CHAPTER 4

\( \Psi \)-BOUNDEDNESS FOR NONLINEAR MATRIX DIFFERENCE EQUATIONS
Chapter 4

Ψ-boundedness for Nonlinear Matrix Difference Equations

Section 4.1.

The aim of this chapter is to present sufficient conditions for the nonlinear matrix difference equation

\[ X(n+1) = A(n)X(n)B(n) + F(n, X(n)) \]  \hspace{1cm} (4.1.1)

has a unique Ψ-bounded solution on \( \mathbb{Z} \), where \( A \in \mathbb{R}^{m\times m} \) a nonsingular matrix, \( F : \mathbb{R} \times \mathbb{R}^{m\times m} \to \mathbb{R}^{m\times m} \) and \( F(n,0) = 0 \). The primary difficulty is to determine the necessary and sufficient conditions for the existence of a solution with some specified boundedness conditions. The standard results of this kind, for system of differential equations were presented by Coppel [15] and for linear and nonlinear difference equations were studied by Agarwal [1]. The authors [21, 24, 37] studied the existence of Ψ-bounded solutions for system of linear difference equations. Recently, Suresh Kumar et al. [80] studied Ψ-bounded solutions of linear matrix difference equations. In [21], Diamandescu proved a necessary and sufficient condition for the existence of Ψ-bounded solutions for the nonhomogeneous linear difference
equation on \( \mathbb{Z} \). Suresh Kumar et al. \cite{80} extended these results to matrix difference equations, using technique of Kronecker product of matrices.

In this chapter, we present sufficient conditions for the existence and uniqueness of \( \Psi \)-bounded solutions for the nonlinear difference equation

\[
x(n + 1) = A(n)x(n) + f(n, x(n))
\]  

(4.1.2)

on \( \mathbb{Z} \). Further, we establish sufficient conditions for the existence and uniqueness of \( \Psi \)-bounded solutions for nonlinear matrix difference equation (4.1.1) on \( \mathbb{Z} \) with the help of Kronecker product of matrices.

In Section 4.2 we present conversion of matrix difference equation to Kronecker product vector difference equation and obtain a result relating to the relationship between \( \Psi \)-bounded solutions of these equations. And also prove some results which are useful in proving main results.

In Section 4.3 we obtain sufficient conditions for the existence and uniqueness of \( \Psi \)-bounded solution of the nonlinear difference equation (4.1.2), using Banach contraction principle. Applying technique of Kronecker product of matrices sufficient conditions are obtained for the existence and uniqueness of \( \Psi \)-bounded solution of the nonlinear matrix difference equation (4.1.1).

The results of this chapter extends the results of Diamandescu\cite{24} and Suresh Kumar et al. \cite{80} to nonlinear difference equations on \( \mathbb{Z} \).
Section 4.2.

In this section we present some notations and results which are useful for later discussion.

Let \( \Psi_i : \mathbb{Z} \to (0, \infty), \ i = 1, 2, \ldots, m \) be functions, and define a matrix function

\[
\Psi = \text{diag}[\Psi_1, \Psi_2, \ldots, \Psi_m].
\]

Then \( \Psi(n) \) is an nonsingular matrix function on \( \mathbb{Z} \).

Taking the vectorization operator on both sides of equation (4.1.1) and using Lemma 2.1.2, we get the following Kronecker product vector difference equation

\[
\hat{X}(n+1) = H(n)\hat{X}(n) + \hat{F}(n, \hat{X}(n)), \quad (4.2.1)
\]

where \( H(n) = (B^T(n) \otimes A(n)) \in \mathbb{R}^{m^2 \times m^2} \) and \( \hat{F}(n, \hat{X}(n)) = \text{Vec}F(n, X(n)) \in \mathbb{R}^{m^2} \).

From the above, the following lemma is immediate.

**Lemma 4.2.1.** If \( X(n) \) is a solution of matrix difference equation (4.1.1) if and only if \( \hat{X}(n) = \text{Vec}X(n) \) is a solution of vector difference equation (4.2.1).

The linear difference equation of (4.2.1) is

\[
\hat{X}(n+1) = H(n)\hat{X}(n). \quad (4.2.2)
\]
The subsequent lemmas perform an important role in the proofs of main results.

**Lemma 4.2.2.** The matrix function $F(n)$ is $\Psi$-bounded on $\mathbb{Z}$ if and only if the vector function $VecF(n)$ is $I_m \otimes \Psi$ - bounded on $\mathbb{Z}$.

*Proof.* It is easily seen that, proof follows from the inequality (3.2.5), for $n \in \mathbb{Z}$. □

The following lemma is well-known and is given in [21], we slightly modify the statement.

**Lemma 4.2.3.** Let $A(n)$ be a nonsingular matrix on $\mathbb{Z}$. Then the fundamental matrix $Y(n)$ of

$$y(n + 1) = A(n)y(n)$$

(4.2.3)

with $Y(0) = I_m$ satisfies the following:

(i) $Y(n) = \begin{cases} A(n-1)A(n-2)\ldots A(1)A(0), & n > 0 \\ I_m, & n = 0 \\ [A(-1)A(-2)\ldots A(n-2)A(n)]^{-1}, & n < 0 \end{cases}$

(ii) $Y(n + 1) = A(n)Y(n)$, for all $n \in \mathbb{Z}$.

(iii) the solution of (4.2.3) with initial condition $x(0) = x_0$ is

$$x(n) = Y(n)x_0, \ n \in \mathbb{Z}.$$
(iv) $Y(n)$ is nonsingular for each $n \in \mathbb{Z}$ and

$$Y^{-1}(n) = \begin{cases} A^{-1}(0)A^{-1}(1)\ldots A^{-1}(n-2)A^{-1}(n-1), & n > 0 \\ I_m, & n = 0 \\ A(-1)A(-2)\ldots A(n-1)A(n), & n < 0. \end{cases}$$

**Lemma 4.2.4.** Let $Y(n)$ be a nonsingular matrix function on $\mathbb{N}$ and let $P$ be a projection. If there exists a positive constant $L > 1$ such that

$$\sum_{\ell=n_0}^{n-1} |\Psi(n)Y(n)PY^{-1}(\ell+1)\Psi^{-1}(\ell+1)| \leq L, \text{ for all } n \in \mathbb{N}(n_0), \quad (4.2.4)$$

then there exists a constant $L_1 > 0$ such that

$$|\Psi(n)Y(n)P| \leq L_1 \left( \frac{L-1}{L} \right)^{n-n_0}, \text{ for all } n \in \mathbb{N}(n_0). \quad (4.2.5)$$

**Proof.** Let $a(n) = |\Psi(n+1)Y(n+1)P|^{-1}$. From the identity

$$\Psi(n)Y(n)P \left( \sum_{\ell=n_0}^{n-1} a(\ell) \right) = \sum_{\ell=n_0}^{n-1} \Psi(n)Y(n)PY^{-1}(\ell+1)\Psi^{-1}(\ell+1) \Psi(\ell+1)Y(\ell+1)Pa(\ell),$$

it follows that

$$|\Psi(n)Y(n)P| \left( \sum_{\ell=n_0}^{n-1} a(\ell) \right)$$
\[
\leq \sum_{\ell=n_0}^{n-1} |\Psi(n)Y(n)PY^{-1}(\ell + 1)\Psi^{-1}(\ell + 1)| |\Psi(\ell + 1)Y(\ell + 1)P|a(\ell) \leq L.
\]

(4.2.6)

Setting \( b(n) = \sum_{\ell=n_0}^{n-1} a(\ell) \), we obtain

\[
b(n) - b(n-1) = |\Psi(n)Y(n)P|^{-1},
\]

for \( n \in \mathbb{N}(n_0 + 1) \). After substituting in (4.2.6), we have

\[
b(n) - b(n-1) \geq \frac{b(n)}{L} \text{ and } b(n) \geq \frac{L}{L-1} b(n-1),
\]

which implies that

\[
b(n) \geq \left( \frac{L}{L-1} \right)^{n-n_0} b(n_0 + 1),
\]

for all \( n \in \mathbb{N}(n_0 + 1) \).

From (4.2.6), we get

\[
|\Psi(n)Y(n)P|b(n) \leq L,
\]

it implies that

\[
|\Psi(n)Y(n)P| \leq L(b(n))^{-1}
\leq L \left( \frac{L - 1}{L} \right)^{n-n_0-1} b^{-1}(n_0 + 1)
\]
\[ L \left( \frac{L - 1}{L} \right)^{n-n_0-1} |\Psi(n_0 + 1)Y(n_0 + 1)P|, \]

If we choose

\[ L_1 = \max \left\{ |\Psi(n_0)Y(n_0)P|, \frac{L^2}{L-1}|\Psi(n_0 + 1)Y(n_0 + 1)P| \right\}, \]

then (4.2.5) follows. \(\square\)

**Lemma 4.2.5.** Let \( Y(n) \) be a nonsingular matrix which is defined on \( \mathbb{N} \) and let \( P \) be a projection. If there exists a constant \( L > 0 \) such that

\[ \sum_{\ell=n}^{\infty} |\Psi(n)Y(n)PY^{-1}(\ell + 1)\Psi^{-1}(\ell + 1)| \leq L, \text{ for all } n \in \mathbb{N}, \quad (4.2.7) \]

then for any vector \( \xi \in \mathbb{R}^m \) such that \( P\xi \neq 0 \),

\[ \limsup_{n \to \infty} \|\Psi(n)Y(n)P\xi\| = \infty. \quad (4.2.8) \]

**Proof.** For any \( n \in \mathbb{N}(n_0) \), we have \( \|\Psi(n+1)Y(n+1)P\xi\| > 0 \). Then, from

\[ \sum_{\ell=n}^{n_1} \|\Psi(\ell + 1)Y(\ell + 1)P\xi\|^{-1}\Psi(n)Y(n)P\xi \]

\[ = \sum_{\ell=n}^{n_1} \|\Psi(\ell + 1)Y(\ell + 1)P\xi\|^{-1}\Psi(n)Y(n)PY^{-1}(\ell + 1)\Psi^{-1}(\ell + 1) \]

\[ \Psi(\ell + 1)Y(\ell + 1)P\xi \]

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and (4.2.7), we get

\[ \| \Psi(\ell + 1)Y(\ell + 1)P\xi \| \sum_{\ell=n}^{n_1} \| \Psi(\ell + 1)Y(\ell + 1)P\xi \|^{-1} \leq L, \]

for \( n_1 \geq n, n, n_1 \in \mathbb{N}(n) \). Therefore,

\[ \sum_{\ell=n}^{\infty} \| \Psi(\ell + 1)Y(\ell + 1)P\xi \|^{-1} \]

exists and so

\[ \limsup_{n \to \infty} \| \Psi(n + 1)Y(n + 1)P\xi \|^{-1} = 0, \]

or

\[ \limsup_{n \to \infty} \| \Psi(n + 1)Y(n + 1)P\xi \| = \infty. \]

\[ \square \]

Section 4.3.

In this section, we obtain sufficient conditions for the existence and uniqueness of \( \Psi \)-bounded solution of the nonlinear difference equations (4.1.2) and (4.1.1), using Banach contraction principle and Kronecker product of matrices.

Let \( X_- \), \( X_0 \), \( X_+ \) be the three subspaces of \( \mathbb{R}^m \) such that a solution \( y(n) \) of (4.2.3) is \( \Psi \)-bounded on \( \mathbb{N}=0, 1, 2, \ldots \) if and only if \( y(0) \in X_0 \) and \( \Psi \)-bounded on \( \mathbb{Z} \) if and only if \( y(0) \in X_- \oplus X_0 \). The matrices \( P_{-1}, P_0, P_1 \) represent corresponding projections of \( \mathbb{R}^m \) onto \( X_- \), \( X_0 \), \( X_+ \) respectively.
In the general case where \((P_0 \neq 0)\), the solution for (4.1.2) is as follows

\[
x(n) = \sum_{\ell=-\infty}^{n-1} Y(n) P_{-1} Y^{-1}(\ell + 1) f(\ell, x(\ell)) + \sum_{\ell=0}^{n-1} Y(n) P_0 Y^{-1}(\ell + 1) f(\ell, x(\ell)) - \sum_{\ell=n}^{\infty} Y(n) P_1 Y^{-1}(\ell + 1) f(\ell, x(\ell)).
\] (4.3.1)

For simplicity assume that the linear equation (4.2.3) has no non-trivial \(\Psi\)-bounded solution \((P_0 = 0)\).

**Theorem 4.3.1. (Existence and Uniqueness)** Suppose that there exist supplementary projections \(P_{-1}, P_1\) and a positive constant \(K\) such that

\[
\sum_{\ell=-\infty}^{n-1} \|\Psi(n) Y(n) P_{-1} Y^{-1}(\ell + 1) \Psi^{-1}(\ell)\| + \sum_{\ell=n}^{\infty} \|\Psi(n) Y(n) P_1 Y^{-1}(\ell + 1) \Psi^{-1}(\ell)\| \leq K. 
\] (4.3.2)

Let \(f(n, x)\) be a vector function such that

\[
\|\Psi(n)[f(n, x) - f(n, y)]\| \leq \alpha \|\Psi(n)(x - y)\|, 
\] (4.3.3)

for \(n \in \mathbb{Z}, \|\Psi x\| \leq \rho, \|\Psi y\| \leq \rho\), where \(\alpha K < 1\), then the equation (4.1.2) has a unique \(\Psi\)-bounded solution \(x(n)\) for which \(\|\Psi x\| \leq \rho\).

**Proof.** From Lemmas [4.2.4] and [4.2.5] the condition (4.3.2) implies that \(|\Psi(n) Y(n) P_{-1} \xi|\) is unbounded for \(n \leq 0\) if \(P_{-1} \xi \neq 0\) and bounded for
\( n \geq 0 \), and that \(|\Psi(n)Y(n)P_1\xi|\) is unbounded for \( n \geq 0 \) if \( P_1\xi \neq 0 \) and bounded for \( n \leq 0 \). Hence the linear equation \([4.2.3]\) has no non-trivial \( \Psi \)-bounded solution.

Let \( x(n) \) be the solution of \([4.1.2]\), then from \([4.3.2]\) and \([4.3.3]\) the function

\[
y(n) = x(n) - \sum_{\ell=-\infty}^{n-1} Y(n)P_1Y^{-1}(\ell + 1)f(\ell, x(\ell)) + \sum_{\ell=n}^{\infty} Y(n)P_1Y^{-1}(\ell + 1)f(\ell, x(\ell)),
\]

exists and is \( \Psi \)-bounded for all \( n \in \mathbb{Z} \). Moreover, it follows that

\[
y(n+1) = x(n+1) - \sum_{\ell=-\infty}^{n} Y(n+1)P_1Y^{-1}(\ell + 1)f(\ell, x(\ell)) + \sum_{\ell=n+1}^{\infty} Y(n+1)P_1Y^{-1}(\ell + 1)f(\ell, x(\ell))
\]

\[= A(n)x(n) + f(n, x(n)) - Y(n+1)P_1Y^{-1}(n + 1)f(n, x(n)) - \sum_{\ell=-\infty}^{n-1} A(n)Y(n)P_1Y^{-1}(\ell + 1)f(\ell, x(\ell))
\]

\[- Y(n+1)P_1Y^{-1}(n + 1)f(n, x(n)) + \sum_{\ell=n}^{\infty} A(n)Y(n)P_1Y^{-1}(\ell + 1)f(\ell, x(\ell))
\]

\[= A(n) \left[ x(n) - \sum_{\ell=-\infty}^{n-1} Y(n)P_1Y^{-1}(\ell + 1)f(\ell, x(\ell)) - \sum_{\ell=n}^{\infty} Y(n)P_1Y^{-1}(\ell + 1)f(\ell, x(\ell)) \right]
\]
\[ + f(n, x(n)) - f(n, x(n)) = A(n)y(n). \]

Therefore, \( y(n) \) is a \( \Psi \)-bounded solution of (4.1.2). Thus \( y(n) = 0 \) that is

\[
x(n) = \sum_{\ell=-\infty}^{n-1} Y(n)P_{-1}Y^{-1}(\ell + 1)f(\ell, x(\ell))
- \sum_{\ell=n}^{\infty} Y(n)P_{1}Y^{-1}(\ell + 1)f(\ell, x(\ell)). \tag{4.3.5}
\]

Define \( C_\Psi = \{ x : \mathbb{R} \to \mathbb{R}^m : x \text{ is } \Psi\text{-bounded functions on } \mathbb{Z} \text{ such that} \right\} \) and

\[
\| x \|_\Psi = \sup_{n \in \mathbb{Z}} \| \Psi(n)x(n) \|.
\]

Clearly this defines a norm on \( C_\Psi \) and \((C_\Psi, \| \cdot \|_\Psi)\) is a Banach space. Let \( T \) be a mapping defined by

\[
Tx(n) = \sum_{\ell=-\infty}^{n-1} Y(n)P_{-1}Y^{-1}(\ell + 1)f(\ell, x(\ell))
- \sum_{\ell=n}^{\infty} Y(n)P_{1}Y^{-1}(\ell + 1)f(\ell, x(\ell)), \tag{4.3.6}
\]

for all \( x \in C_\Psi \). Consider

\[
\| \Psi(n)Tx(n) \| = | \sum_{\ell=-\infty}^{n-1} \Psi(n)Y(n)P_{-1}Y^{-1}(\ell + 1)\Psi^{-1}(\ell)\Psi(\ell)f(\ell, x(\ell))
- \sum_{\ell=n}^{\infty} \Psi(n)Y(n)P_{1}Y^{-1}(\ell + 1)\Psi^{-1}(\ell)\Psi(\ell)f(\ell, x(\ell)) |.
\]
\[
\leq \left[ \sum_{\ell=-\infty}^{n-1} |\Psi(n)Y(n)P_{-1}Y^{-1}(\ell + 1)\Psi^{-1}(\ell)| + \sum_{\ell=n}^{\infty} |\Psi(n)Y(n)P_{1}Y^{-1}(\ell + 1)\Psi^{-1}(\ell)| \right] \|\Psi(\ell)f(\ell, x(\ell))\| \\
\leq K\alpha \|\Psi(\ell)x(\ell)\| \\
\leq K\alpha \rho < \rho.
\]

which implies \(Tx(n) \in C_{\Psi}\) and hence \(T : C_{\Psi} \to C_{\Psi}\).

Now, we show that \(T\) is a contraction mapping on \(C_{\Psi}\). Consider

\[
\|\Psi(n)(Tx - Ty)(n)\| = \| \sum_{\ell=-\infty}^{n-1} \Psi(n)Y(n)P_{-1}Y^{-1}(\ell + 1)\Psi^{-1}(\ell)\Psi(\ell)f(\ell, x(\ell)) \\
- \sum_{\ell=n}^{\infty} \Psi(n)Y(n)P_{1}Y^{-1}(\ell + 1)\Psi^{-1}(\ell)\Psi(\ell)f(\ell, x(\ell)) \\
- \left[ \sum_{\ell=-\infty}^{n} \Psi(n)Y(n)P_{-1}Y^{-1}(\ell + 1)\Psi^{-1}(\ell)\Psi(\ell)f(\ell, y(\ell)) \\
- \sum_{\ell=n}^{\infty} \Psi(n)Y(n)P_{1}Y^{-1}(\ell + 1)\Psi^{-1}(\ell)\Psi(\ell)f(\ell, y(\ell)) \right] \| \\
\leq \left[ \sum_{\ell=-\infty}^{n} \Psi(n)Y(n)P_{-1}Y^{-1}(\ell + 1)\Psi^{-1}(\ell) \right] \\
+ \sum_{\ell=n}^{\infty} \|\Psi(n)Y(n)P_{1}Y^{-1}(\ell + 1)\Psi^{-1}(\ell)\| \\
\|\Psi(\ell)f(\ell, x(\ell)) - f(\ell, y(\ell))\| \\
\leq K\alpha \|\Psi(\ell)(x(\ell) - y(\ell))\|.
\]

Thus \(\|Tx - Ty\|_{\Psi} \leq K\alpha \|x - y\|_{\Psi}\).

Therefore \(T\) is a contraction mapping on \(C_{\Psi}\). Hence by Banach contrac-
tion principle, $T$ has a unique fixed point $x(n)$ on $C_\Psi$. Thus, the nonlinear difference equation (4.1.2) has a unique fixed point for which $\|\Psi x\| \leq \rho$.

Conversely, if $x(n)$ is a solution of (4.1.2) such that $\|\Psi x\| \leq \rho$, then $y = x - Tx$ is a $\Psi$-bounded solution of the linear equation (4.2.3), therefore $y = 0$. 

**Example 4.3.1.** Consider the nonlinear difference equation (4.1.2) with

$$A(n) = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \quad \text{and} \quad f(n, x(n)) = \frac{1}{10} \begin{bmatrix} 3^{-|n|} \tan^{-1}(x_1(n)) \\ 5^{-|n|} x_2(n) \frac{1+|n|^2}{1+|n|^2} \end{bmatrix}.$$ 

The fundamental matrix of (4.2.3) is

$$Y(n) = \begin{bmatrix} 2^n & 0 \\ 0 & 3^{-n} \end{bmatrix}.$$ 

Consider

$$\Psi(n) = \begin{bmatrix} 3^{-n} & 0 \\ 0 & 5^n \end{bmatrix}, \quad \text{for all} \; n \in \mathbb{Z}.$$ 

Clearly the linear equation (4.2.3) has no non-trivial $\Psi$-bounded solution on $\mathbb{Z}$. Their exist supplementary projections

$$P_{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad P_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

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such that
\[
\sum_{\ell=-\infty}^{n-1} |\Psi(n)Y(n)P_{-1}Y^{-1}(\ell + 1)\Psi^{-1}(\ell)| = \frac{1}{3} \sum_{\ell=-\infty}^{n-1} \left(\frac{2}{3}\right)^{\ell-(n+1)} = 1,
\]
\[
\sum_{\ell=n}^{\infty} |\Psi(n)Y(n)P_{1}Y^{-1}(\ell + 1)\Psi^{-1}(\ell)| = (3) \sum_{\ell=n}^{\infty} \left(\frac{5}{3}\right)^{n-\ell} = 7.5,
\]
which implies
\[
\sum_{\ell=-\infty}^{n-1} |\Psi(n)Y(n)P_{-1}Y^{-1}(\ell + 1)\Psi^{-1}(\ell)|
\]
\[
+ \sum_{\ell=n}^{\infty} |\Psi(n)Y(n)P_{1}Y^{-1}(\ell + 1)\Psi^{-1}(\ell)| = 8.5.
\]

And also
\[
\|\Psi(n)[f(n, x) - f(n, y)]\| \leq \frac{1}{10} \|\Psi(n)[x - y]\|.
\]

Therefore, all the conditions of Theorem 4.3.1 are satisfied with \(\alpha = 1/10\) and \(K = 8.5\). Hence the nonlinear difference equation (4.1.2) has a unique \(\Psi\)-bounded solution
\[
x(n) = \frac{1}{10} \left[ \sum_{\ell=-\infty}^{n-1} (2)^{n-(\ell+1)}3^{-|\ell|} \tan^{-1}(x_1(\ell)) \right]
\]
\[
\quad + \sum_{\ell=n}^{\infty} 3^{\ell+1-n/5 - |\ell|} \frac{x_2(\ell)}{1+\ell^2}
\]
on \(\mathbb{Z}\).

Now we obtain sufficient conditions for the existence and uniqueness of
the nonlinear matrix difference equation (4.1.1), using Theorem 4.3.1 and
the technique of Kronecker product of matrices.

Let $Y_-, Y_0, Y_+$ be three subspaces of $\mathbb{R}^{m \times m}$ such that a solution $V(n)$ of

$$X(n + 1) = A(n)X(n)B(n) \quad (4.3.7)$$

is a $\Psi$-bounded solution on $\mathbb{Z}$ if and only if $V(0) \in Y_0$ and $\Psi$-bounded
on $\mathbb{N}$ if and only if $V(0) \in Y_- \oplus Y_0$. The matrices $R_{-1}, R_0, R_1$ represent
corresponding projections of $\mathbb{R}^{m \times m}$ onto $Y_-, Y_0, Y_+$ respectively.

Then $S_-, S_0, S_+$ are the subspaces of $\mathbb{R}^{m^2}$ such that a solution $\hat{V}(n) =
VecV(n)$ of (4.2.2) is $(I_m \otimes \Psi)$-bounded on $\mathbb{Z}$ if and only if $\hat{V}(0) \in S_0$
and $(I_m \otimes \Psi)$-bounded on $\mathbb{N}$ if and only if $\hat{V}(0) \in S_- \oplus S_0$. The matrices $Q_{-1}, Q_0, Q_1$ represent corresponding projections of $\mathbb{R}^{m^2}$ onto $S_-, S_0, S_+$
respectively.

In the general case where ($Q_0 \neq 0$), the solution for (4.2.1) is as follows

$$\hat{X}(n) = \sum_{\ell=-\infty}^{n-1} (Z(n) \otimes Y(n))Q_{-1}(Z^{-1}(\ell + 1) \otimes Y^{-1}(\ell + 1))\hat{F}(\ell, \hat{X}(\ell))$$

$$+ \sum_{\ell=0}^{n-1} (Z(n) \otimes Y(n))Q_0(Z^{-1}(\ell + 1) \otimes Y^{-1}(\ell + 1))\hat{F}(\ell, \hat{X}(\ell)) \quad (4.3.8)$$

$$- \sum_{\ell=n}^{\infty} (Z(n) \otimes Y(n))Q_1(Z^{-1}(\ell + 1) \otimes Y^{-1}(\ell + 1))\hat{F}(\ell, \hat{X}(\ell)).$$

For simplicity assume that the linear equation (4.2.2) has no non-trivial
$(I_m \otimes \Psi)$-bounded solution ($Q_0 = 0$).
Theorem 4.3.2. Suppose that there exist supplementary projections \( Q_{-1}, Q_1 \) and a positive constant \( M \) such that

\[
\sum_{\ell = -\infty}^{n-1} |(Z(n) \otimes \Psi(n)) Q_{-1} (Z^{-1}(\ell + 1) \otimes (Y^{-1}(\ell + 1)\Psi^{-1}(\ell)))| \\
+ \sum_{\ell = n}^{\infty} |(Z(n) \otimes \Psi(n)) Q_1 (Z^{-1}(\ell + 1) \otimes (Y^{-1}(\ell + 1)\Psi^{-1}(\ell)))| \leq M,
\]

(4.3.9)

where \( Y(n) \) and \( Z(n) \) are the fundamental matrices of

\[
X(n + 1) = A(n)X(n)
\]

(4.3.10)

and

\[
X(n + 1) = B^T(n)X(n).
\]

(4.3.11)

Let \( F(n, X) \) be a matrix function such that

\[
|\Psi(n)(F(n, U) - F(n, V))| \leq \beta |\Psi(n)(U - V)|
\]

(4.3.12)

for \( n \in \mathbb{Z}, |\Psi U| \leq \gamma, |\Psi V| \leq \gamma \), where \( m\beta M < 1 \), then the equation (4.1.1) has a unique \( \Psi \)-bounded solution \( X(n) \) from which \( |\Psi X| \leq \gamma \).

Proof. Let \( F(n, X) \) be a matrix function satisfies (4.3.12). From the inequality (3.2.4), we have

\[
||(I_m \otimes \Psi(n))(\hat{F}(n, \hat{U}) - \hat{F}(n, \hat{V}))|| \leq |\Psi(n)(F(n, U) - F(n, V))| \\
\leq \beta |\Psi(n)(U - V)|
\]

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\[ \leq m\beta |(I_m \otimes \Psi(n))(\hat{U} - \hat{V})|, \]

for \(\hat{U}, \hat{V} \in \mathbb{R}^{m^2}\). Also

\[ |(I_m \otimes \Psi(n))\hat{U}| \leq |\Psi(n)U(n)| \leq \gamma, \]

\[ |(I(n) \otimes \Psi(n))\hat{V}| \leq |\Psi(n)V(n)| \leq \gamma. \]

From Kronecker product properties, equations (4.3.9) and (4.3.12), we have that the fundamental matrix \(W(n) = (Z(n) \otimes Y(n))\) of (4.2.2) satisfies condition (4.3.2) and the function \(\hat{F}(n, \hat{X})\) satisfies condition (4.3.3) of Theorem 4.3.1 in place of \(\Psi\) as \(I_m \otimes \Psi\). Therefore, from Theorem 4.3.1, the Kronecker product difference equation (4.2.1) has a unique \((I_m \otimes \Psi)\)-bounded solution on \(Z\). From Lemma 4.2.2, the matrix difference equation (4.1.1) has a unique \(\Psi\)-bounded solution on \(Z\). \(\square\)

**Example 4.3.2.** Consider the non linear matrix difference equation (4.1.1) with

\[
A(n) = \begin{bmatrix} 5 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}, \quad B(n) = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix}
\]

and

\[
F(n, X(n)) = \frac{1}{30} \begin{bmatrix} 5^{-|n|}x_1(n) & 3^{-|n|}\tan^{-1}x_3(n) \\ \frac{5^{-|n|}}{1+|n|}x_2(n) & \frac{3^{-|n|}}{1+|n|}\tan^{-1}x_4(n) \end{bmatrix}.
\]

Then

\[
Y(n) = \begin{bmatrix} 5^n & 0 \\ 0 & 3^{-n} \end{bmatrix} \quad \text{and} \quad Z(n) = \begin{bmatrix} 3^n & 0 \\ 0 & 2^n \end{bmatrix}
\]
are fundamental matrices for (4.3.10) and (4.3.11) respectively.

Let

$$\Psi(n) = \begin{bmatrix}
5^{-n} & 0 \\
0 & 3^n
\end{bmatrix}, \text{ for all } n \in \mathbb{Z}.$$}

Then their exist supplementary projections

$$Q_{-1} = \begin{bmatrix}
I_2 & O_2 \\
O_2 & O_2
\end{bmatrix} \text{ and } Q_1 = \begin{bmatrix}
O_2 & O_2 \\
O_2 & I_2
\end{bmatrix},$$

such that condition (4.3.9) and (4.3.12) satisfied with $M = 15/2$ and $\beta = 1/30$. Moreover, $m \beta M = 1/2 < 1$. Therefore all the conditions of Theorem 4.3.2 are satisfied. Hence nonlinear difference equation (4.1.1) has a unique $\Psi$-bounded solution on $\mathbb{Z}$. 

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