<td>1.0</td>
<td>1.90443</td>
<td>-1.6408</td>
<td>-1.7116</td>
</tr>
<tr>
<td>2.0</td>
<td>1.60469</td>
<td>-1.5952</td>
<td>-1.6996</td>
</tr>
</tbody>
</table>

In Figures 5.10 and 5.11, the typical profiles for velocity and temperature for different values of magnetic parameter are presented. Due to the uniform thermophoresis particle deposition, it is clearly shown that the velocity of the fluid decreases and the temperature of the fluid increases whereas the nanoparticle volume fraction of the fluid is not significant with increase of the strength of magnetic field. The effects of a transverse magnetic field to an electrically conducting fluid gives rise to a resistive-type force called the Lorentz force. This force has the tendency to slow down the motion of the fluid and to increase its temperature profiles. This result qualitatively agrees with the expectations, since magnetic field exerts retarding force on the natural convection flow. Application of a magnetic field moving with the free stream has the tendency to induce a motive force which decreases the motion of the fluid and increases its boundary layer. From the Table 5.2, it is observed that the skin friction decreases and the rate of heat and mass transfer increases with increase of Brownian motion parameter.

5.6 CONCLUSION

In this chapter, the problem of steady MHD boundary-layer flow of a nanofluid past a vertical stretching surface in the presence of suction/injection is studied. The model used for the nanofluid incorporates the effects of Brownian motion and thermophoresis.
It is found that the volume fraction of nanoparticles is a key parameter for studying the effect of nanoparticles on flow fields and temperature distributions.

It is noted that the impact of thermophoresis particle deposition in the presence of magnetic field with Brownian motion have a substantial effect on the flow field and, thus, on the heat transfer and nanoparticle volume fraction rate from the sheet to the fluid.

Particularly, the temperature of the fluid increases whereas the nanoparticle volume fraction decreases with increase of Brownian motion and thermophoretic parameter, respectively. Brownian motion of nanoparticles at the molecular and nanoscale levels is a key nanoscale mechanism governing their thermal behavior. In nanofluid systems, due to the size of the nanoparticles Brownian motion and thermophoresis takes place which can affect the heat transfer properties.

The analysis has helped engineers understand the mechanisms that are most important in the deposition process. Free convective flow through porous media is an area of research undergoing rapid growth in the fluid mechanics and heat transfer and nanoparticle volume fraction field due to its broad range of scientific and engineering applications.

One of the technological applications of nanoparticles that hold enormous promise is the use of heat transfer fluids containing suspensions of nanoparticles to confront cooling problems in thermal systems. Hence, the subject of nanofluids is of great interest worldwide for basic and applied research. Nanofluids are important because they can be used in numerous applications involving heat transfer and other applications such as in detergency. Colloids which are also nanofluids have been used in the biomedical field for a long time, and their use will continue to grow. Nanofluids have also been demonstrated for use as smart fluids.