1. Introduction

In the present Chapter we consider the second order semilinear abstract Volterra integrodifferential equation of the form

\begin{align}
(1.1) \quad u''(t) &= Au(t) \\
&+ \int_0^t g(t,s,u(s),u'(s))ds + \int_0^s h(s,z,u(z),u'(z))dzds \\
&+ \int_0^t p(t,s,u(s),u'(s))ds + f(t), t \in [-T,T],
\end{align}

\begin{align}
(1.2) \quad u(0) &= x, \quad u'(0) = y, \quad x, y \in B,
\end{align}

where \( A \) is the infinitesimal generator of a strongly continuous cosine family \( C(t), t \in \mathbb{R} \) of bounded linear operators in Banach space \( B \), the functions \( g, h, p \) are in general nonlinear and unbounded, and \( f : \mathbb{R} \to B \).

There exists a vast literature on the theory of ordinary second order differential equations. Results on second
order differential equations are often obtained by replacing the second order differential equations with equivalent systems of first order differential equations and applying the theory developed for the latter. In the development of the theory of second order abstract differential equations in many cases it is advantageous to treat second order abstract differential equations directly rather than to convert them to first order differential systems (see [74]). There are very few references in the literature which deals directly the fundamental theory of the nonlinear second order abstract differential and integrodifferential equations, although these equations are of great importance both in theory and applications. The only papers known to the author are the recent papers of C.C. Travis and G.F. Webb [73], [74], J.A. Goldstein [24], and the results given in [3, Chapter V] which deals with the fundamental theory of existence, uniqueness and continuation of solutions of abstract second order differential and integrodifferential equations. The purpose of this Chapter is to study the existence, uniqueness and continuation of solutions of the equation (1.1) - (1.2) which in turn contains as a special case the equation studied by Travis and Webb [74]. The main tools employed in our analysis are based on the theory of strongly continuous cosine family, the Bielecki type norm coupled with α-norm and graph norm, Schauder's fixed point theorem and Banach's contraction mapping principle.
The Chapter is organized as follows. In section 2, we present the preliminaries and statements of our main results. In section 3 we prove the Theorems 1, 2 and Corollary 1 by using Schauder's fixed point theorem. Section 4 deals with the proofs of Theorems 3, 4 and Corollary 2 by using Banach's contraction mapping principle. In section 5 we discuss examples to illustrate the hypotheses used in the theorems.

Finally we indicate that the factorising technique recently employed by Sanëe.ur in [64] can be easily extended to study the existence, uniqueness and continuation of the solutions of a more general second order system of integro-differential equation in a Banach space.

2. Preliminaries and statements of results.

Let $B$ be a Banach space with the norm $\| \cdot \|$, and $A$ be the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$ of bounded linear operators in $B$. The associated sine family $S(t)$, $t \in \mathbb{R}$ is defined by

$$S(t)x = \int_0^t C(s)x ds, \quad x \in B, t \in \mathbb{R}.$$ 

There exist constants $M^* > 1$ and $\mu > 0$ such that $|C(t)| \leq M^* e^{\mu |t|}$, $t \in \mathbb{R}$ and

$$|S(t) - S(t_1)| \leq M^* \int_{t_1}^t e^{\mu |s|} ds, \quad t, t_1 \in \mathbb{R} \text{ (see [73]).}$$

For a strongly continuous cosine family we define $E = \{ x \in B : C(t)x$ is once continuously differentiable on $\mathbb{R} \}$. We note that (see [74]) $S(t)B \subseteq E$ for $t \in \mathbb{R}$, $S(t)E \subseteq D(A)$ for $t \in \mathbb{R}$,
where \( D(A) = \{ x \in \mathcal{B} : C(t) x \text{ is twice continuously differentiable function of } t \} \), \( \frac{d}{dt} C(t) x = AS(t) x \) for \( x \in \mathcal{E} \) and \( t \in \mathbb{R} \), \( \frac{d^2}{dt^2} C(t) x = AC(t) x = C(t) A x \) for \( x \in D(A) \) and \( t \in \mathbb{R} \).

In [20] it is proved that for \( 0 \leq \alpha \leq 1 \) the fractional powers \((-A)^{\alpha}\) exist as closed linear operators in \( \mathcal{B} \), \( D((-A)^{\alpha}) \subseteq D((-A)^{\beta}) \) for \( 0 \leq \beta \leq \alpha \leq 1 \), and \((-A)^{\alpha}(-A)^{\beta}\) for \( 0 \leq \alpha + \beta \leq 1 \).

Let \( C(t), t \in \mathbb{R} \), be a strongly continuous cosine family in \( \mathcal{B} \) satisfying, for \( 0 < \alpha \leq 1 \), \((-A)^{\alpha}\) maps onto \( \mathcal{B} \) and is one to one, so that \( D((-A)^{\alpha}) \) is a Banach space denoted by \( \mathcal{B}_\alpha \), when endowed with the norm \( \| x \| = \| (-A)^{\alpha} x \| \), \( x \in \mathcal{B}_\alpha \), and further assume that \( (-A)^{\alpha} \) is compact. In [74, Lemma-2.1] the authors have proved under the above assumptions that the following are true. For \( 0 < \alpha < 1 \), \((-A)^{\alpha}\) is compact if and only if \( (-A)^{\alpha} \) is compact, for \( 0 < \alpha < 1 \) and \( t \in \mathbb{R} \), \((-A)^{\alpha} C(t) = C(t)(-A)^{\alpha}\) and \((-A)^{\alpha} S(t) = S(t)(-A)^{\alpha}\).

For the operator \( A \), let \([D(A)]\) is a Banach space denoted by \( \mathcal{B}_A \) which is the domain of \( A \) with the graph norm \( \| x \|_A = \| x \| + \| Ax \| , x \in D(A) \). Let \( D_1, D_2 \) are the open subsets of \( \mathcal{B}_A \) respectively, let \( \mathcal{L} \) is an open subset of \( \mathcal{B} \). For each \((x, y) \in D_1 \times \mathcal{L}(i=1, 2)\) there exists a neighbourhood \( D_{x, y} \) about \((x, y)\) such that \( D_{x, y} \subseteq D_i \times \mathcal{L} \).

Throughout the chapter we assume that \( f : \mathcal{R} \to \mathcal{B} \) is continuously differentiable. Let \( p : \mathcal{R} \times D_1 \times \mathcal{L} \to \mathcal{B} \) is
continuous and continuously differentiable with respect to the first argument, the partial derivative of $p$ is denoted by $p_1$ and is such that $p_1 : RXRD_1X \to B$ is continuous. The function $h : RXRX_1X \to B$ is continuous. The function $g : RXRX_1X \to B$ is continuous and continuously differentiable with respect to the first argument, the partial derivative of $g$ is denoted by $g_1$ and is such that $g_1 : RXRX_1X \to B$ is continuous. Without further mention we assume that the functions $M_j, N_j \in \mathbb{C}[RXR_*]$, where $(j = 1, \ldots)$

Let $(x, y) \in D_{x, y}$, and $D_{x, y}$ be a neighbourhood about $(x, y)$ and let $U$ be another neighbourhood about $(x, y)$ such that $\bar{U} \subset D_{x, y}$. Let $X_1, X_2$ be the complete metric spaces of continuous functions $\phi : [-T, T]$ to $B_\alpha$ and $\psi : [-T, T]$ to $B_\beta$ respectively, and are continuously differentiable from $[-T, T]$ to $B$;

and which have the metric

$$f(\phi, \bar{\phi}) \overset{\text{def}}{=} \sup_{t \in [-T, T]} \left( \left\| d(t) - \bar{\phi}(t) \right\|_\alpha + \left\| \phi'(t) - \bar{\phi}'(t) \right\|_\alpha \right),$$

and

$$f(\psi, \bar{\psi}) \overset{\text{def}}{=} \sup_{t \in [-T, T]} \left( \left\| \psi(t) - \bar{\psi}(t) \right\|_\beta + \left\| \psi'(t) - \bar{\psi}'(t) \right\|_\beta \right),$$

respectively.
Define

$$X_1 = C([-T,T]; H_\alpha) \cap C'([-T,T]; B)$$

with the norm

$$\| \phi \|_{X_1} \overset{\text{def}}{=} \max_{\phi \in X_1} \left\{ \| \phi(t) \|_{\alpha} + \| \phi'(t) \|_{B} \right\},$$

then \((X_1, \| \cdot \|)\) is a Banach space.

and

$$X_2 = \left\{ \psi \in C([-T,T]; H_\alpha) \cap C'([-T,T]; B); (\psi(t), \psi'(t)) \in \mathcal{U} (t \in [-T,T]) \right\},$$

for \(\psi, \tilde{\psi} \in X_2\),

$$\int (\psi, \tilde{\psi}) = \sup_{t \in [-T,T]} \left( \| \psi(t) - \tilde{\psi}(t) \|_{\alpha} + \| \psi'(t) - \tilde{\psi}'(t) \|_{B} \right).$$

then \((X_2, \int)\) is a complete metric space.

Let \(M_\delta > 0\) and \(\gamma > 0\) are constants satisfying

\[(H_2)' \quad \delta \left\{ \frac{\gamma |t|}{\alpha-1} t \right\} - \| (-A)^{-\alpha-1} t \| \left\{ \right. \| \int_0^t \| C(t-s) \| \left( \hat{P}_1(s,s) + \hat{P}_2(s,s) \right) \| ds \right. \]

\[\left. - \int_0^t \int_0^s \| C(t-s) \| \hat{P}_2(s,s) dsds \right\} ^\alpha \geq \| (-A)^{-\alpha-1} t \left. \| \left. \left[ \| C(t)x \| + \| A_s(t)y \| + \| C(t)f(s) \| + \| f(t) \| \right. \right. \right. \]

\[\left. + \int_0^t \| C(t-s) \| f'(s) ds \right\| + \| C(t)x \| + \| C(t)y \| \int_0^t \| C(t-s) f(s) \| ds \right\| \]

for all \(t \in [-T,T]\), for \(\hat{P}_1, \hat{P}_2, (\text{see } H_2, \text{ P.135})\).
Define

\[(2.1) \quad \mathbb{Z} = \{ \phi \in \mathbb{X}_1; \| \phi(t) \| \leq M_0 e^{-\lambda |t|}, t \in [-T,T] \}\,.
\]

It is easy to observe that \( \mathbb{Z} \) is closed, convex, bounded subset of \( \mathbb{X}_1 \). The norm in \( \mathbb{Z} \) (see, \([57],[62]\)) is defined by

\[(2.2) \quad |\phi|_Z = \sup_{t \in [-T,T]} \left\{ \left( \| \phi(t) \| + \| \phi'(t) \| \right) e^{-\lambda |t|} \right\}.
\]

It is easily seen that \( \mathbb{Z} \) with the norm (2.2) is Banach space.

From (2.1), (2.2) we have

\[(2.3) \quad |\phi|_Z \leq M_0\,.
\]

Let \( \mathbb{G} \) be the subspace of \( \mathbb{X}_2 \) such that

\[(2.4) \quad \mathbb{G} = \{ \psi \in \mathbb{X}_2; M_0 = M_0(\psi) \leq (\| \psi(t) \|_A + \| \psi'(t) \|_A) e^{-\lambda |t|} \leq M_0, t \in [-T,T] \}
\]

for \( \psi, \tilde{\psi} \in \mathbb{G}, d(\psi, \tilde{\psi}) = \sup_{t \in [-T,T]} \left\{ \left( \| \psi(t) - \tilde{\psi}(t) \|_A + \| \psi'(t) - \tilde{\psi}'(t) \|_A \right) e^{-\lambda |t|} \right\},
\]

then \((\mathbb{G}, d)\) is complete metric space. The norm in \( \mathbb{G} \) (see, \([5],[62]\)) is defined by
It is easily seen that $G$ with the norm (2.5) is a Banach space. From (2.4), (2.5) we have

\[(2.6) \quad |U| \leq \mathbb{H}_0.\]

For the sake of simplicity we shall make use of the following notations in our subsequent discussion:

\[
\Phi(t,s,u(s)) = g(t,s,u(s),u'(s), \int_0^s h(s,z,u(z),u'(z))dz),
\]

\[
\Phi^1(t,s,u(s)) = g_1(t,s,u(s),u'(s), u'(s), \int_0^s h(s,z,u(z),u'(z))dz),
\]

\[
\Phi(t,s,u(s)) = p(t,s,u(s),u'(s)),
\]

\[
\Phi^1(t,s,u(s)) = p_1(t,s,u(s),u'(s)).
\]

The functions $p, p_1, h, g, g_1$ defined on their respective domains when $i = 1$ satisfy the following hypotheses.

\[(H_1) \quad \text{For all } t,s \in \mathbb{R}, (x_1,y_1) \in D_{x,y}, (x_1,y_1;z) \in D_{x,y} \]
such that

\[ \| \mathbf{P}(t,s,x^1,y_1) \| \leq M_1(t,s)(\| x_1 \|_\alpha + \| y_1 \|), \]

\[ \| \mathbf{p}_1(t,s,x^1,y_1) \| \leq M_2(t,s)(\| x_1 \|_\alpha + \| y_1 \|), \]

\[ \| \mathbf{h}(t,s,x^1,y_1) \| \leq M_3(t,s)(\| x_1 \|_\alpha + \| y_1 \|), \]

\[ \| g(t,s,x^1,y_1,z) \| \leq M_4(t,s)(\| x_1 \|_\alpha + \| y_1 \| + \| z \|), \]

\[ \| g_1(t,s,x^1,y_1,z) \| \leq M_5(t,s)(\| x_1 \|_\alpha + \| y_1 \| + \| z \|). \]

\((H_2)\) From \((H_2)\) we have for \( L_0 \), \( \alpha \) replaced by \( M_0 \),

\[ \| (-A)^{\alpha-1} \| \left[ \| C(t)Ax \| + \| AS(t)y \| + \| C(t)f(o) \| + \| f(t) \| \right. \]

\[ + \int_0^t (\| C(t-s) \| (\| f'(s) \| + M_0^A P_1(s,s)) + M_0^A P_2(t,s)) ds \right. \]

\[ + \| S(t)Ax \| + \| C(t)y \| + \int_0^t (\| C(t-s) \| (\| f(s) \| \]

\[ + M_0 \int_0^s P_2(s,z) dz) ds \leq L_0 e^{-t}, \]

where,
The functions \( p, p_1, h, s, g_1 \) defined on their respective domains when \( i = 2 \) satisfy the following hypotheses.

\((H_3)\) For all \( t, s \in \mathbb{R}, (x_1, y_1), (x_2, y_2) \in D_{x, y}, (x_1, y_1, z_1), (x_2, y_2, z_2) \in D_{x, y, z} \) such that

\[
\| p(t, s, x_1, y_1) - p(t, s, x_2, y_2) \| \\
\leq N_1(t, s)(\| x_1 - x_2 \|_A + \| y_1 - y_2 \|),
\]

\[
\| p_1(t, s, x_1, y_1) - p_1(t, s, x_2, y_2) \| \\
\leq N_2(t, s)(\| x_1 - x_2 \|_A + \| y_1 - y_2 \|),
\]
\[ \| h(t,s,x_1,y_1) - h(t,s,x_2,y_2) \| \leq N_3(t,s)(\| x_1 - x_2\|_A + \| y_1 - y_2 \|), \]

\[ \| g(t,s,x_1,y_1,z_1) - g(t,s,x_2,y_2,z_2) \| \leq N_4(t,s)(\| x_1 - x_2\|_A + \| y_1 - y_2 \| + \| z_1 - z_2 \|), \]

\[ \| g_1(t,s,x_1,y_1,z_1) - g_1(t,s,x_2,y_2,z_2) \| \leq N_5(t,s)(\| x_1 - x_2\|_A + \| y_1 - y_2 \| + \| z_1 - z_2 \|). \]

\[(H_4)\] There exists a nonnegative constant \( \hat{L}_0^* \) such that

\[ \int_0^t \left[ |c(t-s)| (P_1^*(s,s) + \int_0^s P_1^*(s,z) \, dz) + P_2^*(s,s) \right. \]

\[ + \left. |s(t-s)| \int_0^s P_2(s,z) \, dz \, ds \right] \leq \hat{L}_0^* e^{\lambda |t|}, \]

where \( P_1^* , P_2^* \) are obtained from \( \hat{P}_1 , \hat{P}_2 \) given in \((H_2)\), replacing \( M_j \) by the corresponding continuous functions \( N_j(j=1,\ldots,5) \), and the constant \( \hat{\gamma} \) by \( \lambda \).
There exists a nonnegative constant \( L \) such that

\[
\|C(t)x + S(t)y\|_{A} + \|S(t)A_{x + C(t)y}\| + \|f(t)\| + \|C(t)f(\theta)\|
\]

\[
+ \int_{0}^{t} (|S(t-s)||f(s)| + |C(t-s)||\hat{\Phi}(s,s,\theta)| + \|\hat{\Phi}(s,s,\theta)\|
\]

\[
+ \|f'(s)||f(s)|| + \|\hat{\Phi}(t,s,\theta)\| + \|\hat{\Phi}(t,s,\theta)\|
\]

\[
+ |S(t-s)| \int_{0}^{s} (\|\hat{\Phi}(s,z,\theta)\| + \|\hat{\Phi}(s,z,\theta)\|) \, dz
\]

\[
+ |C(t-s)| \int_{0}^{s} (\|\hat{\Phi}_1(s,z,\theta)\| + \|\hat{\Phi}_1(s,z,\theta)\| + \|\hat{\Phi}_2(s,z,\theta)\|)
\]

\[
+ \|\hat{\Phi}(s,z,\theta)\|) \, dz \, )ds \leq L e^{\lambda|t|},
\]

where \( \theta \) is a zero element in \( B \).

Let \( W_1 \) and \( W_2 \) are bounded sets in \( RXRD_{1,2}^{\mathbb{A}} \) and \( RXRD_{1,2}^{\mathbb{A,B}} \) respectively, the functions \( p, p_i, h \), are continuous on \( W_1 \) and the functions \( g, g_i \) are continuous on \( W_2 \) such that they satisfy \( (H_1) \) and \( (H_2) \) when \( i = 1 \) and \( 2 \) respectively by replacing the corresponding continuous functions \( M_j, N_j : RXR \rightarrow R^+ \) by \( M_j, N_j : RXR \rightarrow R^+(j=1, \ldots, 5) \) for all \((t,s,x_1,y_1) \in W_1, (t,s,x_1,y_1,z) \in W_2 \) and \((t,s,x_1,y_1), (t,s,x_2,y_2) \in W_1, (t,s,x_1,y_1,z_1), (t,s,x_2,y_2,z_2) \in W_2 \).
Our main results are established in the following theorems.

**THEOREM 1.** Let the hypotheses \((H_1)\) and \((H_2)\) hold. Let \((x, y) \in \mathcal{D}_1 X \cap \mathcal{E}, (-A)^{-1} \gamma \in \mathcal{E}\), then there exists a twice continuously differentiable function \(u \in \mathcal{E}\), which is a solution of (1.1) - (1.2).

**THEOREM 2.** Let the hypothesis \((H_6)\) holds for \(i = 1\). If \((x, y) \in \mathcal{D}_1 X \cap \mathcal{E}, (-A)^{-1} \gamma \in \mathcal{E}\), and \(u\) is a solution of (1.1) - (1.2) noncontinuable to the right on \([c, d]\), then either \(d = +\infty\) or given any closed bounded set \(U\) in \(\mathcal{D}_1 X \cap \mathcal{E}\), there is a sequence \(t_n \rightarrow d^{-}\) such that \((u(t_n), u'(t_n)) \notin U\). Similar result holds for solution noncontinuable to the left.

**COROLLARY 1.** Let the hypotheses of Theorem 2 holds.

If \(x \in \mathcal{D}(A), (-A)^{-1} \gamma \in \mathcal{E}\), and \(u\) is a solution of (1.1) - (1.2) non-continuable to the right on \([c, d]\), then either \(d = +\infty\) or \(\lim_{t \rightarrow d^-} (\|u(t)\|_\alpha + \|u'(t)\|) = +\infty\). Similar result holds for a solution noncontinuable to the left.

**THEOREM 3.** Let the hypotheses \((H_3) - (H_5)\) hold. Let \((x, y) \in \mathcal{D}_2 X \cap \mathcal{E}\) and \(L_0 < 1\), then there exists a unique twice continuously differentiable function \(u \in \mathcal{E}\), which is a solution of (1.1) - (1.2).
THEOREM 4. Let the Hypothesis \( (H_i) \) holds for \( i = 2 \). If \((x,y) \in D_2 X \mathcal{L}, y \in E\) and \( u \) is a solution of \((1.1) - (1.2)\) non-continuable to the right on \([0,\alpha)\), then either 
\[ d = +\infty \] or given closed bounded set \( U \) in \( D_2 X \mathcal{L}, \) there is a sequence \( t_n \to d^- \) such that \( (u(t_n), u'(t_n)) \not\in U \).

Similar results hold for a solution non-continuable to the left.

COROLLARY 2. Let the hypotheses of Theorem 4 holds, let 
\[ D_2 X \mathcal{L} = B_\alpha X B. \] If \( x \in D(A), y \in E\) and \( u \) is a solution of \((1.1) - (1.2)\) non-continuable to the right on \([0,\alpha)\), then either 
\[ d = +\infty \] or 
\[ \lim_{t \to d^-} (\|u(t)\| + \|u'(t)\|) = +\infty. \]
Similar result holds for a solution non-continuable to the left.

REMARK 1. It is to be noted that in \([74]\) Travis and Webb have established the results on existence, uniqueness and continuation of solution of the special form of \((1.1) - (1.2)\). Here the problem \((1.1) - (1.2)\) is of more general type and our conditions on the functions involved in \((1.1) - (1.2)\) are different from those used by Travis and Webb in \([74]\).

We require the following lemma proved in \([74]\).

**Lemma 1.** Let \( A \) be the infinitesimal generator of a strongly continuous cosine family \( C(t), t \in \mathbb{R} \) of bounded linear operators in the Banach space \( E \). Let \( v : \mathbb{R} \to \mathbb{B} \) such that \( v \) is continuously differentiable and let,
q(t) = \int_0^t S(t-s)v(s)ds. Then q is twice continuously differentiable and for t \in \mathbb{R}, q(t) \in D(A), q'(t) = \int_0^t C(t-s)v(s)ds, and q''(t) = \int_0^t C(t-s)v'(s)ds + C(t)v(s) = Aq(t) + v(t) for 0 \leq \alpha \leq 1 and t \in \mathbb{R}, \mathbb{R}^{d-1} q(t) \in E.

3. Proofs of Theorems 1, 2 and Corollary 1

Define the mapping K on Z by

\begin{align}
(3.1) \quad (Ku(t)) &= C(t)x + S(t)y + \int_0^t S(t-s)\int_0^s \hat{\Phi}(s,z,u(z))dzds \\
&\quad + \int_0^t S(t-s)\int_0^s \hat{\Phi}(s,z,u(z))dzds + \int S(t-s)f(s)ds,
\end{align}

for u \in Z, t \in [-T,T], then

\begin{align}
(3.2) \quad (Ku)'(t) &= S(t)x + C(t)y + \int_0^t C(t-s)\int_0^s \hat{\Phi}(s,z,u(z))dzds \\
&\quad + \int_0^t C(t-s)\int_0^s \hat{\Phi}(s,z,u(z))dzds \\
&\quad + \int C(t-s)f(s)ds,
\end{align}

u \in Z, t \in [-T,T]. From (3.1) and using Lemma 1 and Hypothesis (H_1) we have
\[(3.3) \| (Ku)(t) \|_\alpha \leq \| (-A)^{\alpha-1} \| \| C(t)Ax + AS(t)y \|
\]
\[
+ \int_0^t C(t-s) (\Phi(s, s, u(s)) + \int_0^s \Phi_1(s, z, u(z)) dz) ds
\]
\[
- \int_0^t \Phi(t, s, u(s)) ds + \int_0^t C(t-s) (\Phi(s, s, u(s))
\]
\[
+ \int_0^s \Phi_1(s, z, u(z)) dz ds
\]
\[
- \int_0^t \Phi(t, s, u(s)) ds
\]
\[
+ \int_0^t C(t-s) f'(s) ds + C(t) f(0) - f(t)
\]
\[
\leq \| (-A)^{\alpha-1} \| \left[ \| C(t)Ax \| + \| AS(t)y \| + \| \int_0^t C(t-s) f'(s) ds \| \right]
\]
\[
+ \| C(t) f(0) \| + \| f(t) \|
\]
\[
+ \int_0^t \| C(t-s) |M_4(s, s)( \| u(s) \|_\alpha
\]
\[
+ \| u'(s) \|
\]
\[
+ \int_0^s |M_3(s, z) ( \| u(z) \|_\alpha + \| u'(z) \|) dz ds
\]
\[
+ \int_0^t |C(t-s) |M_5(s, z)( \| u(z) \|_\alpha + \| u'(z) \|)
\]
\[
+ \int_0^z M_2(z, r)( \| u(r) \| + \| u'(r) \|) dr dz ds
\]
Similarly from (3.2) and using Hypothesis \((H_1)\) we have

\[
(3.4) \quad \| (Ku)'(t) \| \leq \| S(t)Ax \| + \| C(t)y \| + \int_0^t |C(t-s)| \int_0^s M_4(s,z)(\| u(z) \| + |u'(z)|) \, dz \, ds \\
+ \int_0^t \int C(t-s) |C(t-s)| M_3(s,z)(\| u(z) \| \| u'(z) \| + |u'(z)|) \, dz \, ds
\]

From (3.3), (3.2) and using (2.3) and Hypothesis \((H_2)\) we have

\[
(3.5) \quad (\| (Ku)(t) \| + \| (Ku)'(t) \| ) \leq L_0 e^{\frac{\gamma}{t}},
\]
which implies

\[ |Ku|_2 \leq L_2. \tag{3.6} \]

This shows that $K$ maps $Z$ into itself.

We next show the continuity of $K$, since $g, g_1, h, p, p_1$ are continuous, we have for a given $\epsilon > 0$ there exists a $\delta > 0$ such that for $u_1, u_2 \in Z$, and $\max \{ |u_1 - u_2|, |u_1' - u_2'| \} < \delta$, we have for $t, s \in [-T, T]$,

\[
\sup_{s \in [-T, T]} \| \hat{E}(t, s, u_1(s)) - \hat{E}(t, s, u_2(s)) \| < \epsilon, \tag{3.7}
\]

\[
\sup_{s \in [-T, T]} \| \hat{E}_1(t, s, u_1(s)) - \hat{E}_1(t, s, u_2(s)) \| < \epsilon,
\]

\[
\sup_{s \in [-T, T]} \| \hat{E}(t, s, u_1(s)) - \hat{E}(t, s, u_2(s)) \| < \epsilon.
\]

From (3.1), and using Lemma 1 and (3.7) we have for $u_1, u_2 \in Z$, $t \in [-T, T]$.

\[
\| (Ku_1)(t) - (Ku_2)(t) \|_\alpha \leq 2\epsilon \|(A) \|_{-1} \left[ \int_0^t (Me^{-\mu|t-s|} (1+s) + 1)ds \right], \tag{3.8}
\]

and from (3.2)
Thus from (3.8), (3.9) we have

\[( \|Ku_1(t) - (Ku_2)(t)\|_\alpha + \|Ku_1'(t) - (Ku_2)'(t)\| ) \leq \epsilon_1, \]

where \[\epsilon_1 = 2\epsilon \left[ \left\| (\Lambda) \right\| \left\| \int_0^t (M e^{\mu(t-s)}(1+s)^{\gamma-1}) \, ds \right\| \right] \]

This shows the continuity of \( K \).

To show the equicontinuity of \( \{ Ku : u \in \mathcal{Z} \} \), we have by (3.1) for \(-T \leq t \leq t_1 \leq T\)

\[(3.10) \quad \|Ku(t) - (Ku)(t_1)\|_\alpha \leq \epsilon_1 \left[ \int_0^t (M e^{\mu(t-s)}(1+s)^{\gamma-1}) \, ds \right] \]

\[+ \left\| \int_0^t C(t-s) \bar{\Phi}(s,s,u(s)) \, ds \right\| \]

\[+ \left\| \int_0^t C(t-s) \bar{\Phi}(s,s,u(s)) \, ds \right\| \]

\[+ \left\| \int_0^t C(t-s) \bar{\Phi}_1(s,z,u(z)) \, dz \right\| \]

\[+ \left\| \int_0^t C(t-s) \bar{\Phi}_1(s,z,u(z)) \, dz \right\| \]
where \( R_1, R_2, \ldots, R_{11} \) are the first, second, \ldots, eleventh terms on the right side of (3.10) respectively. Now we shall obtain the estimate for \( R_3 \). Using (2.3) and Hypothesis (H₁) we have
\[ R_3 \leq \left\| \int_0^t (C(t-s)-C(t_1-s)) \Phi(s,u(s)) ds \right\| \\
\quad + \left\| \int_{t_1}^t C(t_1-s) \Phi(s,u(s)) ds \right\| \\
\leq \int_0^t \left| C(t-s)-C(t_1-s) \right| (M_4(s,s)(\|u(s)\| + \|u'(s)\|) \\
\quad + \int_0^s M_2(s,z)(\|u(z)\| + \|u'(z)\|) dz) ds \right| ds \\
\quad + \int_{t_1}^t \left| C(t_1-s) \right| (M_4(s,s)(\|u(s)\| \\
\quad + \|u'(s)\| + \int_0^s M_3(s,z)(\|u(z)\| + \|u'(z)\|) dz) ds \right| ds. \\
\]

Similarly we can obtain the estimates on \( R_4, R_6, R_7 \). Using (2.3) and Hypothesis (H_4) we have

\[ R_5 \leq \left\| \int_0^t \Phi(s,s,u(s)) ds \right\| + \left\| \int_0^t \int_0^s \Phi(s,z,u(z)) dz ds \right\| \\
\leq \int_0^t (M_4(s,s)M_0(e + \int_0^s M_3(s,z)e^{\int z} dz) ds \\
\quad + \int_{t_1}^t (M_5(s,z)M_0(e + \int_0^z M_3(s,z) e^{\int z} dz) dr) dz ds. \]
Similarly we can obtain the estimate for $R_7$. It is easy to see that

$$R_9 \leq \| \int_0^t (S(t-s)-S(t_1-s))f'(s)ds \| + \| \int_0^t S(t_1-s)f'(s)ds \| .$$

From (3.2) we have for $-T \leq t \leq t_1 \leq T$

$$\| (Ku)'(t) - (Ku)'(t_1) \|$$

$$\leq \| (S(t)-S(t_1))A x \| + \| (C(t) - C(t_1))y \|$$

$$+ \| \int_0^t C(t-s) \int_0^s \mathcal{D}(s,z,u(z))dzds$$

$$- \int_0^t C(t_1-s) \int_0^s \mathcal{D}(s,z,u(z))dzds \|$$

$$+ \| \int_0^t C(t-s) \int_0^s \mathcal{D}(s,z,u(z))dzds$$

$$- \int_0^t C(t_1-s) \int_0^s \mathcal{D}(s,z,u(z))dzds \|$$

$$+ \| \int_0^t C(t-s)f(s)ds - \int_0^t C(t_1-s)f(s)ds \| = \sum_{n=1}^\infty R_n^*$$

We can estimate $R_7^*$, $R_4^*$, $R_5^*$ in the similar fashion as in $R_7$ and $R_9$. Thus from (3.10) and (3.11) we have

$$\| (Ku)(t) - (Ku)(t_1) \| + \| (Ku)'(t) - (Ku)'(t_1) \|$$

$$\leq \left[ R_1^* + R_2^* + R_3^* + R_4^* + R_5^* \right]$$

$$+ Q_1 \int_0^t \left| C(t-s) - C(t_1-s) \right| ds$$

$$+ Q_1 \int_0^t \left| C(t_1-s) \right| ds + Q_2 |t-t_1| \rightarrow 0,$$
as \(|t-t_1| \to 0\), uniformly, where

\[
Q_1 = \max_{s \in [-T,T]} \left[ \|(-A)^{-1} \|^\alpha M_0^\wedge P_1(s,s) \|f'(s)\| + \int_0^s \int P_2(s,z) M_0 dz + \|f(s)\| \right],
\]

and

\[
Q_2 = \max_{s \in [-T,T]} \left[ \|(-A)^{-1} \|^\alpha M_0^\wedge P_1(s,s) \right],
\]

since \((-A)^{\alpha^{-1}}\) is compact. Thus the equicontinuity of \(\{Ku: u \in \mathcal{Z}\}\) follows.

Finally, we shall show that for each fixed \(t \in [-T,T]\), the set \(\{Ku: u \in \mathcal{Z}\}\) is precompact in \(B_\alpha\) since \((-A)^\alpha: B \to B_\alpha\) is compact it will be sufficient if we show that \(\{(-A)^\alpha(Ku(t)): u \in \mathcal{Z}\}\) is bounded.

From (3.1), (3.2) we obtain for \(\alpha\) as in (3.6)

\[
(3.13) \quad |Ku|_{\mathcal{Z}} \leq \hat{L}_0^\wedge,
\]

where \(\hat{L}_0\) is obtained from \(L_0\) in \((H_2)\). This shows the boundedness of \(\{(-A)^\alpha(Ku(t)): u \in \mathcal{Z}\}\). By Schauder's fixed point theorem \(K\) has a fixed point in \(\mathcal{Z}\) which is however a solution of (1.1) - (1.2). This completes the proof of Theorem 1.
Suppose that \( d < \infty \) and the conclusion of Theorem 2 fails, then there is a closed bounded set \( U \) in \( D^X \land \) such that \( (u(t),u'(t)) \in U \) for \( \hat{t} \leq t < d \), where \( 0 \leq \hat{t} < d \).

Let \( u(t) \) satisfies the equation

\[
(3.14) \quad u(t) = C(t)x + S(t)y + \int_{0}^{t} S(t-s) \int_{0}^{s} \Phi(s,z,u(z))dzds
\]

\[
+ \int_{0}^{t} C(t-s) \int_{0}^{s} \Gamma(s,z,u(z))dzds + \int_{0}^{t} C(t-s)f(s)ds,
\]

for \( 0 \leq t \leq d \). For \( \hat{t} < t < t_1 < d \), and using Hypothesis \( (H_6) \) we have as in (3.12).

\[
(3.15) \quad \| u(t) - u(t_1) \|_\alpha + \| u'(t) - u'(t_1) \|_\alpha \leq \left[ \sum_{j=1}^{5} R_1 + R_2 + R_0 + R_1^* + R_2^* + \hat{Q}_1 \int_{0}^{t} |C(t-s) - C(t_1-s)|ds \right. \\
\left. + \hat{Q}_1 \int_{t}^{t_1} |C(t_1-s)|ds + \hat{Q}_2 |t-t_1| \right],
\]

where \( \hat{Q}_1, \hat{Q}_2 \) are obtained from \( Q_1, Q_2 \) in (3.12) replacing the continuous functions \( M_j : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+ \) by the corresponding continuous functions \( \hat{M}_j : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+ \) \((j=1, \ldots, 5)\). By (3.15) we see that

\[
\lim_{t \to t_1} \left( \| u(t) - u(t_1) \|_\alpha + \| u'(t) - u'(t_1) \|_\alpha \right) = 0.
\]

For \( 0 \leq t < d \), let
\( 3.16 \) \[ \hat{x} = C(d)x + S(d)y + \int_0^d S(d-s) \int_0^s \hat{\Phi}(s,z,u(z))dzds \]
\[ + \int_0^d S(d-s) \int_0^s \hat{\Phi}(s,z,u(z))dzds \]
\[ + \int_0^d S(d-s)f(s)ds, \]
and
\( 3.17 \) \[ \hat{y} = S(d)Ax + C(d)y + \int_0^d C(d-s) \int_0^s \hat{\Phi}(s,z,u(z))dzds \]
\[ + \int_0^d C(d-s) \int_0^s \hat{\Phi}(s,z,u(z))dzds \]
\[ + \int_0^d C(d-s)f(s)ds. \]

Then \( \lim_{t \to \alpha} (\|u(t)-x\| + \|u'(t)-y\|) = 0 \). By Lemma 1, \( \alpha \in \mathbb{E} \), thus by Theorem 1, we can find a solution for \( \gamma \leq t \leq \alpha \) to the equation
\( 3.18 \) \[ \nu(t) = C(t)\hat{x} + S(t)\hat{y} + \int_0^t S(t-s) \int_0^s \left( \hat{\Phi}(s+d,z+d,u(z+d)) \right. \]
\[ + \left. \hat{\Phi}(s+d,z+d,v(z+d)) \right)dzds \]
\[ + \int_0^t S(t-s) \left[ \int_{-d}^0 \left( \hat{\Phi}(s+d,z+d,u(z+d)) \right. \]
\[ + \left. \hat{\Phi}(s+d,z+d,v(z+d)) \right)dz + f(s+d) \right]ds. \]
Extending $u$ to $[0, d+d_1]$ by defining $u(t) = v(t-d)$, for $d \leq t \leq d + d_1$, then for $d \leq t \leq d + d_1$, we have

$$u(t) = C(t-d) + S(t-d) \int_0^t S(t-s) \int_0^s (\Phi(s+d, z+d, u(z+d)) \, d\phi \, ds \, dz \, ds$$

It is easy to see that the equation (3.19) satisfies the equation (3.14) for $0 < t \leq d + d_1$. This contradicts the noncontinuability assumption and the proof of Theorem 2 is complete.

Assume $\lim_{t \to d^-} (\|u(t)\| + \|u'(t)\|) < \infty$. Let $U$ be the closure in $B_{\alpha}X$ of $\{u(t): 0 \leq t < d\}$, then $U$ is closed and bounded and $U \subset D_{\alpha}X$, then by applying Theorem 2 we have $d = + \infty$. This completes the proof of Corollary 1.
4. Proofs of Theorems 3, 4 and Corollary 2

Define the mapping $H$ on $G$ by

$$
(4.1) \quad (Hu)(t) = G(t)x + S(t)y + \int_0^t S(t-s) \int_0^s \Phi(s,z,u(z)) \, dz \, ds
$$

$$
+ \int_0^t S(t-s) \int_0^s \Phi(s,z,u(z)) \, dz \, ds + \int_0^t S(t-s) f(s) \, ds,
$$

for $u \in G$, $t \in [-T,T]$. Since $h$ is continuous, $g$, $p$, $f$ are continuously differentiable, we define the functions $F$ and $F_1$ by

$$
F(s) = \int_0^s \Phi(s,z,u(z)) \, dz \quad \text{and} \quad F_1(s) = \int_0^s \Phi(s,z,u(z)) \, dz, \quad -T \leq s, z < T,
$$

where $F(s)$, $F_1(s)$ are continuously differentiable from $[-T,T] \to \mathbb{R}$, and

$$
F'(s) = \Phi(s,s,u(s)) + \int_0^s \Phi_1(s,z,u(z)) \, dz,
$$

$$
F_1'(s) = \Phi(s,s,u(s)) + \int_0^s \Phi_1(s,z,u(z)) \, dz.
$$

Using Lemma 1, we have for $u \in G$, $-T \leq t \leq T$, $(Hu)(t) \in D(A)$, and by (4.1)
(4.2) \((AHu)(t) = G(t)Ax + AS(t)y\)

\[
+ \int_0^t C(t-s) [\Phi(s,s,u(s)) + \int_0^s \Phi_1(s,z,u(z))dz]ds
\]

\[
- \int_0^t \Phi(t,s,u(s))ds
\]

\[
+ \int_0^t C(t-s) [\hat{\Phi}(s,s,u(s)) + \int_0^s \hat{\Phi}_1(s,z,u(z))dz]ds
\]

\[
- \int_0^t \hat{\Phi}(t,s,u(s))ds
\]

\[
+ \int_0^t C(t-s) f'(s)ds - f(t) + C(t)f(0).
\]

Thus for \(u \in G\), \(Hu\) is continuous from \([-T,T]\) to \(B_A\). Further

(4.3) \((Hu)'(t) = S(t)Ax + C(t)y + \int_0^t C(t-s) \int_0^s \hat{\Phi}(s,z,u(z))dzds\)

\[
+ \int_0^t C(t-s) \int_0^s \hat{\Phi}_1(s,z,u(z))dzds
\]

\[
+ \int_0^t C(t-s)f(s)ds, u \in G, t \in [-T,T].
\]

Thus, for \(u \in G\), \(Hu\) is continuously differentiable from \([-T,T]\) to \(B\).

Now we show that \(H\) maps \(G\) into itself. From (4.1) and Hypothesis \((H_2)\) we have for \(u \in G\), \(-T \leq t \leq T\),
$$\leq \| \mathcal{S}(t)\tau + S(t)y \|
+ \| \int_0^t S(t-s) \int_0^s \left( \dot{h}(s,z,\theta) - \dot{h}(s,z,\theta) \right) dz \, ds \|
+ \| \int_0^t S(t-s) \int_0^s \left( \dot{\gamma}(s,z,u(z)) - \dot{\gamma}(s,z,u(z)) \right) dz \, ds \|
+ \| \int_0^t S(t-s) f(s) ds \|
+ \| \mathcal{S}(t)\tau + S(t)y \|
+ \int_0^t \int_0^s \left( \| u(r) \|_A + \| u'(r) \| \right) dr dz \, ds
+ \int_0^t \int_0^s \| \mathcal{E}(s,z,\theta) \| dz \, ds
+ \int_0^t \int_0^s \left( \| u(z) \|_A + \| u'(z) \| \right) dz \, ds
+ \int_0^t \int_0^s \| \mathcal{E}(s,z,\theta) \| dz \, ds
+ \int_0^t \| f(s) \| ds.$$
Similarly from (4.2) we have

\[(4.5) \| (AHu)(t) \| \leq \| C(t)Ax + AS(t)y \| \]

\[
+ \int_0^t |C(t-s)|N_4(s,s)(\| u(s) \|_A + \| u'(s) \|) ds
\]

\[
+ \int_0^s N_3(s,z)(\| u(z) \|_A + \| u'(z) \|) \|dz \| ds
\]

\[
+ \int_0^t |C(t-s)| |\tilde{\Phi}(s,s,\theta)\| ds
\]

\[
+ \int_0^t |C(t-s)| \int_0^s \| \tilde{\Phi}_1(s,z,\theta) \| dz ds
\]

\[
+ \int_0^t |C(t-s)|N_1(s,s)(\| u(s) \|_A + \| u'(s) \|) ds
\]

\[
+ \int_0^t |C(t-s)| |\tilde{\Phi}(s,s,\theta)\| ds
\]

\[
+ \int_0^t |C(t-s)| \int_0^s \| \tilde{\Phi}_1(s,z,\theta) \| dz ds
\]

\[
+ \int_0^t |C(t-s)| \int_0^s \| \tilde{\Phi}_2(s,z,\theta) \| dz ds
\]

\[
+ \int_0^t N_4(t,s)(\| u(s) \|_A + \| u'(s) \|)
\]
\[ + \int_0^t N_2(s,z)(\|u(z)\|_A + \|u'(z)\|)dz ds + \int_0^t \| \Phi(t,s,\theta) \| ds \]

\[ + \int_0^t \| N_1(t,s)(\|u(s)\|_A + \|u'(s)\|)ds \]

\[ + \int_0^t \| \Phi(t,s,\theta) \| ds \]

\[ + \int_0^t |C(t-s)| \| f'(s) \| ds + \| f(t) \| + \| C(t) f(\circ) \| \]

and from (4.3) we have

\[ (4.6) \| (Hu)'(t) \| \leq \| S(t) A x + C(t) y \| \]

\[ + \int_0^t |C(t-s)| \int_0^s N_4(s,z)(\|u(z)\|_A + \|u'(z)\|)dz ds \]

\[ + \int_0^t \| \Phi(s,z,\theta) \| dz ds \]

\[ + \int_0^t |C(t-s)| \int_0^s N_1(s,z)(\|u(z)\|_A + \|u'(z)\|)dz ds \]

\[ + \int_0^t \| \Phi(s,z,\theta) \| dz ds \]

\[ + \int_0^t |C(t-s)| \| f(s) \| ds. \]
From $(H_4)$, $(H_5)$, (2.6), (4.4), (4.5) and (4.6) we obtain

$$(\| (Hu)(t) \|_A + \| (Hu)'(t) \|) \leq (L + N \circ L^*) e^{\lambda |t|},$$

and hence

$$(4.7) \quad |Hu| \leq (L + N \circ L^*).$$

This shows that $H$ maps $G$ into itself.

Next we show that $H$ is a contraction map. Let $u_1, u_2 \in G$, then by (4.1), (4.2), (4.3) and $(H_4)$ we have

$$(4.8) \quad (\| (Hu_1)(t) - (Hu_2)(t) \|_A + \| (Hu_1)'(t) - (Hu_2)'(t) \|)$$

$$\leq L^* |u_1 - u_2| e^{\lambda |t|},$$

which implies

$$(4.9) \quad |Hu_1 - Hu_2| \leq L^* |u_1 - u_2|.$$  

Since $L^* < 1$, the mapping $H$ is a contraction on $G$, it follows from Banach's fixed point theorem that the operator $H$ has a unique fixed point in $G$. The fixed point of $G$ is however a solution of (1.1) - (1.2).

This completes the proof of Theorem 3.
Suppose $d<\infty$ and the conclusion of the Theorem 4 fails. Then there is a closed bounded set $U$ in $D_{2X\mathcal{N}}$ such that 

$$ (u(t), u'(t)) \in U \quad \text{for} \quad t_1 < t < d, \quad \text{where} \quad 0 < t_1 < d. $$

For $t_1 < t < t + a < d$ we have

\begin{equation}
(4.10) \quad u(t+a) = G(t+a)x + S(t+a)y
\end{equation}

\[ + \int_{t}^{t+a} S(t+s) \int_{s}^{t+a} \Phi(s, z, u(z)) dz ds \]

\[ + \int_{t}^{t+a} S(t+s) \int_{s}^{t+a} \Phi(s, z, u(z)) dz ds \]

\[ + \int_{t}^{t+a} S(t+s)f(s)ds. \]

Thus for $t_1 < t < t + a < d$, we have from (3.14) and (4.10)

\begin{equation}
(4.11) \quad \| u(t+a) - u(t) \| < \| (G(t+a) - G(t))x \| + \| (S(t+a) - S(t))y \|
\end{equation}

\[ + \| \int_{t-a}^{t} S(t-s) \int_{s}^{s+a} \Phi(s, z, u(z)) dz ds \|
\]

\[ + \| \int_{t-a}^{t} S(t-s) \int_{s}^{s+a} \Phi(s, z, u(z)) dz ds \|
\]

\[ + \| \int_{t}^{\infty} S(t-s)f(s+a)ds \|
\]

\[ + \| \int_{t-a}^{t} S(t-s) \int_{s}^{s+a} \Phi(s, z+a, u(z+a)) dz ds \|. \]
where $R_1, R_2, \ldots, R_{10}$ are first, second, ..., tenth terms on the right side of (4.11) respectively. Now we shall obtain the estimates for $R_g$ and $R_{10}$. Let

$p^*(t) = (\| u(t+a) - u(t) \| + \| u'(t+a) - u'(t) \|)$. Using the Hypothesis (H) we have

$$R_g \leq \| \int_0^t S(t-s) \int_0^s \Phi(s+a, z+a, u(z+a))$$

$$\quad - \Phi(s, z, u(z+a)))dzds\|$$

$$+ \int_0^t |S(t-s)| \int_0^s \mathcal{L}_1(s, z)$$

$$\int_a \| h(z+a, r+a, u(r+a), u'(r+a))\| dr$$
\begin{align*}
+ \int_0^z h(z+a, r+a, u(r+a), u'(r+a))
- h(s, r, u(r+a), u'(r+a))\|dr \right)dzds \\
+ \int_0^t |S(t-s)| \int_0^s S\tilde{N}_1(s, z)(p^*(z)+\int_0^s S\tilde{N}_2(z, r)p^*(r)dr)dzds,
\end{align*}

and

\begin{align*}
\hat{R}_{40} &\leq \| \int_0^t |S(t-s)| \int_0^s \hat{N}_1(s, z+a, u(z+a)) - \hat{N}(s, z, u(z+a)) \right)dzds \\
+ \int_0^t |S(t-s)| \int_0^s S\tilde{N}_1(s, z)p^*(z)dzs.
\end{align*}

Thus there exists a continuous function \( N_1 : \mathbb{R}^+ \to \mathbb{R}^+ \) satisfying

\( N_1(0) = 0 \) such that

\begin{equation}
(4.12) \| u(t+a) - u(t) \| \leq N_1(a) + \int_0^t |S(t-s)| \int_0^s N_4(s, z)(p^*(z)
+ \int_0^z N_3(z, r)p^*(r)dr)dzds
+ \int_0^t |S(t-s)| \int_0^s N_1(s, z)p^*(z)dzs.
\end{equation}

Similarly there exist continuous functions \( N_2, N_3 : \mathbb{R}^+ \to \mathbb{R}^+ \) satisfying \( N_2(0) = 0, N_3(0) = 0 \) such that
\[(4.13) \quad \| A(u(t+a) - u(t)) \| \]

\[< N_2(a) + \int_0^t |C(t-s)| \tilde{N}_4(s,s)(p^*)(s) \]

\[+ \int_0^{s_{21}} N_3(s,z) p(z) dz \] \[ds \]

\[+ \int_0^t |C(t-s)| \int_0^{s_{21}} N_5(s,z)(p^*(z)) \]

\[+ \int_0^{z_{21}} N_2(z,r)p^*(r) dr \] \[dz \] \[ds \]

\[+ \int_0^{t_{21}} N_4(t,s)p^*(s) + \int_0^{s_{21}} N_2(s,z)p^*(z)dz \] \[ds \]

\[+ \int_0^t |C(t-s)| \tilde{N}_1(s,s)p(s) ds \]

\[+ \int_0^t |C(t-s)| \int_0^{s_{21}} N_2(s,z)p(z)dz \] \[ds \]

\[+ \int_0^{t_{21}} N_4(t,s)p^*(s) ds, \]

and
Let $N(a) = N_1(a) + N_2(a) + N_3(a)$, then from (4.12), (4.13) and (4.14) we have

\[ p^*(t) \leq N(a) + \int_0^t |C(t-s)| \left( \sum N_4(s,z) + \sum N_1(s,z) \right) \] 

\[ + \int_0^t \int_0^t \int_0^t \left( |S(t-s)| \left( \sum N_4(s,z) + \sum N_1(s,z) \right) \right) p^*(z)dzdz \] 

\[ + \int_0^t \int_0^t \int_0^t \left( |S(t-s)| \left( \sum N_4(s,z) + \sum N_1(s,z) \right) \right) p^*(z)dzdz. \]
There exist constants $L_1, L_2, L_3$ such that

\begin{equation}
(4.15) \quad p(t) \leq N(a) + \int_0^t p(s) ds,
\end{equation}

where $L_4 = L_1 + L_2 + L_3$.

Using Gronwall's lemma to (4.15), we have for $t_1 < t < +a < a$,

\begin{equation}
(4.16) \quad p(t) \leq N(a) \exp(L_4 t).
\end{equation}

Thus, $\lim_{t \to d^-} (u(t), u'(t))$ exists in $U$. Using Theorem 3 and argument of Theorem 2, we see that the solution $u$ can be continued past $d$, contradicting the noncontinuability assumption and the proof of Theorem 4 is complete.

The proof of Corollary 2 is immediate in view of the proof of Corollary 1.

5. Examples

In this section we consider the partial integro-differential equation of the form
\begin{equation}
 u_{tt}(t,x) = u_{xx}(t,x) + \int_0^t \delta_1(t,s,u_{xx}(s,x), u_t(s,x)) ds + \int_0^t \delta_2(t,s,u_{xx}(s,x), u_t(s,x)) ds + \delta_3(t,x), 0 < x < \pi, t \in \mathbb{R},
\end{equation}

with the given initial and boundary conditions

\begin{equation}
 u(t,0) = u(t,\pi) = 0, t \in \mathbb{R}, u(0,x) = u_0(x), u_t(0,x) = u_1(x).
\end{equation}

We first reduce the problem (5.1) - (5.2) to the form (1.1) - (1.2) by making suitable choice of \( A, f, p, h, \gamma \) and then illustrate the hypotheses of our main results established in section 2. Our examples are the modifications of examples given by Travis and Webb in \[74\]. For the study of equations of the type (5.1) - (5.2) see \[18\], \[74\].

**Example 5.1**: Let \( B = L^2[0,\pi] \) and \( A: B \to B \) is the infinitesimal generator (see \[74\]) of a strongly continuous cosine family \( C(t), t \in \mathbb{R} \) in \( B \) and is defined by

\( Az = z'' \), \( D(A) = \{ z \in B, z, z' \text{ are absolutely continuous} \} \). Let \( \delta : \mathbb{R}^\times \to \mathbb{R} \), \( \delta' : \mathbb{R}^\times \to \mathbb{R} \) are continuous and continu...
differentiable with respect to the first argument, $\mathcal{C}_1 : \mathbb{R}^{n} \times \mathbb{R} \to \mathbb{R}$ is continuous. Let the functions $f : \mathbb{R} \to \mathbb{B}$, $p : \mathbb{R}^{n} \times \mathbb{R} \to \mathbb{B}$, $h : \mathbb{R}^{n} \times \mathbb{R} \times \mathbb{R} \to \mathbb{B}$, $g : \mathbb{R}^{n} \times \mathbb{R}^{n} \times \mathbb{R} \to \mathbb{B}$ are defined as follows

$$f(t)(x) = \mathcal{C}_3(t,x), \quad x \in [a, T], \quad t \in \mathbb{R},$$

$$(p(t,s,z_1,z_2)(x) = \mathcal{C}_2(t,s,z_1(x),z_2(x)),$$

$$(t,s,z_1,z_2) \in \mathbb{R}^{n} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R},$$

$$(5.3) \quad (h(t,s,z_1,z_2)(x)) = \mathcal{C}_1(t,s,z_1(x),z_2(x)),$$

$$(g((t,s,z_1,z_2)(x),\int_0^s h(s,r,z_1,z_2)(x)dx))$$

$$= \mathcal{C}_2(t,s,z_1(x),z_2(x),\int_0^s \mathcal{C}_1(s,r,z_1(x),z_2(x))dx).$$

With the above choice of $A$, $f$, $h$, $p$, $g$ equation (1.1) - (1.2) becomes the abstract formulation of (5.1).

To show that $(H_4)$ is satisfied, for the continuous functions $N_4$, $N_2$ there exists a constant $N$ such that for $t,s,r,z_1,z_2 \in \mathbb{R}$
\[ |\delta(t, s, z_1^*, z_2^*, \int_0^s \delta_1(s, r, z_1^*, z_2^*) dr)| \]

\[ \leq (M_4(t, s) + M_4(t, s) \int_0^s N_3(s, r)(|z_1^*| + |z_2^*|) dr \]

\[ \leq N_1(|z_1^*| + |z_2^*|). \]

Similarly other conditions of \((H_4)\) and \((H_6)\) can be verified.

**EXAMPLE 5.2.** Let \( p, h: R^2 \times B^3 \rightarrow B \), \( g: R^2 \times B^3 \rightarrow B \), and \( B, A, f, \delta, \sigma_1, \sigma_2, \sigma_3 \) are the same as in Example 5.1. For \((t, s, z_1^*, z_2^*) \in (R^2 \times B^3)\) and \( x \in [0, \pi] \), let \( f, p, h, g, \delta, \sigma_1, \sigma_2, \sigma_3 \) be as in \((5.3)\) then equation \((1.1) - (1.2)\) is the abstract formulation of \((5.1) - (5.2)\).

To show that \((H_2)\) is satisfied, for the continuous functions \( N_4, N_2 \) there exists a constant \( N_1^* \) such that for \( t, s, r, z_1^*, z_2^*, l_1^*, l_2^* \in R \),

\[ |\delta(t, s, z_1^*, l_1^*, \int_0^s \delta_1(s, r, z_1^*, l_1^*) dr) \]

\[ - \delta(t, s, z_2^*, l_2^*, \int_0^s \delta_1(s, r, z_2^*, l_2^*) dr)| \]

\[ \leq (N_4(t, s) + N_4(t, s) \int_0^s N_2(s, r)(|z_1^* - z_2^*| + |l_1^* - l_2^*|) dr \]

\[ \leq N_1^* (|z_1^* - z_2^*| + |l_1^* - l_2^*|). \]

Similarly other conditions of \((H_3)\) and \((H_6)\) can be verified.
REMARK 2. We note that many important problems in the theory of nonlinear abstract differential equations of hyperbolic type have been studied by J.L. Lions in [44]. Many interesting results have been obtained by H. Bronis [8], F.E. Browder [7], V. Parbu [3, Chapter V] and others. Here our problem (1.1) - (1.2) and the method used to study the existence, uniqueness and continuation of solutions are different from the above mention authors.

In a recent paper [54] J.T. Sandefur has used the factorising technique to study the existence, uniqueness and continuation of the solutions of second order nonlinear differential equation of the form

\[(5.4) \quad u''(t) + Mu'(t) + Nu(t) = f(t,u(t)),\]

in an arbitrary Banach space \( B \), where the operators \( M, N \) are in general unbounded in \( B \). To study the equation (5.4) Sandefur has considered the Cauchy problem of the form

\[(5.5) \quad u''(t) - (A_1 + A_2)u'(t) + A_1 A_2 u(t) = f(t,u(t)),\]

\[(5.6) \quad u(0) = 0, \quad u'(0) = 1,\]

where \( A_1, A_2 \) are linear (possibly unbounded) operators on \( B \). The operators \( A_j (j=1,2) \) generates the \((C^0)\) - semi-groups \( S_j (j=1,2) \) and such that \( A_1 A_2 = M, A_2 A_1 = N \). The
solution $u$ is said to be a mild solution of (5.5) - (5.6) if it satisfies

$$
(5.7) \quad u(t) = S_1(t)\phi + \int_0^t S_1(t-z)S_2(z)(\Psi - A_1\phi)dz \\
+ \int_0^t \int_0^z S_1(t-z)S_2(z-s)f(s,u(s))dsdz,
$$

where $\phi \in D(A_1)$. The idea of integral equation (5.7) came from studying (5.4) in the factored form

$$
(5.8) \quad (\partial/\partial t - A_1)(\partial/\partial t - A_2) u(t) = f_1(t,u(t)).
$$

The results established by J.T. Sandefur in [64] for the equation (5.4) can be very easily extended to the form

$$
(5.9) \quad u''(t) + Mu'(t) + u(t) = \int_0^t \left[ a(t,s)g_0(s,u(s)) \\
+ g_1(t,s,u(s)) \right] ds + f_1(t,u(t)) + f_2(t),
$$

$$
(5.10) \quad u(0) = \phi, \ u'(0) = \Psi, \ \phi, \ \Psi \in B,
$$

where $B$, $M$ and $\Psi$ are the same as mentioned above. The mappings $g_0$, $g_1$ and $f_1$ are nonlinear on $B$ and functions $a$ and $f_0$ are Hölder continuous on $B$. The Cauchy problem corresponding to (5.9) - (5.10) can be written as
\[(5.11)\quad u''(t) - (A_1 + A_2)u'(t) + A_2 A_1 u(t) = \int_0^t [a(t,s)g_0(s,u(s)) + g_1(t,s,u(s))ds + f_1(t,u(t)) + f_0(t), \quad t \geq 0,\]

\[(5.12)\quad u(0) = \phi, \quad u'(0) = \psi,\]

where \(A_1\) and \(A_2\) are the same as mentioned above. The mild solution of (5.11) - (5.12) can be written as, see, \(|1, 64|,\]

\[(5.13)\quad u(t) = S_1(t) \phi + \int_0^t S_1(t-z)S_2(z)(\psi - A_1 \phi)dz + \int_0^t \int_0^z S_1(t-z)S_2(z-s) \int_0^s [a(s,r)g_0(r,u(r)) + g_1(s,r,u(r))]drdsdz + \int_0^t \int_0^z S_1(t-z)S_2(z-s)[f_1(s,u(s)) + f_0(s)]dsdz,\]

where \(\phi \in D(A_1), \psi \in \mathbb{R}.\)

In order to establish results on existence, uniqueness, and continuation of solutions by successive approximation, we need the following assumptions.
(H_1) The nonlinear mappings \( g_0, g_1, \) and \( f_1 \) are such that \( g_0, f_1 \) maps from \( R_+ \oplus B \) into \( B \) and \( g_1 \) maps from \( R_+ \oplus R_+ \oplus B \) into \( B \). There are positive nondecreasing functions \( L_i (i=1, 2, 3): [0, \infty) \to (0, \infty) \) such that

\[
\| g_0(t, \phi) \| < L_1(x),
\]

(II)

\[
\| g_1(t, s, \phi) \| < L_2(x),
\]

\[
\| f_1(t, \phi) \| < L_3(x),
\]

and

\[
\| g_0(t, \phi) - g_0(t, \psi) \| < L_1(x) \| \phi - \psi \|,
\]

(II)

\[
\| g_1(t, s, \phi) - g_1(t, s, \psi) \| < L_2(x) \| \phi - \psi \|,
\]

\[
\| f_1(t, \phi) - f_1(t, \psi) \| < L_3(x) \| \phi - \psi \|,
\]

for all \( t \in [0, T], T > 0, \phi, \psi \in B \) and if \( \| \phi \|, \| \psi \| < x \).

(H_2) The functions \( a : B \times B \to B, f_0 : R_+ \to B \), are Folder continuous such that

\[
\| a(t, s) - a(t, s') \| < M (|t - t'|^{\gamma} + |s - s'|^{\gamma}),
\]

\[
\| f_0(t) - f_0(s) \| < M_1 |t - s|^{\beta},
\]

for \( t, s \in [0, T] \), \( ^*, ^* \), \( ^* \) are positive constants and \( \gamma, \beta \) are Holder exponents.

(H_3) The nonlinear mappings \( g_0, g_1, \) and \( f_1 \) are jointly continuous such that \( g_0, f_1 \) maps from \( R_+ \times B \) into \( B \) and \( g_1 \) maps from \( R_+ \times R_+ \times B \) into \( B \). There are positive nondecreasing functions \( L_i (i=1, 2, 3): [0, \infty) \to (0, \infty) \) such that the condition (II) of (H_1) holds.
With the above preparation we are now able to state the following theorems on existence, uniqueness and continuation of the solutions of (5.11) - (5.12).

**Theorem 5.** Let \((H_1), (H_2)\) hold and \(A_1, A_2\) are semigroup generators on \(\mathbb{D}\). Assume that \(g_0, g_1, f_1 \in C(R_+, D)\) when \(u(.) \in C(R_+; D)\). Then there exists a unique continuous function \(u\) satisfying (5.13) for \(t \in [0, \infty)\).

**Theorem 6.** Let \((H_2), (H_3)\) hold and \(A_1, A_2\) are semigroup generators on \(\mathbb{D}\). Let \(\phi \in D(A_1) \cap D(A_2)\) then there exists a unique continuous solution to (5.13) on \(R_+\).

**Theorem 7.** Let \(A_1\) and \(A_2\) are semigroup generators on \(\mathbb{D}\), \(g_0, g_1, f_1 \in C(R_+, D)\) when \(u(.) \in C(R_+; D)\) and \(\phi \in D(A_1) \cap D(A_2)\). Let \(T_0 > 0\) be such that there exists a solution \(u\) to (5.13) on \([0, T_0]\) but that \(u\) can not be continued beyond \([0, T_0]\). Then either \(T_0 = \infty\) or \(\lim \sup_{t \to T_0} \|u(t)\| = +\infty\).

The details of the proofs of Theorems are very close to that given in [64] with suitable modifications, therefore we omit the details.