Chapter 1

Introduction

Random matrix theory (RMT) was introduced by Wigner, Dyson, Mehta and others in the late fifties [1-10] for statistical study of spectra of complex quantum systems. At that time the highly excited resonance data of complex nuclei obtained from the neutron scattering were explained by RMT while microscopic theory of nuclear shell model could successfully explain low lying excitations [11]. Later analysis involving the whole body of resonance data showed good agreement with RMT [12-14]. Subsequently, RMT has found applications in explaining the universal energy-level fluctuations in other complex systems such as atoms and molecules, and also model simple quantum systems with few degrees of freedom, which exhibit chaos in classical limit (quantum chaotic systems) e.g., billiards, kicked rotors and tops [3, 11, 15-18]. In recent years, RMT has been used extensively in the study of disordered and mesoscopic systems [19-27] where it explains the universality of conductance fluctuations. Furthermore, RMT has found applications in quantum field theory [16, 28-33].

1.1 Energy Level Fluctuations and Random Matrix Ensemble

The energy level fluctuations are studied in terms of the measures of departures from the spectrum with average spacing unity, i.e. from the unfolded spectra. Universality of fluctuations in random matrix spectra comes about, because for large matrices the
local statistics, viz. properly rescaled (unfolded) correlation functions of the eigenvalues, belong to one of the three universality classes for a large class of random matrices. The three classes are defined by the invariance of the ensemble under orthogonal, unitary or symplectic transformations and are applicable to the three types of physical systems in the presence or lack of time reversal and rotational symmetries \([4,10]\). The orthogonal ensembles are applicable to systems for which time reversal and space rotation symmetries are both valid. The orthogonal ensembles also applies to systems with integral spin, where time reversal symmetry is good but rotation symmetry is not good. Symplectic ensembles apply to system with half integral spin, where time reversal symmetry is again good but rotation symmetry is not good. Finally, unitary ensembles are applicable to systems without time reversal symmetry. The corresponding ensembles with Gaussian weight function are called as Gaussian orthogonal ensemble (GOE), Gaussian unitary ensemble (GUE) and Gaussian symplectic ensemble (GSE) respectively.

The earlier work was concerned with neutron excitation spectra of heavy nuclei \([11]\). These are many-particle systems whose interaction is so complex that the Hamiltonian representing the system should behave like a large matrix whose elements are random. Initially it was believed that random matrix behaviour in complex nuclei comes from many particle nature of the system. Later it was discovered that even simple one-particle quantum systems exhibit random matrix statistics, if the classical limit of the system is chaotic \([17]\). Thus it is conjectured that the statistics of energy levels are typically described by random matrix theory, when the classical limit of the quantum dynamics is chaotic \([15,17]\) as for example in Sinai billiard, Stadium billiard and kicked rotors etc \([15]\). In contrast, if the underlying dynamics is regular, the energy levels will not follow random matrix statistics but will rather behave like independent random variables or a Poisson process \([34]\). An example of such a regular system is an electron confined to a circular domain and its level spacing distribution is the exponential distribution of a Poisson process. Thus random matrix behavior of the eigenvalues can be identified with chaos in underlying classical dynamics and can be taken as signature of quantum chaos.

Another study concerns the zeros of Riemann zeta function, which is defined for
\Re(z) > 1 \text{ as } \zeta(z) = \sum_1^\infty n^{-z}, \text{ where } z \text{ is complex. By analytic continuation } \zeta(z) \text{ is defined in the entire complex plane. The points } z = -2, -3, -4, \ldots \text{ are the trivial zeros of the Riemann zeta functions. According to Riemann hypothesis [3, 35], all the other zeros are on the straight line } z = 1/2 + i\gamma_n, \text{ where } \gamma_n \text{ is real; these are called nontrivial zeros of } \zeta(z). \text{ If } z \text{ is the zero of } \zeta(z) \text{ then } z^* \text{ is also a zero of Riemann zeta function since } \zeta(z^*) \text{ is the complex conjugate of } \zeta(z). \text{ Thus the nontrivial zeros of the Riemann zeta function lie symmetrically about the real line and also there are infinite number of them. The position of the zeros is of crucial importance in numerous problems in analytic number theory e.g., how the largest gap between two consecutive primes } \leq x \text{ increases with } x? \text{ For such questions, one needs to know also the distribution of } \gamma_n. \text{ It is found that the nontrivial Riemann zeros have the same level statistical properties as eigenvalues of large random matrices from unitary ensembles [3, 36].}

Spectral fluctuation properties of amorphous clusters are also explained by RMT [37, 38]. Unlike the crystalline states, in amorphous systems the interatomic distance is varying throughout the systems. Since the force constant between the atoms is a function of the interatomic distance, a large number of configurations, varying exponentially with the size of the system, exist which correspond to the minimum energy. Thus inherent configuration of the amorphous system is disordered and this gives rise to randomness in the Hessian matrix, viz. the second derivative of the potential energy of the system. The calculations of vibrational frequencies is done by solving a suitable eigenvalue problem involving the Hessian matrix of the potential energy function of the corresponding inherent structure. The fluctuations of the vibrational spectra are consistent with the universal random matrix results [37, 38].

The applications of RMT can be divided in two broad classes, viz. autonomous systems and systems with time-periodic Hamiltonians. For autonomous systems, Hamiltonians are modeled by random matrix ensembles of Hermitian matrices [3]. The time periodically perturbed systems such as quantum chaotic maps, arising from time periodic Hamiltonians, are characterized by ensembles of unitary matrices - introduced by Dyson; they represent evolution operators for the propagation of wave functions over one period of time. Ensembles of unitary matrices are also useful for investigating scattering in chaotic systems; these are described by scattering matrices which are
unitary [4–8]. In these ensembles one studies the statistics of the phases of the eigenvalues, also called eigenangles. These ensembles are called circular ensembles - COE CUE and CSE respectively for invariances under orthogonal, unitary and symplectic transformations. See Sec. 1.4 ahead.

A unique feature of the eigenvalues and eigenangles of the two classes of random matrices is the repulsion of neighboring eigenvalues. See Fig. 1.1 for a demonstration of the level repulsion as a parameter of the system is varied. Systems with and without the symmetries exhibit different degrees of repulsion. For both types of ensembles, the degree of the repulsion depends on the symmetry of the system. Systems with time reversal invariance and rotation invariance or integral spin display linear repulsion, and a quadratic repulsion is obtained for systems with broken time reversal symmetry. Systems with half-integer spins, having time reversal invariance are found to have quartic repulsion of the neighboring eigenvalues, provided Kramers degeneracy is not taken into account.

The property of level repulsion is well depicted by the Wigner surmise. The probability density to find nearest neighbor spacing between s and \( s + ds \) is given by

\[
P(s) = A s^\beta \exp(-Bs^2) \quad s \geq 0,
\]

where \( \beta \) is the symmetry parameter and denotes the degree of repulsion which takes the values 1, 2 and 4 for orthogonal, unitary and symplectic ensembles respectively. Here the constants \( A \) and \( B \) can be worked out by the normalizing \( P(s) \) to unity, i.e. \( \int P(s)ds = 1 \) and by making average spacing unity by the integral \( \int sp(s)ds = 1 \). Thus \( A \) is given by \( \pi/2, 32/\pi^2 \) and \( (64/9\pi)^3 \) and \( B \) is given by \( \pi/4, 4/\pi \) and \( 64/9\pi \) respectively for \( \beta = 1, 2 \) and 4. Note that Wigner surmise formula for spacing distribution is exact for 2-dimensional random matrices but gives an excellent approximation for large dimensional random matrices [9]. Also, for the Poisson spectra \( P(s) = \exp(-s) \), where level clustering is the salient feature. For Poisson spectra \( P(0) = 1 \) while for the random matrix spectra \( P(0) = 0 \). See Fig. 1.2.

Apart from the spacing distribution, one also studies the number variance \( \Sigma^2(r) \), viz. the variance of number of levels in intervals containing on the average \( r \) levels.
Figure 1.1: Energy vs. magnetic field for the hydrogen atom in magnetic field which shows the level repulsion. Figure is taken from [19,23].
Figure 1.2: Nearest neighbour spacing distribution for the Nuclear Data Ensemble (NDE) comprising 1726 spacings (histogram). The experimental data is compared with the RMT prediction of GOE and the results of Poisson distribution. Figure is taken from [12].

Figure 1.3: Number Variance for the Nuclear Data Ensemble (NDE) which matches with the GOE of RMT. Figure is taken from [12,13].
For Poisson spectra $\Sigma^2(r) = r$, while for the random matrices
\[ \Sigma^2(r) = \frac{2}{\beta \pi^2} \log(r) + C_\beta, \] (1.1.2)
for $r \geq 1$. Here $C_\beta = 0.44, 0.34$ and $0.23$ respectively for $\beta = 1, 2$ and $4$. See Fig. 1.3 for a demonstration of $\Sigma^2(r)$ for the nuclear spectra. In comparison with the Poisson result (1.1.2) implies long-range order in the spectrum.

1.2 Ensembles of Hermitian Matrices

Probability distribution for $N$-dimensional Hermitian matrix, $H$ is given as
\[ P_{N\beta}(H)dH = C'_{N\beta} \exp[-\text{tr } u(H)]dH \] (1.2.1)
where $C'_{N\beta}$ is a normalization constant and $u(H)$ is a function of $H$. The parameter $\beta$ has values 1, 2 and 4, and $\beta$ denotes the number of real sites in the off diagonal matrix elements of $H$. Here $H$ is real symmetric, Hermitian and self-dual Hermitian respectively for $\beta = 1, 2$ and $4$. These ensembles, as mentioned above, are invariant under orthogonal, unitary and symplectic transformations respectively for $\beta = 1, 2$ and $4$ and are referred to as orthogonal, unitary and symplectic ensembles respectively.

From the invariance of ensembles the joint probability density (jpd) of the eigenvalues $x_1, x_2, \ldots, x_N$ of $H$ can be written as [3]
\[ P_{N\beta}(x_1, x_2, \ldots, x_N) = C_{N\beta} \prod_{j>k} |x_j - x_k|^\beta \prod_{j=1}^N w(x_j), \] (1.2.2)
where $w(x) = \exp(-u(x))$ is the weight function and $C_{N\beta}$ is the normalization constant. The fluctuations in the eigenvalue spectra are studied in terms of the $n$-level correlation function
\[ R_{n\beta}(x_1, x_2, \ldots, x_n) = \frac{N!}{(N-n)!} \int dx_{n+1} \ldots \int dx_N P_{N\beta}(x_1, x_2, \ldots, x_N). \] (1.2.3)

Gaussian ensembles were introduced by Wigner [10] and study of statistical properties of the nuclear spectra has been done using these ensembles. For such ensembles, Gaussian weight function is used, i.e. $u(x) = x^2/2v^2$, $v$ being a scale parameter, and also there are no correlations among the matrix elements. As mentioned above, there
are three types of Gaussian ensembles, viz. Gaussian orthogonal ensemble (GOE), Gaussian unitary ensemble (GUE) and Gaussian symplectic ensemble (GSE) which are invariant under orthogonal, unitary and symplectic transformations respectively. The eigenvalue density in these ensembles is given by the Wigner’s semicircle,

\[ R_1(x) = \frac{1}{\pi \beta v^2} \sqrt{2N \beta v^2 - x^2}. \] (1.2.4)

The unfolded spectra exhibit the spacing distribution \( p(s) \) and number variance \( \Sigma^2(r) \) as given in (1.1.1, 1.1.2).

The concept of non-Gaussian ensembles has been introduced by Dyson [39] and Mehta [3, 40]. In the non-Gaussian ensembles, the level density is different from the Wigner’s semicircle and can even exhibit banded behaviour for some weight functions; see e.g., [41]. It has been shown [3, 39, 42, 43] that all the correlation functions, (1.2.3) can be written for \( \beta = 2 \) in terms of orthogonal polynomials \( p_\mu(x) \),

\[
\int p_\mu(x)p_\nu(x)w(x)dx = \delta_{\mu \nu},
\] (1.2.5)

where \( \mu, \nu = 0, 1, 2, \ldots \). For \( \beta = 1 \), certain skew-orthogonal polynomials \( q_\mu(x) \) with the condition

\[
\int \int q_\mu(x)q_\nu(y)e(x-y)w(x)w(y)dxdy = Z_{\mu \nu},
\] (1.2.6)

are needed. Here

\[
Z_{j,k} = \begin{cases} 
1 & \text{if } k = j + 1, \text{ and } j \text{ is even} \\
-1 & \text{if } k = j - 1, \text{ and } j \text{ is odd} \\
0 & \text{otherwise.}
\end{cases}
\] (1.2.7)

Finally for \( \beta = 4 \) another type of skew-orthogonal polynomials \( r_\mu(x) \) are needed,

\[
\int \left[ r'_\mu(x)r_\nu(x) - r_\mu(x)r'_\nu(x) \right] w(x)dx = Z_{\mu \nu}.
\] (1.2.8)

The skew-orthogonal polynomials for Jacobi class of weight functions have been worked out explicitly while for other weight functions the asymptotic forms of the polynomials are established to work out the correlation functions [42, 43]. The local correlation functions are found to be universal for different weight functions and even for the cases where the density develops the banded structure [41].
1.3 Quantum Chaos and Maps

In classical mechanics the dynamics of physical systems are described by Hamilton's equations of motion. If the typical wavelength in quantum problem is very small compared to all length scales of the system then classical description is not sufficient. There is a region of crossover from the quantum regime to the classical regime and is called the semiclassical regime [44,45]. In the semiclassical regime, quantum effects are important while the classical description is relevant as well. The quantum description of chaotic classical dynamics in semiclassical regime is termed as quantum chaos. The positive value of Lyapunov exponent, measuring the exponential separation of trajectories, is a good indicator for classical chaotic systems [45,46]. While in the quantum chaotic systems level repulsion between the energy levels is an important feature.

For time-independent Hamiltonian systems the energy is a constant of motion. The Schrödinger equation of the bound system gives a complete, denumerable set of eigenfunctions with corresponding energy levels, $E$, in the semiclassical regime. The density of states $\rho(E)$ is defined in terms of delta functions as

$$\rho(E) = \sum_j \delta(E - E_j), \quad (1.3.1)$$

where $E_j \leq E_{j+1}$. The cumulative density, the number of states with energy less than some value $E$, will be $\int_{-\infty}^E \rho(E')dE'$ and increases in steps of one as $E$ passes through the energy level. In the semiclassical limit one can introduce a smoothed density of states $\bar{\rho}(E)$,

$$\bar{\rho}(E) = \frac{1}{2\Delta} \int_{E-\Delta}^{E+\Delta} \rho(E')dE' \quad (1.3.2)$$

and corresponding smoothed cumulative density can be defined. The expression for $\bar{\rho}(E)$ is provided by Weyl’s formula [44]. RMT concerns the study of fluctuations in the energy level spectra from the smoothed cumulative density. The fluctuations are studied using spacing distribution and the number variance. See [17] for excellent demonstration of the RMT results in such chaotic systems.

For systems where Hamiltonian varies periodically with time the energy is not conserved. In such systems also the interplay of classical chaos and quantum mechanics is important, as e.g., in atoms and molecules in electromagnetic fields [23]. An exten-
sively studied system is kicked rotors problem which, in the classical case, leads to the
standard map. The Hamiltonian of the system in terms of position coordinate (angle)
\(\psi\) and the corresponding canonical momentum (angular momentum) \(p\) is [45]
\[
H = \frac{p^2}{2I} + \alpha \cos \psi \sum_n \delta(t - n\tau),
\]
where the system can be considered as a pendulum of moment of inertia \(I\), driven
by periodic impulsive force of strength \(\alpha\) in intervals of period \(\tau\). Without loss of
generality we can take \(I = 1\) and \(\tau = 1\). The corresponding one dimensional classical
map for the system can be obtained using Hamilton's equation of motions. We have
\[
p_{n+1} - p_n = \alpha \sin \psi_{n+1}, \\
\psi_{n+1} - \psi_n = p_n \text{ (modulo } 2\pi),
\]
where \(p_n\) and \(\psi_n\) denote the momentum and position after the \(n\)th kick. The corre­
sponding quantum map is discussed in Chapter 7. Thus the dynamics of the system
depends only on the kicking strength, \(\alpha\). By varying the normalized kicking parame­
ter, the dynamics changes from integrable \((\alpha = 0)\) to near integrable \((\alpha \approx 1)\) to fully
chaotic \((\alpha \gg 1)\). Thus corresponding quantum dynamics, as described in terms of
the eigenangle of the evolution operator, makes the transition from Poisson to RMT
behaviour [18,47].

1.4 Circular Ensembles

As mentioned above, since eigenvalues of the unitary matrices lie on the unit circle, the
ensembles of unitary matrices are called circular ensembles. As in the case of ensembles
of Hermitian matrices, there are three universality classes of circular ensembles, viz.
circular orthogonal ensemble (COE), circular unitary ensemble (CUE) and circular
symplectic ensemble (CSE) according to the parameter values, \(\beta = 1, 2\) and \(4\) respec­
tively. The three ensembles, COE, CUE and CSE, are ensembles of symmetric unitary,
general unitary and self-dual unitary matrices. The transformation \(WUW^{-1}\), with \(W\)
as orthogonal \((\beta = 1)\), unitary \((\beta = 2)\), and symplectic \((\beta = 4)\) transformations leaves
the ensembles invariant [3,48]. Here, unlike the case of Gaussian ensembles, the matrix
elements are correlated because of the unitarity conditions.
We consider the ensemble of unitary matrices with the probability density distribution,

\[ P_{N\beta}(U) = c'_{N\beta} \left| \det D(U) \right|^2. \]  (1.4.1)

Here \( c'_{N\beta} \) is a constant and \( D(U) \) defines the weight function,

\[ w(\theta) = |D(e^{i\theta})|^2. \]  (1.4.2)

The joint probability density of eigenangles \( \theta_1, \theta_2, \ldots, \theta_N \) of \( U \) is

\[ P_{N,\beta}(\theta_1, \ldots, \theta_N) = c_{N\beta} \prod_{j>k} |e^{i\theta_j} - e^{i\theta_k}|^{\beta} \prod_{l} w(\theta_l), \]  (1.4.3)

where \( N \) is the dimensionality of the matrices and \( c_{N\beta} \) is the normalization constant and \( w(\theta) \) is the weight function. We will also write,

\[ w(\theta) = e^{-\beta V(\theta)}, \]  (1.4.4)

where \( V(\theta) \) is a periodic function and referred to as a potential.

For uniform weight function, \( w(\theta) = 1 \), the ensembles are known as Dyson ensembles and have been studied in detail [3–8,48]. However, there have been very few results on ensembles with \( \theta \)-dependent weight functions [49]. The main aim of this thesis is an extensive study of nonuniform circular ensembles, viz. ensembles with nonuniform \( w(\theta) \). We consider various aspects of these ensembles [50]. We study the level-density for a wide class of potentials by investigating hierarchic relations among correlation functions. We study correlation functions analytically by introducing polynomials on the unit circle. We show that the energy level fluctuations are universal with respect to the weight function. Breakdown of universality occurs only for long-range fluctuations. We also study the effect of nonuniformity of the weight function on quantum transport properties of mesoscopic systems, particularly the conductance fluctuations as mentioned in Sec. (1.5) ahead. We verify our results by Monte Carlo simulations of the spectra as well as matrix ensembles. Finally, we apply some of these results in systems of quantum kicked rotors.

We have studied the polynomials on the unit circle for nonuniform weight functions to derive the correlation functions. For CUE the orthogonal polynomials \( \phi_\mu \) on the
unit circle are needed and are defined as [51,52]

$$
\int_{-\pi}^{\pi} \phi_\mu(e^{i\theta}) (\phi_\nu(e^{i\theta}))^* w(\theta) d\theta = g_\mu \delta_{\mu\nu},
$$

(1.4.5)

where $g_\mu$ is the normalization constant, $(\phi_\mu(e^{i\theta}))^*$ is the complex conjugate of $\phi_\mu(e^{i\theta})$ and $\mu, \nu = 0, 1, 2, \ldots$. As discussed in section (1.3), for orthogonal and symplectic ensembles of Hermitian matrices, skew-orthogonal polynomials on the real line are needed. Similarly for COE and CSE ($\beta = 1, 4$) one needs to introduce appropriate skew-orthogonal polynomials on the unit circle. We are able to define the skew-orthogonal polynomials on the unit circle for $\beta = 1, 4$ cases in terms of which we can derive the correlation functions for such ensembles. We have explicitly worked out skew-orthogonal polynomials for a few weight functions. For large $N$ and weak potentials, we have proposed asymptotic forms of the polynomials and thereby investigated the universality of fluctuations.

### 1.5 Conductance Fluctuations and Circular Ensembles

There has been lot of interest in the study of conductance fluctuations in mesoscopic systems and disordered wires [19–22, 24–27]. Mesoscopic systems are those that are larger than atoms and very much smaller than the macroscopic objects, in particular they are made up of about thousand atoms. The experimental observations are done at temperatures around or below 100 mK so that electrons are scattered elastically by the impurities. The system is so small in size that a complete quantum mechanical treatment of electrons is essential at low temperatures [19, 24]. On the other hand, the system is so complex and large that its exact microscopic description becomes complicated and hopeless.

The conductance can be understood as the result of interference due to multiple scattering along different paths through the systems. These paths are much longer than the wavelength associated with charge carriers and the accumulated phase through different paths is random. This results in random interference pattern and fluctuations of conductance around the mean value are of the order of $(e^2/h)$ as shown in Fig. 1.4.
Figure 1.4: Fluctuation of conductance around its average value, $\Delta G$ vs. magnetic field $B$. The fluctuations in the conductance is of the order of $e^2/h$. Figure is taken from [19].

Because of reproducible nature of the fluctuations in conductance this is known as universal conductance fluctuations (UCF). The universality of the conductance fluctuations was discovered by Altshuler, Lee and Stone [22] and is experimentally verified by Beenakker and Houten [26]: The universality has two aspects. The variance of the conductance is of the order of $(e^2/h)^2$, independent of sample size and impurity. Also, variance of conductance decreases precisely by a factor two if time reversal symmetry is broken by a magnetic field [19–21,24].

The electronic conductance in mesoscopic systems at low temperatures is well explained by scattering theory [19,25]. Wigner-Dyson distribution is useful in energy level fluctuations of closed quantum chaotic systems and circular ensemble is useful in the study of statistical properties of open systems such as quantum dots with hole. A mesoscopic conductor is modeled by a phase-coherent disordered region connected by ideal leads, with $N$ propagating modes to two electron reservoirs. Scattering matrix, a $2N \times 2N$ matrix relates the amplitude of outgoing wave to the incoming wave, has
the block structure

\[ U = \begin{pmatrix} r & t \\ t' & r' \end{pmatrix}, \tag{1.5.1} \]

with \( N \times N \) reflection matrices \( r, r' \) and transmission matrices \( t, t' \). Flux conservation implies that matrix \( U \) should be unitary matrix. Transport properties are studied using the transmission eigenvalues \( T_n \). The \( T_n \)'s are not the eigenvalues of \( U \). Instead, \( T_n \) is an eigenvalue of the Hermitian matrix \( tt^\dagger \) where \( \dagger \) denotes the Hermitian adjoint which is positive definite and thus \( T_n \geq 0 \). In units of \((e^2/h)\), conductance \( g \) is the sum of the transmission eigenvalues,

\[ g = \sum T_j. \tag{1.5.2} \]

Depending upon the symmetry of the system, the matrix \( U \) is a member of one of the circular ensembles. We have studied the conductance and its fluctuations using the nonuniform circular ensembles as models for the scattering matrices and investigated the effect of nonuniformity on UCF.

It has been shown that there is a close relation between disordered systems and the quantum systems whose classical analogue is chaotic [16, 53, 54]. As discussed in Sec. 1.3, periodically kicked quantum rotor for large strength of the kicking parameter becomes chaotic. Thus the quantum kicked rotor systems are suitable to investigation of the conductance fluctuation phenomenon [55] also.

### 1.6 Summary and Preview

In Chapter 2, we review the quaternion algebra and Dyson-Mehta theorem for correlation functions. We also review the correlation functions for the Dyson ensembles and discuss the correlations in terms of the long-range and short-range fluctuations. In this thesis we will take the fluctuations on the global scale, viz. \( r = O(N/2) \) as the long-range fluctuations and on the local scale \( (r \ll N) \) as the short-range fluctuations.

In Chapter 3, we illustrate a method to obtain eigenangle density for a given weight function. We discuss the banded as well as non-banded behavior of the density by solving the integral equation of the density for both the cases. We also discuss the
universality of long-range correlations for ensembles with weak periodic potentials by the method of parametric differentiation.

In Chapter 4, we have explored the correlation functions in terms of polynomials on the unit circle for nonuniform circular ensembles. To solve the problem we have used orthogonal polynomials on unit circle for CUE and introduced the skew-orthogonal polynomials on the unit circle for COE and CSE. The universality of the correlation functions is established by proposing asymptotic forms of the polynomials on the unit circle for a class of weight functions. The universality of the correlation functions and asymptotic forms of polynomials are verified with explicit workout of orthogonal and the skew-orthogonal polynomials on the unit circle for a few weight functions. We also propose an integral representation formula for the polynomials for each of the three classes of circular ensembles.

In Chapter 5, we give the details of our Monte Carlo study of nonuniform circular ensembles. We have generated circular ensemble spectra for many weight functions. We have verified the eigenangle density and its fluctuation properties like spacing distribution and number variance for different weight functions.

In Chapter 6, we consider the conductance problem. We review the results for the average and variance of the fluctuations in various cases of Dyson ensembles. Next we consider the conductance and its fluctuations for nonuniform circular ensembles. We explore the extent to which the universality of conductance fluctuations is valid. Our results are verified numerically by MC simulations of scattering matrices for nonuniform weight functions.

In Chapter 7, we discuss the applications of the correlation function results in the system of quantum kicked rotors. We consider singly kicked as well as multiply kicked rotors. We find universality of short-range fluctuations in both the systems, but long-range correlations are universal only for multiply kicked rotors. The evolution operator of the system can be treated as scattering matrices and then conductance and its fluctuations can be studied. We find that the universal conductance fluctuation phenomenon is valid for the multiply kicked rotors, but not for the singly kicked rotors. This justifies the importance of newly introduced long-range fluctuations for the conductance fluctuations study. Finally, we have studied the parametric correlations in the system
which deals with the shift in the eigenangles when a parameter of the system is slowly varied.

Our results are summarized in Chapter 8.

Some of the detailed calculations are given in appendices, A.1 – A.10. In Appendix A.1 we give the derivation of the number variance result. In A.2 we have derived the correlation functions in the determinantal form using the orthogonal polynomials for $\beta = 2$. In A.3 we have derived the asymptotic orthogonal polynomials on the unit circle from Jacobi real orthogonal polynomials. In A.4 we have worked out the relation between skew-orthogonal polynomials and its complex conjugate. In A.5 and A.6 we have derived the determinantal form of the correlations functions for $\beta = 1, 4$ in terms of corresponding skew-orthogonal polynomials. Appendices A.7, A.8 and A.9 concern the proof of integral representations of polynomials for $\beta = 1, 2$ and 4 respectively. In A.10 we have discussed some averages for eigenvalues and eigenvectors which are needed for deriving conductance fluctuation results.

Finally, we mention that we will use the notation $\langle A \rangle$ for $N^{-1} \text{tr} A$, $\bar{F}$ for the ensemble average of $F$ and $f^*$ for the complex conjugate of $f$. 

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