CHAPTER 5

Ψ-STABILITY FOR NONLINEAR MATRIX DIFFERENCE EQUATIONS
Chapter 5

Ψ-Stability for Nonlinear Matrix Difference Equations

Section 5.1.

Stability theory deals with the stability of solutions of dynamical systems under small perturbations of initial conditions. System stable means the state of the system do not change too much under small perturbations. Classical results relating to stability, asymptotic stability for system of differential equations were studied by many authors [6, 10, 15, 73]. Recently, Diamandescu [20, 23], Murty and Suresh Kumar [60] studied Ψ-stability for differential equations and matrix Lyapunov systems. Difference equations play an important role in many scientific fields, due to their importance Murty et al. [58] studied existence and uniqueness criterion for first order nonhomogeneous matrix difference equations. In this chapter first we obtain sufficient conditions for the Ψ-stability and Ψ-uniform stability of trivial solution of nonlinear difference equation

\[ x(n+1) = A(n)x(n) + f(n, x(n)) \]  (5.1.1)
as a perturbed equation of

\[ x(n + 1) = A(n)x(n), \quad (5.1.2) \]

where \( A(n) \) is a nonsingular matrix-valued functions and \( f(n, x(n)) \) is a vector-valued function of order \( m \) on \( \mathbb{N} = \{0, 1, 2, \ldots\} \) with \( f(n, 0) = 0 \). Here we develop new difference equations corresponding to (5.1.1) and (5.1.2) which are (uniformly) stable on \( \mathbb{N} \), provided (5.1.1) and (5.1.2) are \( \Psi \)-uniformly stable on \( \mathbb{N} \). We investigate conditions on the fundamental matrix \( Y(n) \) of linear equation (5.1.2) and on the function \( f(n, x) \) under which the trivial solution of (5.1.1) is \( \Psi \)-uniformly stable on \( \mathbb{N} \). Further, we prove (necessary and) sufficient conditions for \( \Psi \)-stability and \( \Psi \)-uniform stability of trivial solution of first order nonlinear matrix difference equation

\[ X(n + 1) = A(n)X(n)B(n) + F(n, X(n)), \quad (5.1.3) \]

as a perturbed equation of the linear equation

\[ X(n + 1) = A(n)X(n)B(n), \quad (5.1.4) \]

where \( A(n), B(n) \) are nonsingular and \( F(n, X(n)) \) is a \( m \times m \) matrix-valued functions on \( \mathbb{N} = \{0, 1, 2, \ldots\} \) with \( F(n, 0) = 0 \). Applying technique of Kronecker product of matrices, we investigate conditions on the fundamental matrices of

\[ X(n + 1) = A(n)X(n), \quad (5.1.5) \]
\[ X(n + 1) = B^T(n)X(n), \quad (5.1.6) \]

and on the matrix function \( F(n, X(n)) \) under which the trivial solution of the equations (5.1.3) and (5.1.4) are \( \Psi \)-stable and \( \Psi \)-uniformly stable on \( \mathbb{N} \).

In Section 5.2 we present some definitions relating to \( \Psi \)-stability and \( \Psi \)-uniform stability and provide a way to construct stable or uniformly stable difference equation from the given equation (vector or matrix difference) using the concept of \( \Psi \)-stability and \( \Psi \)-uniform stability. Further, we establish a relationship between \( \Psi \)-stable and \( \Psi \)-uniform stable solutions of matrix difference equation and Kronecker product vector difference equation.

In Section 5.3 we prove necessary and sufficient conditions for \( \Psi \)-stability and \( \Psi \)-uniform stability of linear difference equation (5.1.2) and also obtain sufficient conditions for \( \Psi \)-stability and \( \Psi \)-uniform stability of nonlinear difference equation (5.1.1).

In Section 5.4 we prove necessary and sufficient conditions for \( \Psi \)-stability and \( \Psi \)-uniform stability of linear matrix difference equation (5.1.4) and also obtain sufficient conditions for \( \Psi \)-stability and \( \Psi \)-uniform stability of nonlinear matrix difference equation (5.1.3).

This chapter extend some of the results of Diamandescu [20, 23], Murty and Suresh Kumar [60] on differential equations to difference equations.

Section 5.2.

This section presents some preliminary definitions and results related to \( \Psi \)-stability and \( \Psi \)-uniform stability for difference equations.
Let $\Psi_r : \mathbb{N} \to (0, \infty)$, $r = 1, 2, \ldots m$, and matrix function defined by

$$\Psi = \text{diag}[\Psi_1, \Psi_2, \ldots, \Psi_m].$$

Clearly $\Psi(n)$ is a nonsingular matrix of order $m$.

Now, we state definitions of various types of $\Psi$-stability on $\mathbb{N}$.

**Definition 5.2.1.** The trivial solution of the vector difference equation $x(n + 1) = f(n, x(n))$ ($x \in \mathbb{R}^m$ and $f$ is a vector function of order $m$) is said to be $\Psi$-stable on $\mathbb{N}$ if for every $\varepsilon > 0$ and every $n_0$ in $\mathbb{N}$, there exists $\delta = \delta(\varepsilon, n_0) > 0$ such that any solution $x(n)$ of the equation which satisfies the inequality $\|\Psi(n_0)x(n_0)\| < \delta$ and also satisfies the inequality $\|\Psi(n)x(n)\| < \varepsilon$ for all $n \geq n_0$.

Similarly we can define above definition for matrix difference equation.

**Definition 5.2.2.** The trivial solution of the matrix difference equation $X(n + 1) = F(n, X(n))$ ($X \in \mathbb{R}^{m \times m}$ and $F$ is a $m \times m$ matrix function) is said to be $\Psi$-stable on $\mathbb{N}$ if for every $\varepsilon > 0$ and every $n_0$ in $\mathbb{N}$, there exists $\delta = \delta(\varepsilon, n_0) > 0$ such that any solution $X(n)$ of the equation which satisfies the inequality $|\Psi(n_0)X(n_0)| < \delta$ and also satisfies the inequality $|\Psi(n)X(n)| < \varepsilon$ for all $n \geq n_0$.

**Definition 5.2.3.** The trivial solution of the vector difference equation $x(n + 1) = f(n, x(n))$ is said to be $\Psi$-uniformly stable on $\mathbb{N}$ if it is $\Psi$-stable on $\mathbb{N}$ and in the Definition 5.2.1 $\delta$ is independent of $n_0$.

Similarly we can define above definition for matrix difference equation.
**Definition 5.2.4.** The trivial solution of the matrix difference equation $X(n + 1) = F(n, X(n))$ is said to be $\Psi$-uniformly stable on $\mathbb{N}$ if it is $\Psi$-stable on $\mathbb{N}$ and in the Definition 5.2.2 $\delta$ is independent of $n_0$.

**Remark 5.2.1.** For $\Psi_r = 1, r = 1, 2, \ldots, n$, we obtain the notions of classical stability and uniform stability.

Now, we generate a difference equation, by multiplying $\Psi(n+1)$ on both sides of (5.1.1),

\[
\Psi(n+1)x(n+1) = \Psi(n+1)A(n)x(n) + \Psi(n+1)f(n, x(n))
\]

\[
\Psi(n+1)x(n+1) = \Psi(n+1)A(n)\Psi^{-1}(n)\Psi(n)x(n)
\]

\[
+ \Psi(n+1)f(n, \Psi^{-1}(n)\Psi(n)x(n))
\]

Taking $u(n) = \Psi(n)x(n)$, the above equation can be written as

\[
u(n + 1) = A_\Psi(n)u(n) + f_\Psi(n, u(n)), \quad (5.2.1)
\]

where $u(n) = \Psi(n)x(n)$, $A_\Psi(n) = \Psi(n+1)A(n)\Psi^{-1}(n)$ and $f_\Psi(n, u(n)) = \Psi(n+1)f(n, \Psi^{-1}(n)u(n))$. The corresponding linear equation of (5.2.1) is

\[
u(n + 1) = A_\Psi(n)u(n). \quad (5.2.2)
\]

Similarly, we can generate the following matrix difference equations from
(5.1.3) and (5.1.4) are

\[ U(n + 1) = A_\Psi(n)U(n)B(n) + F_\Psi(n, U(n)) \quad (5.2.3) \]

and

\[ U(n + 1) = A_\Psi(n)U(n)B(n), \quad (5.2.4) \]

where \( U(n) = \Psi(n)X(n), \ A_\Psi(n) = \Psi(n+1)A(n)\Psi^{-1}(n) \) and \( F_\Psi(n, U(n)) = \Psi(n + 1)f(n, \Psi^{-1}(n)U(n)) \).

From Definitions 5.2.1 to 5.2.3 and Remark 5.2.1, we have the following lemma.

**Lemma 5.2.1.** Let \( \Psi \) be an invertible matrix function on \( \mathbb{N} \). Then

1. the trivial solution of (5.1.1) is \( \Psi \)-uniformly stable on \( \mathbb{N} \) if and only if the trivial solution of (5.2.1) is (uniformly) stable on \( \mathbb{N} \);

2. the trivial solution of (5.1.3) is \( \Psi \)-uniformly stable on \( \mathbb{N} \) if and only if the trivial solution of (5.2.3) is (uniformly) stable on \( \mathbb{N} \).

**Remark 5.2.2.** From the lemma 5.2.1 we infer that

1. if the difference equation (5.1.1) is \( \Psi \)-uniformly stable on \( \mathbb{N} \), then we can generate (uniformly) stable difference equation (5.2.1);

2. if the matrix difference equation (5.1.3) is \( \Psi \)-uniformly stable on \( \mathbb{N} \), then we can generate (uniformly) stable matrix difference equation (5.2.3).
The following lemma represent the relationship between fundamental matrices of (5.1.2) and (5.2.2).

**Lemma 5.2.2.** If \( Y(n) \) is the fundamental matrix of (5.1.2), then \( V(n) = \Psi(n)Y(n) \) is the fundamental matrix of (5.2.2).

**Proof.** Since \( Y(n) \) is the fundamental matrix of (5.1.2), then \( Y(n) \) satisfies (5.1.2) and nonsingular. Consider

\[
V(n+1) = \Psi(n+1)Y(n+1) \\
= \Psi(n+1)A(n)Y(n) \\
= \Psi(n+1)A(n)\Psi^{-1}(n)\Psi(n)Y(n) \\
= A\Psi(n)V(n)
\]

Since \( \Psi(n) \) and \( Y(n) \) are nonsingular matrices, then \( V(n) \) is also nonsingular. Therefore, \( V(n) = \Psi(n)Y(n) \) is a fundamental matrix of (5.2.2). \( \square \)

**Lemma 5.2.3.** If \( X(n) \) is a solution of matrix difference equation (5.1.3) if and only if \( \hat{X}(n) = \text{Vec}X(n) \) is a solution of Kronecker product vector difference equation

\[
\hat{X}(n+1) = H(n)\hat{X}(n) + \hat{F}(n, \hat{X}(n)), \tag{5.2.5}
\]

where \( H(n) = (B^T(n) \otimes A(n)) \in \mathbb{R}^{m^2 \times m^2} \) and \( \hat{F}(n, \hat{X}(n)) = \text{Vec}F(n, X(n)) \in \mathbb{R}^{m^2}. \)
The linear difference equation of (5.2.5) is

\[ \hat{X}(n + 1) = H(n)\hat{X}(n). \]  

(5.2.6)

From the proof of Lemma 2.1.1 we have the following lemma.

**Lemma 5.2.4.** For every matrix function \( F : \mathbb{N} \to \mathbb{R}^{m \times m} \), we have

\[
\frac{1}{m} |\Psi(n)F(n)| \leq \| (I_m \otimes \Psi(n)).VecF(n)\|_{\mathbb{R}^{m^2}} \leq |\Psi(n)F(n)|, 
\]

(5.2.7)

**Lemma 5.2.5.** The trivial solution of matrix difference equation (5.1.3) is \( \Psi \)-stable on \( \mathbb{N} \), then the trivial solution of the corresponding Kronecker product vector difference equation (5.2.5) is \( I_m \otimes \Psi \)-stable on \( \mathbb{N} \).

**Proof.** Suppose that the trivial solution of (5.1.3) is \( \Psi \)-stable on \( \mathbb{N} \). From Definition 5.2.2 for every \( \varepsilon > 0 \) and for any \( n_0 \in \mathbb{N} \), there exists a \( \delta = \delta(\varepsilon, n_0) > 0 \) such that any solution \( X(n) \) of the equation (5.1.3) which satisfies the inequality \( |\Psi(n_0)X(n_0)| < \delta_0 \) and satisfies \( |\Psi(n)X(n)| < \varepsilon \), for \( n \geq n_0 \).

Let \( X \) be a solution of (5.2.5) and choose \( \delta_0(\varepsilon, n_0) = \frac{1}{m} \delta(\varepsilon, n_0) \) such that

\[ \|(I_m \otimes \Psi(n_0))\hat{X}(n_0)\| < \delta_0(\varepsilon, n_0). \]

From Lemma 5.2.3 and Lemma 5.2.4, \( X(n) = Vec^{-1}(\hat{X}(n)) \) is a solution of (5.1.3) such that \( |\Psi(n_0)X(n_0)| < \delta(\varepsilon, n_0) \). It follows that \( |\Psi(n)X(n)| < \varepsilon \) for all \( n \geq n_0 \). Again from Lemma 5.2.4, we have that \( \|(I_m \otimes \Psi(n))\hat{X}(n)\| < \varepsilon \) for all \( n \geq n_0 \). Thus the trivial solution of the equation (5.2.5) is \( I_m \otimes \Psi \).
stable on $\mathbb{N}$.

Conversely, suppose that the trivial solution of the equation (5.2.5) is $I_m \otimes \Psi$-stable on $\mathbb{N}$. From Definition 5.2.1, for every $\varepsilon > 0$ and for any $n_0$ in $\mathbb{N}$, there exist a $\delta = \delta(\varepsilon, n_0) > 0$ such that any solution $\hat{X}(n)$ of the equation (5.2.5) satisfies $\| (I_m \otimes \Psi(n_0)) \hat{X}(n_0) \| < \delta$ and also satisfies $\| (I_m \otimes \Psi(n)) \hat{X}(n) \| < \varepsilon$, for $n \geq n_0$.

Let $X$ be a solution of (5.2.5) and choose $\delta_0(\varepsilon, n_0) = \delta(\frac{\varepsilon}{m}, n_0)$ such that

$$|\Psi(n_0)X(n_0)| < \delta_0.$$

From Lemma 5.2.3 and Lemma 5.2.4, the vector function $\hat{X}(n) = Vec(X(n))$ is a solution of the (5.2.5) on $\mathbb{N}$ such that $\| (I_m \otimes \Psi(n_0)) \hat{X}(n_0) \| < \delta_0$. It implies that $\| (I_m \otimes \Psi(n)) \hat{X}(n) \| < \frac{\varepsilon}{m}$, for all $n \geq n_0$.

From Lemma 5.2.4 we have that $|\Psi(n)X(n)| < \varepsilon$ for all $n \geq n_0$. Thus from Definition 5.2.2 the trivial solution of equation (5.1.3) is $\Psi$-stable on $\mathbb{N}$. \hfill $\Box$

**Lemma 5.2.6.** The trivial solution of matrix difference equation (5.1.3) is $\Psi$-uniformly stable on $\mathbb{N}$, then the trivial solution of the corresponding Kronecker product vector difference equation (5.2.5) is $I_m \otimes \Psi$-uniformly stable on $\mathbb{N}$.

**Proof.** The proof follows similar lines as in Lemma 5.2.5 \hfill $\Box$
Section 5.3.

In this section we obtain sufficient conditions for Ψ-stability and Ψ-uniform stability of trivial solution of linear and nonlinear difference equations (5.1.2) and (5.1.1). The results obtained in this section are illustrated with suitable examples.

In the following theorem we obtain necessary and sufficient condition for Ψ-stability and Ψ-uniform stability of trivial solution of linear equation (5.1.2).

**Theorem 5.3.1.** If the fundamental matrix of (5.1.2) is $Y(n)$, then

(i) the trivial solution of (5.1.2) is Ψ-stable on $\mathbb{N}$ if and only if there exists a positive constant $K$ such that $|\Psi(n)Y(n)| \leq K$ for all $n \in \mathbb{N}$;

(ii) the trivial solution of (5.1.2) is Ψ-uniformly stable on $\mathbb{N}$ if and only if there exists a positive constant $K$ such that

$$|\Psi(n)Y(n)Y^{-1}(\ell)\Psi^{-1}(\ell)| \leq K,$$

for all $\ell \leq n$ and $\ell, n \in \mathbb{N}$.

**Proof.** The solution of (5.1.2) with $x(n_0) = x_0$ is $x(n) = Y(n)Y^{-1}(n_0)x_0$ for $n \in \mathbb{N}$.

Suppose that there exist $K > 0$ such that $|\Psi(n)Y(n)| \leq K$ for $n \in \mathbb{N}$. For $\varepsilon > 0$ and $n_0 \in \mathbb{N}$, let $\delta(\varepsilon, n_0) = \frac{\varepsilon}{2K|Y^{-1}(n_0)\Psi^{-1}(n_0)|}$. For
\[ \| \Psi(n_0)x(n_0) \| < \delta \text{ and } n \geq n_0, \text{ we get} \]

\[ \| \Psi(n)x(n) \| = \| \Psi(n)Y(n)Y^{-1}(n_0)\Psi^{-1}(n_0)\Psi(n_0)x(n_0) \| \]
\[ = K|Y^{-1}(n_0)\Psi^{-1}(n_0)|\delta < \varepsilon, \]

which implies that the trivial solution of (5.1.2) is \( \Psi \)-stable on \( \mathbb{N} \).

Conversely suppose that the trivial solution of (5.1.2) is \( \Psi \)-stable on \( \mathbb{N} \). Then, for \( \varepsilon = 1 \) and \( n_0 = 0 \), there exists \( \delta > 0 \) such that any solution \( x(n) \) of (5.1.2) which satisfies the inequality \( \| \Psi(0)x(0) \| < \delta \) and

\[ \| \Psi(n)Y(n)(\Psi(0)Y(0))^{-1}\Psi(0)x(0) \| < 1 \text{ for } n \in \mathbb{N}. \]

Let \( v \in \mathbb{R}^n \) be such that \( \| v \| \leq 1 \). If we take \( x(0) = \frac{\delta}{2} \Psi^{-1}(0)v \), then we have \( \| \Psi(0)x(0) \| < \delta \). Hence, \( \| \Psi(n)Y(n)(\Psi(0)Y(0))^{-1}\frac{\delta}{2}v \| < 1 \) for \( n \in \mathbb{N} \). Therefore, \( |\Psi(n)Y(n)(\Psi(0)Y(0))^{-1}| \leq 2/\delta \text{ for } n \in \mathbb{N}. \) Hence, \( |\Psi(n)Y(n)| \leq K \), a positive constant, for \( n \in \mathbb{N} \).

Proof of part (ii) is similar. \( \square \)

The following example illustrate Theorem 5.3.1.

**Example 5.3.1.** Consider the linear difference equation (5.1.2) with

\[ A(n) = \begin{bmatrix} 0 & 1 \\ \frac{2}{n+1} - 1 & 0 \end{bmatrix}. \]
Then its fundamental matrix is

\[
Y(n) = \begin{bmatrix}
(n + 1) \cos \frac{n\pi}{2} & (n + 1) \sin \frac{n\pi}{2} \\
-(n + 2) \sin \frac{n\pi}{2} & (n + 2) \cos \frac{n\pi}{2}
\end{bmatrix}.
\]

Clearly \(Y(n)\) is unbounded on \(\mathbb{N}\), it follows that the equation (5.1.2) is not stable on \(\mathbb{N}\).

Now, we construct a difference equation (5.2.2) from (5.1.2), which is uniformly stable on \(\mathbb{N}\) with the help of Theorem 5.3.1 and part (1) of Lemma 5.2.1. Consider

\[
\Psi(n) = \begin{bmatrix}
\frac{1}{n+1} & 0 \\
0 & \frac{1}{n+2}
\end{bmatrix}.
\]

Then for all \(\ell \leq n, \ell, n \in \mathbb{N}\), we get

\[
\Psi(n)Y(n)Y^{-1}(\ell)\Psi^{-1}(\ell) = \begin{bmatrix}
\cos \frac{(n-\ell)\pi}{2} & \sin \frac{(n-\ell)\pi}{2} \\
-\sin \frac{(n-\ell)\pi}{2} & \cos \frac{(n-\ell)\pi}{2}
\end{bmatrix}
\]

and \(|\Psi(n)Y(n)Y^{-1}(\ell)\Psi^{-1}(\ell)| \leq 2\). From Theorem 5.3.1 the trivial solution of linear difference equation (5.1.2) is \(\Psi\)-uniformly stable on \(\mathbb{N}\).

By part (1) of Lemma 5.2.1 the difference equation (5.2.2) with

\[
A_\Psi(n) = \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\]

is uniformly stable on \(\mathbb{N}\).
Remark 5.3.1. Ψ-uniform stability implies Ψ-stability but the converse need not be true. It is shown by the following example.

Example 5.3.2. Consider the linear difference equation (5.1.2) with

\[ A(n) = \begin{bmatrix} \left(\frac{n+2}{n+1}\right)^2 - \left(\frac{n+2}{n+1}\right)^2 (2n+3)e^n \\ \frac{2}{n} e^{-1} \end{bmatrix}. \]

Then the fundamental matrix of (5.1.2) is

\[ Y(n) = \begin{bmatrix} (n+1)^2 & 1 \\ 0 & e^{-n} \end{bmatrix}. \]

If

\[ \Psi(n) = \begin{bmatrix} 1 \frac{1}{(n+1)^2} \\ 0 \end{bmatrix}, \]

then

\[ \Psi(n)Y(n) = \begin{bmatrix} 1 \frac{1}{(n+1)^2} \\ 0 \frac{e^{-n}}{n+2} \end{bmatrix}, \]

Clearly \(|\Psi(n)Y(n)| \leq 2\), for all \(n \in \mathbb{N}\). From Theorem 5.3.1, the equation (5.1.2) is \(\Psi\)-stable on \(\mathbb{N}\). On the other hand

\[ \Psi(n)Y(n)Y^{-1}(\ell)\Psi^{-1}(\ell) = \begin{bmatrix} 1 e^\ell \left(\frac{1}{(n+1)^2} - \frac{1}{(\ell+1)^2}\right) \\ 0 e^{\ell-n} \end{bmatrix}. \]

is unbounded for \(0 \leq \ell \leq n, n, \ell \in \mathbb{N}\). Again from Theorem 5.3.1, the
equation (5.1.2) is not $\Psi$-uniformly stable on $\mathbb{N}$.

Now, we consider the nonlinear difference equation (5.1.1) as the perturbed equation of (5.1.2) and provide sufficient conditions on the nonlinear function $f(n, x(n))$ to obtain $\Psi$-stability and $\Psi$-uniform stability of the trivial solution of (5.1.1).

**Theorem 5.3.2.** Suppose that

(i) The linear equation (5.1.2) is $\Psi$-stable on $\mathbb{N}$.

(ii) There exist a sequence $\varphi : \mathbb{N} \to (0, \infty)$ and a positive constant $L$ such that the fundamental matrix $Y(n)$ of the equation (5.1.2) satisfies the condition

$$\sum_{\ell=0}^{n-1} \varphi(\ell)|\Psi(n)Y(n)Y^{-1}(\ell + 1)\Psi^{-1}(\ell + 1)| \leq L, \quad n \in \mathbb{N}. \quad (5.3.1)$$

(iii) The nonlinear function $f$ satisfies the condition

$$|\Psi(n + 1)f(n, x(n))| \leq \frac{\alpha(n)}{\varphi(n)} \|\Psi(n)x(n)\|, \quad (5.3.2)$$

where $\alpha(n)$ is a nonnegative sequence such that

$$\sup_{n \geq n_0} \frac{\alpha(n)}{\varphi(n)} < \frac{1}{L}, \quad (5.3.3)$$

for all $n, n_0 \in \mathbb{N}$ and $x(n) \in \mathbb{R}^m$.

Then, the trivial solution of nonlinear difference equation (5.1.1) is $\Psi$-stable on $\mathbb{N}$.
Proof. From condition (i) and Theorem 5.3.1, it follows that there exists a positive constant $K$ such that $|\Psi(n)Y(n)| \leq K$, for all $n \in \mathbb{N}$.

From (5.3.3), there exists $\beta$ such that

$$\frac{\alpha(n)}{\varphi(n)} \leq \beta < \frac{1}{L}, \text{ for all } n \in \mathbb{N}. $$

For a given $\varepsilon > 0$ and $n_0 \leq n, n_0 \in \mathbb{N}$, we choose

$$\delta = \min \left\{ \varepsilon, \frac{1 - \beta L}{2 \cdot 2K |Y^{-1}(n_0)\Psi^{-1}(n_0)|} \right\}$$

such that $\|\Psi(n_0)x(n_0)\| < \delta$. By variation of constants formula, the solution of (5.1.1) with $x(n_0) = x_0$ is given by

$$x(n) = Y(n)Y^{-1}(n_0)x(n_0) + \sum_{\ell=n_0}^{n-1} Y(n)Y^{-1}(\ell + 1)f(\ell, x(\ell)), \quad (5.3.4)$$

for all $n_0 \leq n \in \mathbb{N}$. Consider

$$\|\Psi(n)x(n)\| \leq |\Psi(n)Y(n)||Y^{-1}(n_0)\Psi^{-1}(n_0)||\Psi(n_0)x(n_0)||$$

$$+ \sum_{\ell=n_0}^{n-1} \varphi(\ell)|\Psi(n)Y(n)Y^{-1}(\ell + 1)\Psi^{-1}(\ell + 1)|$$

$$\|\Psi(\ell + 1)f(\ell, x(\ell))\|$$

$$\leq K|Y^{-1}(n_0)\Psi^{-1}(n_0)|\delta$$

$$+ \sum_{\ell=n_0}^{n-1} \varphi(\ell)|\Psi(n)Y(n)Y^{-1}(\ell + 1)\Psi^{-1}(\ell + 1)|$$

$$\left(\sup_{\ell \in \mathbb{N}} \frac{\alpha(\ell)}{\varphi(\ell)} \right) \|\Psi(\ell)x(\ell)\|$$

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≤ K|Y^{-1}(n_0)\Psi^{-1}(n_0)|\delta + \beta L\|\Psi(n)x(n)\|.

It implies that
\[
\|\Psi(n)x(n)\| \leq \frac{K}{1 - \beta L}|Y^{-1}(n_0)\Psi^{-1}(n_0)|\delta \\
\leq \frac{\varepsilon}{2} < \varepsilon.
\]

From Definition 5.2.1, the trivial solution of nonlinear difference equation (5.1.1) is \(\Psi\)-stable on \(\mathbb{N}\).

The following example illustrate above theorem.

**Example 5.3.3.** Consider the nonlinear difference equation (5.1.1) with
\[
A(n) = \begin{bmatrix}
\frac{n+2}{n+1} & 0 \\
0 & 2
\end{bmatrix}, \quad f(n, x(n)) = \begin{bmatrix}
5^{-n} \left(\frac{n+2}{n+1}\right) \sin(x_1(n)) \\
3(5^{-n})x_2(n)
\end{bmatrix}.
\]

Then the fundamental matrix of (5.1.2) is
\[
Y(n) = \begin{bmatrix}
n+1 & 0 \\
0 & 2^n
\end{bmatrix}.
\]

If
\[
\Psi(n) = \begin{bmatrix}
\frac{1}{n+1} & 0 \\
0 & 3^{-n}
\end{bmatrix}, \quad \phi(n) = 1, \forall \ n \in \mathbb{N}
\]
then
\[
\Psi(n)Y(n) = \begin{bmatrix} 1 & 0 \\ 0 & \left(\frac{2}{3}\right)^n \end{bmatrix}, \quad |\Psi(n)Y(n)| = \left(\frac{2}{3}\right)^n \leq 1, \forall \ n \in \mathbb{N}.
\]

From Theorem 5.3.1, the linear difference equation (5.1.2) is \(\Psi\)-stable on \(\mathbb{N}\).

And also
\[
\Psi(n)Y(n)Y^{-1}(\ell + 1)\Psi^{-1}(\ell + 1) = \begin{bmatrix} 1 & 0 \\ 0 & \left(\frac{2}{3}\right)^n \left(\frac{3}{2}\right)^{\ell+1} \end{bmatrix},
\]

\[
\sum_{\ell=0}^{n-1} \varphi(\ell)|\Psi(n)Y(n)Y^{-1}(\ell + 1)\Psi^{-1}(\ell + 1)| = \left(\frac{2}{3}\right)^n \sum_{\ell=0}^{n-1} \left(\frac{3}{2}\right)^{\ell+1}
\]
\[
= 3 \left(1 - \left(\frac{2}{3}\right)^n\right) \leq 3,
\]

for \(n \in \mathbb{N}\). Therefore, conditions (i) and (ii) of Theorem 5.3.2 are satisfied.

Consider
\[
\Psi(n + 1)f(n, x(n)) = 5^{-n} \begin{bmatrix} \sin(x_1(n)) \\ \frac{n+1}{3^{-n}x_2(n)} \end{bmatrix},
\]

we have
\[
||\Psi(n + 1)f(n, x(n))|| \leq 5^{-n}||\Psi(n)x(n)||,
\]

for all \(n \in \mathbb{N}\). If \(\alpha(n) = 5^{-n}\), then (5.3.2) and (5.3.3) are satisfied. Therefore, all conditions of Theorem 5.3.2 are satisfied. Thus, the nonlinear difference equation (5.1.1) is \(\Psi\)-stable on \(\mathbb{N}\).
From part (1) of Lemma [5.2.1] the difference equation (5.2.1) with

\[ A_\Psi = \begin{bmatrix} 1 & 0 \\ 0 & 2/3 \end{bmatrix} \quad \text{and} \quad F_\Psi(n, z(n)) = \frac{1}{5^n} \begin{bmatrix} \sin((n+1)z_1(n)) \\ n+1 \\ z_2(n) \end{bmatrix} \]

is stable on $\mathbb{N}$.

**Theorem 5.3.3.** Suppose that :

(i) The fundamental matrix $Y(n)$ of the equation (5.1.2) satisfies

\[ |\Psi(n)Y(n)Y^{-1}(\ell)\Psi^{-1}(\ell)| \leq K, \]

for all $n_0 \leq \ell \leq n$, where $K$ is a positive constant.

(ii) The nonlinear function $f$ satisfies the condition

\[ \|\Psi(n+1)f(n, x(n))\| \leq \alpha(n)\|\Psi(n)x(n)\|, \]

where $\alpha(n)$ is a nonnegative sequence on $\mathbb{N}$ such that

\[ N = \sum_{\ell=0}^{\infty} \alpha(n) < \infty. \]

Then, the trivial solution of the nonlinear difference equation (5.1.1) is $\Psi$-uniformly stable on $\mathbb{N}$.

**Proof.** Let $n_0 \in \mathbb{N}$ and $x(n_0) = x_0 \in \mathbb{N}$. Then the solution of (5.1.1) with $x(n_0) = x_0$ is given by (5.3.4). Let $\varepsilon > 0$ and $\delta(\varepsilon) = \frac{e^{-KN}}{2K} \varepsilon > 0$ such that
\[ \| \Psi(n_0) x(n_0) \| < \delta. \] Consider

\[ \| \Psi(n) x(n) \| \leq |\Psi(n) Y(n) Y^{-1}(n_0) \Psi^{-1}(n_0) \| \| \Psi(n_0) x(n_0) \| \]
\[ + \sum_{\ell=n_0+1}^{n} |\Psi(n) Y(n) Y^{-1}(\ell) \Psi^{-1}(\ell) \| \| \Psi(\ell) f(\ell-1, x(\ell-1)) \| \]
\[ \leq K \delta + K \sum_{\ell=n_0}^{n-1} \| \Psi(\ell+1) f(\ell, x(\ell)) \| \]
\[ \leq K \delta + K \sum_{\ell=n_0}^{n-1} \alpha(\ell) \| \Psi(\ell) x(\ell) \|. \]

By Grownwall’s inequality

\[ \| \Psi(n) x(n) \| \leq K \delta \prod_{\ell=n_0}^{n-1} (1 + K \alpha(\ell)) \]
\[ \leq K \delta e^{K \sum_{\ell=n_0}^{n-1} \alpha(\ell)} \]
\[ \leq K \delta e^{K N} \leq \frac{\varepsilon}{2} < \varepsilon. \]

From Definition 5.2.2, the trivial solution of nonlinear difference equation (5.1.1) is \( \Psi \)-uniformly stable on \( \mathbb{N} \).

\[ \square \]

**Example 5.3.4.** Consider the nonlinear difference equation (5.1.1) with

\[ A(n) = \begin{bmatrix} 0 & 1 \\ -\frac{n+3}{n+1} & 0 \end{bmatrix} \text{ and } f(n, x(n)) = \begin{bmatrix} 2^{-n \left( \frac{n+2}{n+1} \right)} x_1(n) \\ 2^{-n \left( \frac{n+3}{n+2} \right)} \sin(x_2(n)) \end{bmatrix}, \]
for all $n \in \mathbb{N}$. The fundamental matrix of (5.1.2) is

$$Y(n) = \begin{bmatrix} (n + 1) \cos \frac{n\pi}{2} & (n + 1) \sin \frac{n\pi}{2} \\ -(n + 2) \sin \frac{n\pi}{2} & (n + 2) \cos \frac{n\pi}{2} \end{bmatrix}.$$

If

$$\Psi(n) = \begin{bmatrix} \frac{1}{n+1} & 0 \\ 0 & \frac{1}{n+2} \end{bmatrix} \quad \text{for} \quad n \in \mathbb{N},$$

then from Example 5.3.1 we have

$$|\Psi(n)Y(n)Y^{-1}(\ell)\Psi^{-1}(\ell)| \leq 2,$$

for $n_0 \leq \ell \leq n$. Consider

$$\Psi(n + 1)f(n, x(n)) = 2^{-n} \begin{bmatrix} \frac{x_1}{n+1} \\ \frac{x_2(n)}{n+2} \end{bmatrix},$$

it follows that, condition (ii) of Theorem 5.3.3 satisfied with $\alpha(n) = 2^{-n}$ (nonnegative sequence) and

$$N = \sum_{\ell=0}^{\infty} \alpha(\ell) = \sum_{\ell=0}^{\infty} 2^{-n} = 2 < \infty.$$

Therefore, from Theorem 5.3.3 the trivial solution of (5.1.1) is $\Psi$-uniformly stable on $\mathbb{N}$. 

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From part (1) of Lemma 5.2.1, the difference equation (5.2.1) with

$$A_\Psi = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and

$$F_\Psi(n, z(n)) = \frac{1}{2^n} \begin{bmatrix} z_1(n) \\ \sin((n + 2)z_2(n)) \\ \frac{n + 2}{n + 2} \end{bmatrix}$$

is uniformly stable on $\mathbb{N}$.

Section 5.4.

In this section, we obtain conditions for $\Psi$-stability and $\Psi$-uniform stability for trivial solutions of linear and nonlinear matrix difference equations (5.1.4) and (5.1.3). These conditions are expressed in terms of fundamental matrices of the matrix difference equations (5.1.5) and (5.1.6).

First, we prove necessary and sufficient conditions for $\Psi$-stability and $\Psi$-uniform stability of trivial solution of linear matrix difference equation (5.1.4).

**Theorem 5.4.1.** Let $Y(n)$ and $Z(n)$ be fundamental matrices of (5.1.5) and (5.1.6). Then,

(i) the trivial solution of (5.1.4) is $\Psi$-stable on $\mathbb{N}$ if and only if there exists a positive constant $M$ such that $|Z(n) \otimes \Psi(n)Y(n)| \leq M$ for all $n \in \mathbb{N}$.

(ii) the trivial solution of (5.1.4) is $\Psi$-uniformly stable on $\mathbb{N}$ if and only
if there exists a positive constant $M$ such that

$$|(Z(n)Z^{-1}(\ell)) \otimes (\Psi(n)Y(n)Y^{-1}(\ell)\Psi^{-1}(\ell))| \leq M,$$

for all $\ell \leq n$ and $\ell, n \in \mathbb{N}$.

**Proof.** Suppose that the trivial solution of (5.1.4) is $\Psi$-stable on $\mathbb{N}$. From Lemma 5.2.3 it follows that the trivial solution of the corresponding Kronecker product vector difference equation (5.2.6) is $I_m \otimes \Psi$-stable on $\mathbb{N}$. From Theorem 5.3.1 Lemma 3.2.4 and Lemma 2.1.2 it follows that there exists a positive constant $M$ such that the fundamental matrix $W = Z(n) \otimes Y(n)$ of (5.2.6) satisfies the condition

$$|(I_m \otimes \Psi(n))W(n)| = |Z(n) \otimes (\Psi(n)Y(n))| \leq M,$$  \hspace{1cm} (5.4.1)

for all $n \in \mathbb{N}$.

Conversely, suppose that there exists a positive constant $M$ such that $|Z(n) \otimes (\Psi(n)Y(n))| \leq M$ for all $n \in \mathbb{N}$. Again from Lemma 2.1.2 the fundamental matrix $W(n) = Z(n) \otimes Y(n)$ of the (5.2.6) satisfies the condition (5.4.1). From Theorem 5.3.1 it follows that the trivial solution of the equation (5.2.6) is $I_m \otimes \Psi$-stable on $\mathbb{N}$. Thus from Lemma 5.2.3 the trivial solution of the (5.1.4) is $\Psi$-stable on $\mathbb{N}$.

The proof of part (ii) is similar, hence we omit it. \qed
Example 5.4.1. Consider the linear matrix difference equation (5.1.4) with

\[
A(n) = \begin{bmatrix}
\frac{n+2}{n+1} & 0 \\
0 & 2
\end{bmatrix}
\quad \text{and} \quad
B(n) = \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}.
\]

Then the fundamental matrices of (5.1.5) and (5.1.6) are

\[
Y(n) = \begin{bmatrix}
\frac{n+1}{n+1} & 0 \\
0 & 2^n
\end{bmatrix}
\quad \text{and} \quad
Z(n) = \begin{bmatrix}
\cos\left(\frac{n\pi}{2}\right) & \sin\left(\frac{n\pi}{2}\right) \\
-\sin\left(\frac{n\pi}{2}\right) & \cos\left(\frac{n\pi}{2}\right)
\end{bmatrix}.
\]

The fundamental matrix of (5.2.6) is

\[
Z(n) \otimes Y(n) = \begin{bmatrix}
(n+1) \cos\left(\frac{n\pi}{2}\right) & 0 & (n+1) \sin\left(\frac{n\pi}{2}\right) & 0 \\
0 & 2^n \cos\left(\frac{n\pi}{2}\right) & 0 & 2^n \sin\left(\frac{n\pi}{2}\right) \\
-(n+1) \sin\left(\frac{n\pi}{2}\right) & 0 & (n+1) \cos\left(\frac{n\pi}{2}\right) & 0 \\
0 & -2^n \sin\left(\frac{n\pi}{2}\right) & 0 & 2^n \cos\left(\frac{n\pi}{2}\right)
\end{bmatrix}
\]

Clearly, the difference equation (5.2.6) is \(\Psi\)-unstable on \(\mathbb{N}\).

Consider

\[
\Psi(n) = \begin{bmatrix}
\frac{1}{n+1} & 0 \\
0 & 2^{-n}
\end{bmatrix}, \quad \text{for} \quad n \in \mathbb{N}.
\]

Then

\[
Z(n) \otimes (\Psi(n)Y(n)) = \begin{bmatrix}
\cos\left(\frac{n\pi}{2}\right) & 0 & \sin\left(\frac{n\pi}{2}\right) & 0 \\
0 & \cos\left(\frac{n\pi}{2}\right) & 0 & \sin\left(\frac{n\pi}{2}\right) \\
-\sin\left(\frac{n\pi}{2}\right) & 0 & \cos\left(\frac{n\pi}{2}\right) & 0 \\
0 & -\sin\left(\frac{n\pi}{2}\right) & 0 & \cos\left(\frac{n\pi}{2}\right)
\end{bmatrix}
\]
and $|Z(n) \otimes (\Psi(n)Y(n))| \leq 2$, for all $n \in \mathbb{N}$. From Theorem 5.4.1, the trivial solution of linear matrix difference equation (5.1.4) is $\Psi$-stable on $\mathbb{N}$.

Moreover, for $\ell \leq n, \ell, n \in \mathbb{N}$, we get

$$(Z(n)Z^{-1}(\ell)) \otimes (\Psi(n)Y(n)Y^{-1}(\ell)\Psi^{-1}(\ell))$$

and $$|(Z(n)Z^{-1}(\ell)) \otimes (\Psi(n)Y(n)Y^{-1}(\ell)\Psi^{-1}(\ell))| \leq 2,$$ for all $n \in \mathbb{N}$. Again from Theorem 5.4.1, the trivial solution of linear matrix difference equation (5.1.4) is $\Psi$-uniformly stable on $\mathbb{N}$.

**Remark 5.4.1.** $\Psi$-uniform stability implies $\Psi$-stability but the converse need not be true for matrix difference equation (5.1.4). It is shown by the following example.

**Example 5.4.2.** Consider the linear difference equation (5.1.5) with

$$A(n) = \begin{bmatrix} 0 & 1 \\ -\left(\frac{n+3}{n+1}\right) & 0 \end{bmatrix} \quad \text{and} \quad B(n) = \begin{bmatrix} 1 - \frac{2n}{(n+1)(n+2)} \\ 0 & 1/2 \end{bmatrix}.$$ 

Then the fundamental matrices of (5.1.5) and (5.1.6) are

$$Y(n) = \begin{bmatrix} (n+1)\cos\left(\frac{n\pi}{2}\right) & (n+1)\sin\left(\frac{n\pi}{2}\right) \\ -(n+2)\sin\left(\frac{n\pi}{2}\right) & (n+2)\cos\left(\frac{n\pi}{2}\right) \end{bmatrix} \quad \text{and} \quad Z(n) = \begin{bmatrix} 1 & 1/(n+1) \\ 0 & 2^{-n} \end{bmatrix}.$$
Clearly, \( Z(n) \otimes Y(n) \) is unbounded and hence the trivial solution of (5.1.4) is unstable on \( \mathbb{N} \).

If
\[
\Psi(n) = \begin{bmatrix}
\frac{1}{n+1} & 0 \\
0 & \frac{1}{n+2}
\end{bmatrix}, \text{ for } n \in \mathbb{N}
\]
then
\[
Z(n) \otimes (\Psi(n)Y(n)) = \begin{bmatrix}
\cos\left(\frac{n\pi}{2}\right) & \sin\left(\frac{n\pi}{2}\right) & \frac{1}{n+1} \cos\left(\frac{n\pi}{2}\right) & \frac{1}{n+1} \sin\left(\frac{n\pi}{2}\right) \\
-\sin\left(\frac{n\pi}{2}\right) & \cos\left(\frac{n\pi}{2}\right) & -\frac{1}{n+1} \sin\left(\frac{n\pi}{2}\right) & \frac{1}{n+1} \cos\left(\frac{n\pi}{2}\right) \\
0 & 0 & 2^{-n} \cos\left(\frac{n\pi}{2}\right) & 2^{-n} \sin\left(\frac{n\pi}{2}\right) \\
0 & 0 & -2^{-n} \sin\left(\frac{n\pi}{2}\right) & 2^{-n} \cos\left(\frac{n\pi}{2}\right)
\end{bmatrix}
\]
and \( |Z(n) \otimes (\Psi(n)Y(n))| \leq 4 \), for all \( n \in \mathbb{N} \). From Theorem 5.4.1, the trivial solution of linear matrix difference equation (5.1.4) is \( \Psi \)-stable on \( \mathbb{N} \).

On the other hand, for \( 0 \leq \ell \leq n \),
\[
(Z(n)Z^{-1}(\ell)) \otimes (\Psi(n)Y(n)Y^{-1}(\ell)\Psi^{-1}(\ell))
\]
\[
= \begin{bmatrix}
\cos\left(\frac{(n-\ell)\pi}{2}\right) & \sin\left(\frac{(n-\ell)\pi}{2}\right) & \frac{2^\ell(n-\ell)}{(\ell+1)(n+1)} \cos\left(\frac{(n-\ell)\pi}{2}\right) & \frac{2^\ell(n-\ell)}{(\ell+1)(n+1)} \sin\left(\frac{(n-\ell)\pi}{2}\right) \\
-\sin\left(\frac{(n-\ell)\pi}{2}\right) & \cos\left(\frac{(n-\ell)\pi}{2}\right) & -\frac{2^\ell(n-\ell)}{(\ell+1)(n+1)} \sin\left(\frac{(n-\ell)\pi}{2}\right) & \frac{2^\ell(n-\ell)}{(\ell+1)(n+1)} \cos\left(\frac{(n-\ell)\pi}{2}\right) \\
0 & 0 & 2^{\ell-n} \cos\left(\frac{(n-\ell)\pi}{2}\right) & 2^{\ell-n} \sin\left(\frac{(n-\ell)\pi}{2}\right) \\
0 & 0 & -2^{\ell-n} \sin\left(\frac{(n-\ell)\pi}{2}\right) & 2^{\ell-n} \cos\left(\frac{(n-\ell)\pi}{2}\right)
\end{bmatrix}
\]
is unbounded on \( \mathbb{N} \). From Theorem 5.4.1, the difference equation (5.1.4) is not \( \Psi \)-uniformly stable on \( \mathbb{N} \).

Now, we consider the nonlinear matrix difference equation (5.1.3) as a
perturbed equation of (5.1.4) and obtain sufficient conditions for Ψ-stability of (5.1.3).

**Theorem 5.4.2.** Suppose that:

(i) the linear matrix difference equation (5.1.4) is Ψ-stable on \( \mathbb{N} \),

(ii) there exist a sequence \( \varphi : \mathbb{N} \to (0, \infty) \) and a positive constant \( L \) such that the fundamental matrices \( Y(n) \), \( Z(n) \) of (5.1.5), (5.1.6) satisfies the condition

\[
\sum_{\ell=0}^{n-1} \varphi(\ell)|Z(n)Z^{-1}(\ell+1)\otimes(Y(n)Y^{-1}(\ell+1)\Psi^{-1}(\ell+1))| \leq L, \quad n \in \mathbb{N},
\]

(ii) the nonlinear matrix function \( F(n, X(n)) \) satisfies

\[
|\Psi(n + 1)F(n, X(n))| \leq \frac{\alpha(n)}{m\varphi(n)}|\Psi(n)X(n)|, \quad (5.4.2)
\]

where \( \alpha(n) \) is a non-negative sequence such that

\[
\sup_{n \geq n_0} \frac{\alpha(n)}{\varphi(n)} < \frac{1}{L}, \quad (5.4.3)
\]

for all \( n, n_0 \in \mathbb{N} \) and \( X(n) \in \mathbb{R}^{m \times m} \).

Then, the trivial solution of nonlinear matrix difference equation (5.1.3) is Ψ-stable on \( \mathbb{N} \).

**Proof.** From condition (5.4.2) and Lemma 5.2.4, we have

\[
\|(I_m \otimes \Psi(n + 1))\tilde{F}(n, \tilde{X}(n))\| \leq |\Psi(n + 1)F(n, X(n))|
\]
\[ \leq \frac{\alpha(n)}{m\varphi(n)} |\Psi(n)X(n)| \]
\[ \leq \frac{\alpha(n)}{\varphi(n)} \| (I_m \otimes \Psi(n))(\hat{F}(n, \hat{X}(n))) \|. \]

for all \( n, n_0 \in \mathbb{N} \) and \( \hat{X}(n) \in \mathbb{R}^{m^2} \). From Lemma 2.1.2 and Theorem 5.3.2, it follows that the trivial solution of the corresponding Kronecker product vector difference equation (5.2.5) is \( I_m \otimes \Psi \)-stable on \( \mathbb{N} \). Thus, from Lemma 5.2.5, the trivial solution of matrix difference equation (5.1.3) is \( \Psi \)-stable on \( \mathbb{N} \).

\[ \square \]

**Example 5.4.3.** Consider the nonlinear matrix difference equation (5.1.3) with

\[
A(n) = \begin{bmatrix}
\frac{n^2+2n+2}{n^2+1} & 0 \\
0 & \frac{n+2}{n+1}
\end{bmatrix}, \quad B(n) = \begin{bmatrix}
\frac{1}{3} & 0 \\
0 & \frac{1}{2}
\end{bmatrix}
\]

and

\[
F(n, X(n)) = \frac{1}{3^{n+1}} \begin{bmatrix}
sin(x_1(n)) & x_3(n) \\
x_2(n) & \tan^{-1}(x_4(n))
\end{bmatrix}.
\]

Then fundamental matrices of (5.1.5) and (5.1.6) are

\[
Y(n) = \begin{bmatrix}
n^2+1 & 0 \\
0 & n+1
\end{bmatrix} \quad \text{and} \quad Z(n) = \begin{bmatrix}
3^{-n} & 0 \\
0 & 2^{-n}
\end{bmatrix}, \quad \text{for all} \quad n \in \mathbb{N}.
\]

If we take

\[
\Psi(n) = \begin{bmatrix}
\frac{1}{n+1} & 0 \\
0 & \frac{1}{n+1}
\end{bmatrix}, \quad \text{for all} \quad n \in \mathbb{N},
\]
then $\left( Z(n)Z^{-1}(\ell + 1) \right) \otimes (\Psi(n)Y(n)Y^{-1}(\ell + 1)\Psi^{-1}(\ell + 1)) = $

$$
\begin{bmatrix}
3^{\ell+1-n} & 0 & 0 & 0 \\
0 & 3^{\ell+1-n} & 0 & 0 \\
0 & 0 & 2^{\ell+1-n} & 0 \\
0 & 0 & 0 & 2^{\ell+1-n}
\end{bmatrix}.
$$

such that

$$
\sum_{\ell=0}^{n-1} \varphi(\ell) |(Z(n)Z^{-1}(\ell+1)) \otimes (\Psi(n)Y(n)Y^{-1}(\ell+1)\Psi^{-1}(\ell+1))| = \frac{3}{2} \left[ 1 - \frac{1}{3^n} \right] \leq \frac{3}{2}.
$$

Therefore, condition (i) of Theorem 5.4.2 satisfied with $L = \frac{3}{2}$, $\varphi(n) = 1$. Consider

$$
\Psi(n + 1)F(n, X(n)) = \frac{1}{3^{n+1}} \begin{bmatrix}
\frac{\sin(x_1(n))}{n^2+2n+2} & \frac{x_3(n)}{n^2+2n+2} \\
\frac{x_2(n)}{n+2} & \frac{\tan^{-1}(x_4(n))}{n+2}
\end{bmatrix},
$$

which implies that

$$
|\Psi(n + 1)F(n, X(n))| \leq \frac{1}{3^{n+1}} |\Psi(n)X(n)|, \text{ for all } n \in \mathbb{N}.
$$

Therefore, condition (ii) of Theorem 5.4.2 satisfied with $\alpha(n) = \frac{2}{3^{n+1}}$, $m = 2$. Thus, all conditions of Theorem 5.4.2 are satisfied. Hence the trivial solution of matrix difference equation (5.1.3) is $\Psi$-stable on $\mathbb{N}$.

In the following theorem, we obtain sufficient conditions for $\Psi$-uniform stability for trivial solution of nonlinear matrix difference equation (5.1.3).
Theorem 5.4.3. Suppose that:

(i) the fundamental matrices $Y(n)$ and $Z(n)$ of (5.1.5) and (5.1.6) satisfies

$$|(Z(n)Z^{-1}(\ell))\otimes(\Psi(n)Y(n)Y^{-1}(\ell)\Psi^{-1}(\ell))| \leq M, \quad \text{for all } n_0 \leq \ell \leq n,$$

where $M$ is a positive constant,

(ii) the nonlinear function $F$ satisfies the condition

$$|\Psi(n+1)F(n,X(n))| \leq \frac{\alpha(n)}{m}|\Psi(n)X(n)|, \quad (5.4.4)$$

where $\alpha(n)$ is a non-negative sequence on $\mathbb{N}$ such that

$$q = \sum_{\ell=0}^{\infty} \alpha(\ell) < \infty.$$

Then, the trivial solution of nonlinear matrix difference equation (5.1.3) is $\Psi$-uniformly stable on $\mathbb{N}$.

Proof. From condition (5.4.4) and Lemma 5.2.4, we have

$$\|(I_m \otimes \Psi(n+1))\hat{F}(n,\hat{X}(n))\| \leq |\Psi(n+1)F(n,X(n))|$$

$$\leq \frac{\alpha(n)}{m}|\Psi(n)X(n)|$$

$$\leq \alpha(n)\|(I_m \otimes \Psi(n))(\hat{F}(n,\hat{X}(n)))\|.$$

for all $n, n_0 \in \mathbb{N}$ and $\hat{X}(n) \in \mathbb{R}^{m^2}$. From Lemma 2.1.2 and Theorem 5.3.3.
it follows that the trivial solution of the corresponding Kronecker product vector difference equation (5.2.5) is $I_m \otimes \Psi$-uniformly stable on $\mathbb{N}$. Thus, from Lemma 5.2.6, the trivial solution of matrix difference equation (5.1.3) is $\Psi$-uniformly stable on $\mathbb{N}$. 

**Example 5.4.4.** Consider the nonlinear matrix difference equation (5.1.3) with $A(n), B(n)$ as in Example 5.4.1 and

$$F(n, X(n)) = \frac{1}{2^{n+1}} \begin{bmatrix} \tan^{-1}(x_1(n)) & x_2(n) \\ \sin(x_3(n)) & x_4(n) \end{bmatrix}.$$ 

If we take $\Psi(n)$ as in Example 5.4.1, Condition (i) of Theorem 5.4.3 satisfied with $M = 2$. Consider

$$\Psi(n+1)F(n, X(n)) = \frac{1}{3^n} \begin{bmatrix} 2^{-n}x_1(n) & 2^{-n}\tan^{-1}x_2(n) \\ 5^{-n}\sin(x_3(n)) & 5^{-n}\sin(x_4(n)) \end{bmatrix},$$

which implies that

$$|\Psi(n+1)F(n, X(n))| \leq \frac{1}{3^{n+1}}|\Psi(n)X(n)|, \text{ for all } n \in \mathbb{N}.$$ 

Therefore, condition (ii) of Theorem 5.4.3 satisfied with $\alpha(n) = \frac{1}{2^n}$, and

$$N = \sum_{\ell=0}^{\infty} \frac{1}{2^{\ell+1}} = 2 < \infty.$$ 

Therefore, all conditions of Theorem 5.4.3 are satisfied. Hence the trivial solution of difference equation (5.1.3) is $\Psi$-uniformly stable on $\mathbb{N}$. 105