CHAPTER 2

PRELIMINARIES
In this chapter we introduce the notation used in this thesis, present some standard results from books as well as from recent papers. Proofs are hinted for a few frequently used results.

**Section 2.1.**

Set theoretic symbols like $\in$, $\subseteq$, $\cap$, $\cup$, $\rightarrow$ are freely used. Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{N}$ are represent the set of all real, nonnegative real, integers, positive integers, nonnegative integers respectively and $\mathbb{N}(n_0) = \{n_0, n_0 + 1, n_0 + 2, \ldots\}$, where $n_0 \in \mathbb{Z}$. The $m$-dimensional Euclidean space is denoted by $\mathbb{R}^m$. Let $\mathbb{R}^{p \times q}$ denotes the space of all $p \times q$ matrices whose elements are real numbers. Unless otherwise stated vectors in $\mathbb{R}^m$ are denoted by small Roman letters and the matrices in $\mathbb{R}^{p \times q}$ are denoted by capital Roman letters.

The transpose of a matrix ‘$A$’ is denoted by $A^T$. If ‘$A$’ is a nonsingular matrix, $A^{-1}$ stands for inverse of $A$. The unit matrix of order ‘$m$’ is denoted by $I_m$ (whose order can be specified by the context in which it occurs) and the matrix all of whose elements are zero’s is denoted by 0. Though the
symbol 0 is used in several senses, no confusion arises, since the context in which it occurs clearly indicates what it stands for.

By a $p \times q$ matrix we mean a matrix with $p$-rows and $q$-columns and an $m \times m$ matrix merely said to be of order $m$ (or square). A matrix function $A : n \rightarrow A(n)$ is denoted by $A(n)$ or $[a_{ij}(n)]$, where $a_{ij}(n)$ stands for the $i$-th row $j$-th column element which is a function of $n$. A summation of a matrix $A$ is the matrix obtained by summation of each component of $A$.

\[ \sum_{k=r}^{s} A(k) = \left[ \sum_{k=r}^{s} a_{ij}(k) \right]. \]

In the sequel, we use the following standard norms for vectors and matrices namely for $x \in \mathbb{R}^m$,

\[ \|x\| = \max_i |x_i| \]

and for $A = [a_{ij}] \in \mathbb{R}^{p\times q}$,

\[ |A| = \sup_{\|x\| \leq 1} \|Ax\|. \]

**Definition 2.1.1.** [1] A matrix $P$ is said to be a projection if $P^2 = P$. If $P$ is the projection, then $I - P$ is also a projection. Two such projections, whose sum is $I$ and hence whose product is zero are said to be supplementary.

Kronecker product also known as a direct product or a tensor product is a concept having its origin in group theory and has important applications in particle physics. This technique has been successfully applied in various
fields of matrix theory.

**Definition 2.1.2.** [35] Let \( S \in \mathbb{R}^{p \times q} \) and \( T \in \mathbb{R}^{r \times s} \) then the Kronecker product of \( S \) and \( T \) written \( S \otimes T \) is defined to be the partitioned matrix

\[
S \otimes T = \begin{bmatrix}
    s_{11}T & s_{12}T & \cdots & s_{1q}T \\
    s_{21}T & s_{22}T & \cdots & s_{2q}T \\
    \vdots & \vdots & \ddots & \vdots \\
    s_{p1}T & s_{p2}T & \cdots & s_{pq}T
\end{bmatrix}
\]

is a \( pr \times qs \) matrix and is in \( \mathbb{R}^{pr \times qs} \).

**Definition 2.1.3.** [35] Let \( S = [s_{ij}] \in \mathbb{R}^{p \times q} \), then the vectorization (Vec) operator

\[
Vec : \mathbb{R}^{p \times q} \to \mathbb{R}^{pq},
\]

defined and denoted by

\[
\hat{S} = VecS = \begin{bmatrix}S_{1} \\
S_{2} \\
\vdots \\
S_{q}\end{bmatrix}, \text{ where } S_{j} = \begin{bmatrix}s_{1j} \\
s_{2j} \\
\vdots \\
s_{pj}\end{bmatrix} \quad (1 \leq j \leq q).
\]

**Lemma 2.1.1.** The vectorization operator \( Vec : \mathbb{R}^{m \times m} \to \mathbb{R}^{m^2} \), is a linear and bijection operator and also \( Vec \) and its inverse operator \( Vec^{-1} \) are continuous.

**Proof.** From the Definition 2.1.3 it is clear that the \( Vec \) operator is linear

20
and bijection. Now, for $S = [s_{ij}] \in \mathbb{R}^{m \times m}$, we have

$$\|Vec(S)\| = \max_{1 \leq i, j \leq m} \{|s_{ij}|\} \leq \max_{1 \leq i \leq m} \left\{ \sum_{j=1}^{m} |s_{ij}| \right\} = |S|.$$  

It follows that the $Vec$ operator is continuous and $\|Vec\| \leq 1$.

If $S = I_m$, then

$$\|Vec(I_m)\| = 1 = |I_m|$$

and $\|Vec\| = 1$.

The inverse of $Vec$ operator $Vec^{-1} : \mathbb{R}^{m^2} \to \mathbb{R}^{m \times m}$, is defined as

$$Vec^{-1}(v) = \begin{bmatrix} v_1 & v_{m+1} & \ldots & v_{m^2-m+1} \\ v_2 & v_{m+2} & \ldots & v_{m^2-m+2} \\ \vdots & \vdots & \ddots & \vdots \\ v_m & v_{2m} & \ldots & v_{m^2} \end{bmatrix}.$$  

Where $v = (v_1, v_2, v_3, \ldots, v_{m^2})^T \in \mathbb{R}^{m^2}$. We have

$$|Vec^{-1}(v)| = \max_{1 \leq i \leq m} \left\{ \sum_{j=0}^{m-1} |v_{mj+i}| \right\} \leq m \cdot \max_{1 \leq i \leq m} \{|v_i|\} = m \cdot \|v\|.$$  

It follows that $Vec^{-1}$ is a continuous operator. Also, if we take $v = VecS$ in the above inequality, then

$$|S| \leq m\|VecS\|, \quad \text{for every} \quad S \in \mathbb{R}^{m \times m}.$$  

21
In the following lemma, we state some properties of Kronecker product and Vec operator.

**Lemma 2.1.2.** The following properties and rules are true, provided that the dimension of the matrices are such that the various expressions exist.

1. \((P \otimes Q)^T = P^T \otimes Q^T\).
2. \((P \otimes Q)^{-1} = P^{-1} \otimes Q^{-1}\).
3. \(|P \otimes Q| \leq |P||Q|\).
4. \(P \otimes (Q \otimes R) = (P \otimes Q) \otimes R\).
5. \(P \otimes (Q + R) = (P \otimes Q) + (P \otimes R)\).
6. \((P + Q) \otimes R = (P \otimes R) + (Q \otimes R)\).
7. \((P \otimes Q)(R \otimes S) = (PR \otimes QS)\).
8. If \(P, Q, X \in \mathbb{R}^{m \times m}\), then
   
   (i) \(\text{Vec}(PXQ) = (Q^T \otimes P)\text{Vec}X\),
   
   (ii) \(\text{Vec}(PX) = (I_m \otimes P)\text{Vec}X\),
   
   (iii) \(\text{Vec}(XQ) = (Q^T \otimes I_m)\text{Vec}X\).
9. There exists a zero element \(0_{mn} = 0_m \otimes 0_n\).
10. There exists a unit element \(I_{mn} = I_m \otimes I_n\).
Section 2.2.

The problem

\[ y(n + 1) = A(n)y(n), \quad (2.2.1) \]

\[ y(n_0) = y_0, \quad (2.2.2) \]

where \( A = [a_{ij}] \) is a nonsingular matrix of order \( m \), is called a initial value problem.

**Theorem 2.2.1.** \([31]\) For each \( y_0 \in \mathbb{R}^m \) and \( n_0 \in \mathbb{N} \) there exists a unique solution \( y(n, n_0, y_0) \) of the initial value problem \((2.2.1)\) with \((2.2.2)\).

Any set of \( m \)-linearly independent solutions \( y_1, y_2, \ldots, y_m \) of \((2.2.1)\) is called a fundamental set of solutions and the matrix with \( y_1, y_2, \ldots, y_m \) as its columns is called a fundamental matrix for the equation \((2.2.1)\) and is denoted by \( Y \). The fundamental matrix \( Y \) is nonsingular.

**Theorem 2.2.2.** \([31]\) If \( Y(n) \) is fundamental matrix of \((2.2.1)\) if and only if \( Y(n) \) satisfies \((2.2.1)\) and nonsingular.

**Theorem 2.2.3.** \([31]\) If \( Y \) is a fundamental matrix for the equation \((2.2.1)\), then for any constant \( n \)-vector \( c \), \( Yc \) is a solution of \((2.2.1)\) and every solution of \((2.2.1)\) is of this form.

The equation

\[ y(n + 1) = A(n)y(n) + f(n), \quad (2.2.3) \]

where \( A \in \mathbb{R}^{m \times m} \) nonsingular and \( f \in \mathbb{R}^m \) is termed as a nonhomogeneous equation. If \( f = 0 \), then it is called homogeneous equation.
Theorem 2.2.4. [31] If \( y_p \) is any particular solution of the nonhomogeneous equation (2.2.3) and \( Y \) is a fundamental matrix for the corresponding homogeneous equation (2.2.1), then \( y \) defined by

\[
y = y_p + Yc
\]

is a solution of (2.2.3) for every constant \( m \)-vector \( 'c' \) and every solution of (2.2.3) is of this form, where

\[
y_p(n) = Y(n) \sum_{k=n_0}^{n-1} Y^{-1}(k+1)f(k)
\]

is a particular solution of (2.2.3).

Proof. Clearly \( y \) in (2.2.4) is a solution of (2.2.3). If \( u \) is any other solution of (2.2.3), then \( (u - y_p)(n + 1) = A(n)(u - y_p)(n) \) so that \( (u - y_p) \) is a solution of (2.2.1). Hence \( u - y_p = Yc \) or \( u = y_p + Yc \). \( \square \)

Let \( X \) be a non-empty set. If \( d \) is a metric for \( X \), then the ordered pair \((X, d)\) is called a metric space and \( d(x, y) \) is called the distance between \( x \) and \( y \).

Definition 2.2.1. [14] A metric space \((X, d)\) is said to be complete if and only if every Cauchy sequence in \( X \) has a limit point in \( X \).

Definition 2.2.2. [14] A normed linear space is a vector space over the field of real numbers or complex numbers in which a real valued function \( \|x\| \) is defined with the following properties:
(i) \( (a) \|x\| \geq 0, \quad (b) \|x\| = 0 \) if and only if \( x = 0 \),

(ii) \( \|\alpha x\| = |\alpha|\|x\| \),

(iii) \( \|x_1 + x_2\| \leq \|x_1\| + \|x_2\| \).

It may be noted that \( d(x, y) = \|x - y\| \) is a metric for the space. It is called the natural metric.

**Definition 2.2.3.** \([14]\) A normed linear space which is complete with respect to its natural metric is called a Banach space.

**Theorem 2.2.5.** \([14]\) The space \( C(X) \) of all bounded real functions defined on a metric space \((X, d)\) forms a Banach space with the norm given by \( \|f\| = \sup |f(n)| \).

One of the simplest and useful tools of the nonlinear analysis is the principle of contraction mapping.

**Definition 2.2.4.** \([74]\) Let \( \mathbb{B} \) be a Banach space. Let \( f : \mathbb{B} \to \mathbb{B} \) be a mapping. If there exists an \( \alpha \in (0, 1) \) such that

\[
\|fx_1 - fx_2\| \leq \alpha \|x_1 - x_2\|, \quad \forall \quad x_1, x_2 \in \mathbb{B}.
\]

Then ‘\( f \)’ is called a contraction mapping and ‘\( \alpha \)’ is called contraction constant of the mapping.

It is evident that every contraction mapping is continuous.

**Definition 2.2.5.** \([74]\) A point \( x \) in \( \mathbb{B} \) is called a fixed point of the mapping \( f \) if \( fx = x \).
Theorem 2.2.6. [74] (Banach Fixed Point Theorem or Contraction Mapping Theorem) Every contraction mapping defined on a Banach space has one and only one fixed point.

Lemma 2.2.1. [74] (Mazur’s Lemma) Let $B$ be a Banach space and let \( \{u_m\}_{m \in \mathbb{N}} \) be a sequence in $B$ that converges weakly to some $u_0$ in $B$. Then there exists a function $p : \mathbb{N} \to \mathbb{N}$ and a sequence of sets of real numbers \( \{\alpha(m)_k/k = m, \ldots, p(m)\} \) such that $\alpha(m)_k \geq 0$ and $\sum_{k=m}^{p(m)} \alpha(m)_k = 1$ such that the sequence \( \{v_m\}_{m \in \mathbb{N}} \) defined by the convex combination $v_m = \sum_{k=m}^{p(m)} \alpha(m)_k u_k$ converges strongly in $B$ to $u_0$.

Theorem 2.2.7. [1] (Gronwall Inequalities) Let for all $n \in \mathbb{N}(n_0)$ the following inequality be satisfied

$$u(n) \leq p(n) + q(n) \sum_{k=n_0}^{n-1} f(k)u(k).$$

Then, for all $n \in \mathbb{N}(n_0)$

$$u(n) \leq p(n) + q(n) \sum_{k=n_0}^{n-1} p(k)f(k) \prod_{\tau=k+1}^{n-1} (1 + q(\tau)f(\tau)).$$

Corollary 2.2.1. [1] Let in Theorem 2.2.7 $p(n) = p$ and $q(n) = q$ for all $n \in \mathbb{N}(n_0)$. Then, for all $n \in \mathbb{N}(n_0)$

$$u(n) \leq p \prod_{\tau=n_0}^{n-1} (1 + qf(\tau)).$$
Section 2.3.

Let the solution \( x(n) = x(n, n_0, x_0) \) of

\[
x(n + 1) = f(n, x(n))
\]  

(2.3.1)

with \( x(n_0) = x_0 \) exists for all \( n \in \mathbb{N}(n_0) \).

**Definition 2.3.1.** A sequence \( \phi : \mathbb{N}(n_0) \to \mathbb{R}^m \) is said to be bounded on \( \mathbb{N}(n_0) \) if there exists \( M > 0 \) such that \( \|\phi(n)\| \leq M \), for all \( n \in \mathbb{N}(n_0) \).

**Theorem 2.3.1.** Let \( Y(n) \) be a fundamental matrix for (2.2.1). Then all solutions of (2.2.1) are bounded on \( \mathbb{N}(n_0) \) if and only if there exists a positive constant \( M \) such that

\[
|Y(n)| \leq M, \quad \text{for all } n \geq n_0 \geq 0;
\]

Now we define various concepts of stability.

**Definition 2.3.2.** The solution \( x(n) \) is said to stable, if for each \( \varepsilon > 0 \), there exists \( \delta = \delta(\varepsilon, n_0) > 0 \) such that any solution \( \pi(n) = x(n, n_0, \pi_0) \) of (2.3.1), the inequality \( \|\pi_0 - x_0\| < \delta \) implies \( \|\pi(n) - x(n)\| < \varepsilon \) for all \( n \in \mathbb{N}(n_0) \).

**Definition 2.3.3.** The solution \( x(n) \) is said to be unstable, if it is not stable.

**Definition 2.3.4.** The solution \( x(n) \) is said to be uniformly stable, if it is stable and \( \delta \) is independent of \( n_0 \).
Definition 2.3.5. The solution \(x(n)\) is said to be asymptotically stable, if it is stable and in addition, there exists a \(\delta = \delta(n_0) > 0\) such that for any solution \(x(n) = x(n, n_0, x_0)\) of (2.3.1), the inequality \(|x_0 - x|| < \delta\), implies \(|x(n) - x(n)| \to 0\) as \(n \to \infty\).

Now we state some results relating to stability of linear difference equation (2.2.1) for \(n \geq n_0\).

**Theorem 2.3.2.** All solutions of the difference system (2.2.1) are stable if and only if they are bounded on \(\mathbb{N}(n_0)\).

**Theorem 2.3.3.** Let \(Y(n)\) be a fundamental matrix of (2.2.1). Then

(i) the trivial solution of (2.2.1) is stable if and only if there exists a positive constant \(M\) such that

\[|Y(n)| \leq M, \text{ for all } n \geq n_0 \geq 0;\]

(ii) the trivial solution of (2.2.1) is uniformly stable if and only if there exists a positive constant \(M\) such that

\[|Y(n)Y^{-1}(k)| \leq M, \text{ for all } n_0 \leq k \leq n < \infty.\]

**Theorem 2.3.4.** Let \(Y(n)\) be a fundamental matrix of (2.2.1). Then, the trivial solution of (2.2.1) is asymptotically stable if and only if \(\lim_{n \to \infty} |Y(n)| = 0\).
Lemma 2.3.1. [1] Let $V(n)$ be nonsingular matrix function on $\mathbb{N}(n_0)$ and let $P$ be a projection. If there exists a positive constant $c > 1$ such that
\[
\sum_{k=n_0}^{n-1} \|V(n)PV^{-1}(k+1)\| \leq c, \quad \text{for all } n \in \mathbb{N}(n_0),
\]
then there exists a constant $c_1$ such that
\[
\|V(n)P\| \leq c_1 \left(\frac{c-1}{c}\right)^{n-n_0}, \quad \text{for all } n \in \mathbb{N}(n_0).
\]

Lemma 2.3.2. [1] Let $V(n)$ be an invertible matrix which is defined on $\mathbb{N}(n_0)$ and $P$ be a projection. If there exists a constant $c > 0$ such that
\[
\sum_{k=n}^{\infty} \|V(n)PV^{-1}(k+1)\| \leq c, \quad \text{for all } n \in \mathbb{N}(n_0),
\]
then for any vector $\xi \in \mathbb{R}^m$ such that $P\xi \neq 0$, $\limsup_{n \to \infty} \|V(n)P\xi\| = \infty$.

Consider the nonlinear difference equation
\[
y(n+1) = A(n)y(n) + f(n, y(n)), \quad (2.3.2)
\]
where $f : \mathbb{N} \to \mathbb{N} \times \mathbb{R}^m$ is the perturbed equation of the linear equation [2.2.1].

Theorem 2.3.5. [1] Let for all $(n, y) \in \mathbb{N}(n_0) \otimes \mathbb{R}^m$, the function $f(n, y(n))$ satisfy
\[
\|f(n, y(n))\| \leq h(n)\|y\|,
\]
where $h(n)$ is a non-negative function defined on $\mathbb{N}(n_0)$ and $\sum_{k=n_0}^{\infty} h(k) < \infty$.

Then, the trivial solution $y(n, n_0, 0) = 0$ of (2.3.2) is uniformly (asymptotically) stable provided the trivial solution $y(n, n_0, 0) = 0$ of (2.2.1) is uniformly (asymptotically) stable.

**Theorem 2.3.6.** Suppose that there exist a constant $c > 1$ such that for all $n \in \mathbb{N}(n_0)$

$$\sum_{k=n_0}^{n-1} |Y(n)Y^{-1}(k + 1)| \leq c,$$

where $Y(n)$ is the fundamental matrix of linear equation (2.2.1). Further, suppose that for all $(n, y) \in \mathbb{N}(n_0) \times \mathbb{R}^m$ the function $f(n, y(n))$ satisfy the inequality

$$\|f(n, y(n))\| \leq \alpha \|y\|,$$

with $\alpha < 1$. Then, the trivial solution of (2.3.2) is asymptotically stable.

**Section 2.4.**

Let $\mathbb{P}_k(\mathbb{R}^m)$ denotes the family of all non-empty compact convex subsets of $\mathbb{R}^m$. Define the addition and scalar multiplication in $\mathbb{P}_k(\mathbb{R}^m)$ as usual. Then Radstrom [72] states that $\mathbb{P}_k(\mathbb{R}^m)$ is a commutative semi-group under addition, which satisfies the cancellation law. Moreover, if $a, b \in \mathbb{R}$ and $P, Q \in \mathbb{P}_k(\mathbb{R}^m)$, then

$$a(P + Q) = aP + aQ, \quad a(bP) = (ab)P, \quad 1P = P$$
and if $a, b \geq 0$, then $(a + b)P = aP + bP$. The Hausdorff metric is the distance between $P$ and $Q$ and is defined by

$$d(P, Q) = \inf\{\epsilon : P \subset N(Q, \epsilon), Q \subset N(P, \epsilon)\},$$

where

$$N(P, \epsilon) = \{x \in \mathbb{R}^m : ||x - y|| < \epsilon, \text{ for some } y \in P\}.$$

Let $J = [c, d] \subset \mathbb{R}$ be a compact interval and denote

$$\mathbb{E}^m = \{v : \mathbb{R}^m \to [0, 1] \text{ satisfies (i)-(iv) below}\},$$

where

(i) $v$ is normal, i.e. there exists a $y_0 \in \mathbb{R}^m$ such that $v(y_0) = 1$;

(ii) $v$ is fuzzy convex, i.e. for $z_1, z_2 \in \mathbb{R}^m$ and $0 \leq \lambda \leq 1$,

$$v(\lambda z_1 + (1 - \lambda)z_2) \geq \min[v(z_1), v(z_2)];$$

(iii) $v$ is upper semi-continuous;

(iv) $[v]^0 = \{y \in \mathbb{R}^m / v(y) > 0\}$ ($\overline{P}$ = closure of set $P$) is compact.

For $0 < \alpha \leq 1$, the $\alpha$-level set is denoted and defined by $[v]^\alpha = \{y \in \mathbb{R}^m / v(y) \geq \alpha\}$. Obviously, $[v]^\alpha \in \mathbb{P}_k(\mathbb{R}^m)$ for all $0 \leq \alpha \leq 1$. 

31
The real numbers can be embedded to $\mathbb{E}^1$ by the correspondence

$$c \rightarrow \tilde{c}(t) = \begin{cases} 
1 & \text{if } t = c, \\
0 & \text{elsewhere}.
\end{cases}$$

It is well known that

$$[w_1 + w_2]^\alpha = [w_1]^\alpha + [w_2]^\alpha, \quad [aw]^\alpha = a[w]^\alpha,$$

for all $w, w_1, w_2 \in \mathbb{E}^m, a \in \mathbb{R}, 0 \leq \alpha \leq 1$.

In the follow up, we require the following well known theorem.

**Theorem 2.4.1.** (Representation Theorem)[67] If $v \in \mathbb{E}^m$, then

1. $[v]^\alpha \in P_k(\mathbb{R}^m)$, for all $0 \leq \alpha \leq 1$,

2. $[v]^{\alpha_2} \subset [v]^{\alpha_1}$, for all $0 \leq \alpha_1 \leq \alpha_2 \leq 1$,

3. If $\{\alpha_k\}$ is a non-decreasing sequence converging to $\alpha > 0$, then

$$[v]^\alpha = \bigcap_{k \geq 1} [v]^{\alpha_k}.$$

Conversely, if $\{P^\alpha : 0 \leq \alpha \leq 1\}$ is a family of subsets of $\mathbb{R}^m$ satisfying (1)-(3), then there exists a $v \in \mathbb{E}^m$ such that $[v]^\alpha = P^\alpha$ for $0 < \alpha \leq 1$ and

$$[v]^0 = \bigcup_{0 < \alpha \leq 1} P^\alpha \subset P^0.$$

Define $D : \mathbb{E}^m \times \mathbb{E}^m \rightarrow [0, \infty)$ by the equation

$$D(w_1, w_2) = \sup\{d([w_1]^\alpha, [w_2]^\alpha) / \alpha \in [0, 1]\},$$
where \( d \) is the Hausdorff metric. From the outcomes of \([16, 70]\), it is easily shown that the metric space \((\mathbb{E}^m, D)\) is complete, but it is not a locally compact. In addition, the distance \( D \) satisfies the following

1. \( D(x + z, y + z) = D(x, y), \quad x, y, z \in \mathbb{E}^m \),
2. \( D(\lambda x, \lambda y) = |\lambda| D(x, y), \quad x, y \in \mathbb{E}^m, \lambda \in \mathbb{R} \),
3. \( D(w + x, y + z) \leq D(w, y) + D(x, z), \quad w, x, y, z \in \mathbb{E}^m \).

From \([72]\), we notice that \((\mathbb{E}^m, D)\) is not a vector space, but it can be embedded isomorphically as a cone in a Banach space.

**Definition 2.4.1.** \([35]\) A mapping \( F : J \to \mathbb{E}^m \) is strongly measurable if for all \( \alpha \in [0, 1] \) the set-valued mapping \( F_\alpha : J \to \mathbb{P}_k(\mathbb{R}^m) \) defined by \( F_\alpha(t) = [F(t)]^\alpha \) is (Lebesgue) measurable, when \( \mathbb{P}_k(\mathbb{R}^m) \) endowed with the topology generated by the Hausdorff metric \( d \).

Consider the linear difference control system of the form

\[
\begin{align*}
x(n + 1) &= A(n)x(n) + B(n)u(n), \quad x(n_0) = x_0, \quad (2.4.1) \\
y(n) &= C(n)x(n) + D(n)u(n), \quad (2.4.2)
\end{align*}
\]

where \( A(n) \) is a nonsingular matrix, \( B, C, D \) are matrix functions of \( n \) on \( J = [n_0, L] \cap \mathbb{N}, \ L \in \mathbb{N} \). If the inputs \( u(n) \) are crisp, then it is the deterministic difference control system.

**Definition 2.4.2.** \([31]\) System \((2.4.1)\) is said to be completely controllable (or simply controllable) if for any \( n_0 \in J \), any initial state \( x(n_0) = x_0 \),
and any given final state (the desired state) $x_f$, there exists a finite time $N > n_0$ and a control $u(n)$, $n_0 < n \leq N$, such that $x(N) = x_f$.

**Definition 2.4.3.** The input output system (2.4.1) and (2.4.2) is completely observable if for any $n_0 \geq 0$, there exists $N > n_0$ such that the knowledge of $u(n)$ and $y(n)$ for $n_0 \leq n \leq N$ suffices to determine $x(n_0) = x_0$.

**Theorem 2.4.2.** Let $Y(n)$ be the fundamental matrix of (2.2.1). Then the unique solution of the initial value problem (2.4.1) is

$$x(n) = Y(n)Y^{-1}(n_0)x_0 + \sum_{k=n_0}^{n-1} Y(n)Y^{-1}(k + 1)B(k)u(k).$$