CHAPTER 1

INTRODUCTION
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Introduction

Section 1.1.

The importance of difference equations and their occurrence in biology,
digital signal processing, economics and computer sciences are well known.
Matrix difference equations are gaining an important role in dynamical
systems, stability analysis, control systems, finite difference schemes, epi-
demics and competition models. This piece of work mainly concentrates on
the study of first order matrix difference equation of the form

\[ X(n+1) = A(n)X(n)B(n), \]  

(1.1.1)

where \( A(n), B(n) \) are square matrices of order ‘\( m \)’ whose elements \( a_{ij}, b_{ij} \)
are real or complex valued functions of ‘\( n \)’. A matrix difference equation
is a difference equation in which the value of a vector (or sometimes, a
matrix) of variables at one point in time is related to its own value at one
or more previous points in time, using matrices. Occasionally, the time-
varying entity may itself be a matrix instead of a vector. The order of the
equation is the maximum time gap between any two indicated values of the
variable vector (or matrix). To avoid confusion in studying these equations, we call vector difference equation if the time entity is a vector and matrix difference equation if the time entity is a matrix. The vector difference equation is of the form

\[ x(n + 1) = A(n)x(n). \]  

(1.1.2)

Difference equations are the appropriate mathematical representation for discrete processes, which have special importance in areas such as biology, population dynamics, economics, social sciences and neural networks. The theory of difference equations and their applications were presented in the monographs [1, 2, 7, 43, 46, 55, 76]. In biological population, difference equations have been utilized to study the characteristics of structured population models since the exploratory work of Lewis [49] and Leslie [48] in the 1940s. Usually, it is desired to recognize the long term behaviour of population growth. One of the most crucial factors in this regard is ergodicity. A population is said to be ergodic if its inevitable behaviour is independent of its initial state [9]. For an age structured population model with predetermined fertility and mortality rates, it is known that the normalized age distribution approaches a stable age distribution irrespective of the initial population. Such property is well described and often known to as the fundamental theorem of demography or the strong ergodic theorem of demography [9, 12]. Analysis of difference equations models in population dynamics was studied by Fisher [32, 33]. Mathematical models in
biology and epidemiology were presented in [8, 77]. May [51] developed simple mathematical models with very complicated dynamics, using difference equations. Applications of difference equations were also extended to control [5, 40, 47, 68, 84], oscillatory [82], periodic [38], neural networks [56, 71, 86], mechanics [34] and social science [75] problems.

The primary difficulty is to determine the necessary and sufficient conditions for the existence of a solution with some specified boundedness conditions. A traditional outcome of this kind, for system of differential equations is provided by Coppel [15]. A problem of great importance is that of determine the behaviour of a physical system in the neighbourhood of an equilibrium state. If the system returns to this state after being subjected to small disturbances, it is called stable; otherwise it is called unstable. Consequently, when designing a system we would like to have a mathematical criterion for stability. Several classical results relating to stability, asymptotic stability for system of differential equations were quoted in the text books of Bellman [6], Coddington & Levinson [10], Coppel [15], Rao & Ahmad [73], and also the references there in.


$$y(n+1) = A(n)y(n) + f(n, y(n)).$$

(1.1.3)
with almost constant coefficients, where \( A \) is a constant matrix. Hurt \[39\] developed a generalization of the second method of Lyapunov which utilizes certain invariance properties of solutions of ordinary difference equations and these properties are utilized to develop stability theorems. Applying these theorems a region of convergence is derived for the Newton-Raphson and secant iteration methods and also applied to study the effect of round off errors in the Newton-Raphson and Gauss-Seidel iteration methods. Sugiyama \[78\] obtained comparison results on asymptotic behaviour of solutions of difference equations with discrete variable and stated some problems on boundedness and stability which are proved in \[79\], using Lyapunov functions. Fisher \[33\] constructed effective Lyapunov and Lyapunov-like functions for a class of discrete time models of interacting populations with biologically meaningful principle and also obtained stability results. Further, these results were successfully applied to a single species model and a model of competition between two species.

Medina and Pinto \[54\] presented the concept of \( h \)-stability and obtain results on stability for weakly stable difference systems under some perturbations. Further, in \[52\] Medina studied the concept of \( h \)-stability and compared with classical stabilities and also obtained sufficient conditions for stability of linear difference equation \((1.1.2)\) and its nonlinear equation \((1.1.3)\). Furthermore, Medina \[53\] expanded study of exponential stability to a wide range of affordable systems called \( h \)-systems and obtained asymptotic formulae for these systems, which state new results about asymptotic behaviour for perturbed systems under general hypotheses.
Murty et al. [58] studied the existence and uniqueness criteria for first order matrix difference systems. In this paper, authors obtained general solution of nonhomogeneous matrix difference equation

$$X(n+1) = A(n)X(n)B(n) + F(n),$$

(1.1.4)

using method of variation of parameters formula and also constructed a unique solution to the two point boundary value problem associated with (1.1.4), using QR-algorithm and the Bartels-Stewart algorithm.

One common theme of recent work in numerical analysis is the desire to model long term (qualitative) properties of the original problem in the numerical solution. In the long term, errors in numerical solutions grow, and it is not reasonable to demand that global errors in the numerical solution shall converge to zero with small step sizes h. However, it is important that key features of the solution (boundedness, oscillations, periodic or closed orbits, stability) should be preserved. The aim is to identify good numerical methods, which are those that can be relied upon to reproduce faithfully the true qualitative behaviour of solutions to a class of problems. The analysis of numerical methods applied to autonomous linear problems is well-developed. The direct analysis of nonlinear and nonautonomous problems is less well understood and is dependent on the availability of suitable general theorems on the behaviour of solutions to difference equations. Edwards et al. [30] focused on boundedness and stability of solutions of difference equations and present a unified theory that applied both to autonomous
and nonautonomous equations and to nonlinear equations as well as linear equations. Further, they presented some simple examples which illustrate that Lipschitz constants can provide useful insights into the qualitative behaviour of solutions to some nonlinear problems including those arising in numerical analysis.

The problem of solutions being $\Psi$-bounded and $\Psi$-stability for system of ordinary differential equations on $R_+ = [0, \infty)$ had been studied by several authors; namely Akinyele [3], Avramescu [4], Constantin [13], and Diamandescu [17,20]. Akinyele [3] presented the concept of $\Psi$-stability of degree $k$ with respect to a function $\Psi$ is continuous, increasing and differentiable on $\mathbb{R}_+$ such that $\Psi(t) \geq 1$, for $t \geq 0$ and $\lim_{t \to \infty} \Psi(t) = b$, where $b \in [1, \infty)$. Later Constantin [13] introduced the notion of degree of stability and boundedness for solutions of ordinary differential equations, with respect to a continuous positive and nondecreasing function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$. Further, Morchalo [57] presented the notion of $\Psi$-stability, $\Psi$-uniform stability and $\Psi$-asymptotic stability for trivial solution of the nonlinear differential equation $x' = f(t, x)$, where $\Psi$ is a scalar continuous function. He also obtained numerous new sufficient conditions for various types of $\Psi$-stability for the linear differential equation $x' = A(t)x(t)$. Moreover, sufficient conditions are obtained for uniform Lipschitz stability for the nonlinear perturbed differential equation $x' = f(t, x) + g(t, x)$. Diamandescu proven a necessary and sufficient condition for the existence of $\Psi$-bounded solution of nonhomogeneous linear differential equation $x' = A(t)x + f(t)$ on $\mathbb{R}_+$ in [18,19] and on $\mathbb{R}$ in [22]. He also obtained [17,20] several sufficient conditions for
various types of $\Psi$-stability for the trivial solution of the nonlinear Volterra integro-differential system, under the assumption that the function $\Psi$ is a continuous matrix valued function. Recently, Murty and Suresh Kumar \cite{60, 63} and Diamandescu \cite{23} extended the concept of $\Psi$-boundedness and $\Psi$-stability to matrix Lyapunov differential equations.

The behaviour of solutions of difference equations has been paid much more attention by mathematicians and scientists, and the boundedness of solutions is closely related to the investigation of numerical discretization for differential equations \cite{1, 2}. Han and Hong \cite{37} defined $\Psi$-boundedness of solutions for difference equations via $\Psi$-bounded sequences and established existence criteria for $\Psi$-bounded solutions of nonhomogeneous difference equation

$$y(n+1) = A(n)y(n) + f(n),$$

(1.1.5)

where $f$ is a $\Psi$-summable sequence on $\mathbb{Z}^+$. Further, Diamandescu \cite{21} proved a necessary and sufficient condition so that the nonhomogeneous linear difference equation (1.1.5) have at least one $\Psi$-bounded solution on $\mathbb{Z}$ for every $\Psi$-summable function $f$ on $\mathbb{Z}$. Furthermore, Diamandescu \cite{24} established a necessary and sufficient condition for the existence of $\Psi$-bounded solutions for the difference equation (1.1.5) on $\mathbb{Z}^+$, where $f$ is $\Psi$-bounded sequence on $\mathbb{Z}^+$. Recently, Suresh Kumar et al. \cite{80} extended the concept of $\Psi$-boundedness to first order nonhomogeneous matrix difference equation (1.1.4) on $\mathbb{Z}$.

Recently, control systems have believed a more and more part in the development and progression of modern society and technological innovation.
Essentially every aspect of our day-to-day activities is affected by some
type of control systems. Control systems are found in variety in all areas
of industry, such as quality control of produced products, automated set
up line, machine-tool control, computer control, power systems, transport
systems, space technological innovation and weapon systems, robotics and
many others. Even the control of inventory, social and economic systems
may be approach from the concept of automatic control.

In most branches of applied mathematics, the aim being is to analyze
a given situation. Its main aim being to compel or control a system to
behave in some desired fashion. Here ‘system’ is used to mean a collection
of objects, which are related by interactions and produce various outputs
in response to different inputs. The main interest is to control the system
automatically, without direct human intervention.

The objective of the control is to transfer the state of the system to a
desirable state from the initial state using the given input $u(t)$. However,
the existence of such an input should be assured; this is the controllability
condition. On the other hand it is sometimes necessary to know all state
variables from measurement of the output $y(t)$, whose dimension is less than
that of the state. The observability condition assures the construction of
the state from the output. These properties are intrinsic for the systems
and play an important role in the theory of linear systems.

Barnett and Cameron [5] studied controllability and observability for
continuous first order systems, and also similar type of discrete systems.
Results on control theory for discrete cases are given in [46 68 83]. In [81],
Weiss presented necessary and sufficient conditions for complete reachability and complete observability of a linear time-varying discrete-time system and also obtained sufficient conditions for local controllability of nonlinear discrete-time systems relating reachability to the concept of discrete Pfaffian systems. Further, established a minimal-dimension difference equation (with possibly variable coefficients) from a given input/output function of a system. Murty, Rao and Suresh Kumar \[59\] obtained necessary and sufficient conditions for complete controllability, complete observability, and realizability associated with first order matrix Lyapunov systems under certain smoothness conditions.

Generally, several systems are mostly related to uncertainty and inaccuracy. The problem of inaccuracy is considered in general an exact science and that of uncertainty is considered as vague or fuzzy and accidental. Since 1965, when Zadeh published his pioneering paper \[85\], hundreds of examples have been supplied where the nature of uncertainty in the behaviour of a system possesses fuzzy rather than stochastic nature. Non-stationary fuzzy systems described by fuzzy processes look as their natural extension into the time domain. From different viewpoints they were carefully studied in \[42\], \[50\].

Most popular fuzzy logic systems discussed in the literature may be classified into three types: pure fuzzy systems, Takagi and Sugeno’s fuzzy systems, and fuzzy logic systems using fuzzifiers and defuzzifiers. We know that fuzzy logic systems are very useful in helping to establish an intelligent control theory. There are two main approaches to intelligent control: one
that combines differential equations with expert systems in artificial intelligence, the so called expert control, and the other that combines differential equations with discrete event systems or Markov Chain. The first approach is practically useful, but it is difficult to analyze because the formulations of differential equations are based on mathematical formulas and the other is based on symbolic artificial intelligence. The second approach is mathematically well developed, but the theory is complex and applications to real practical problems are not easy.

Ding and Kandel\cite{26,28}, Ding et al.\cite{29} provided a way to incorporate differential equations with fuzzy sets or fuzzy IF-THEN rules to form a new fuzzy logic system, called fuzzy dynamical system, which can be regarded as a new approach to intelligent control. In\cite{26,29} they studied the controllability and observability properties of linear system modelled by

\begin{align}
    x'(t) &= A(t)x(t) + B(t)u(t), \quad x(0) = x_0 \\
    y(t) &= C(t)x(t) + D(t)u(t),
\end{align}

where $A, B, C, D$ are continuous matrix functions and $u(t)$ is a fuzzy set. Further, in\cite{27,28} they continued their study on controllability and observability of fuzzy dynamical systems\cite{1.1.6} with the input $u(t)$ in another form namely as a product of ‘$n$’ fuzzy sets defined in real intervals. Recently, Murty and Suresh Kumar\cite{61,62}, Murty et al.\cite{65} extended these concepts and obtained sufficient conditions for the controllability and observability of fuzzy dynamical matrix Lyapunov systems.
Section 1.2.

In this thesis we study some qualitative properties such as $\Psi$-boundedness, $\Psi$-stability, $\Psi$-asymptotic stability, controllability and observability of first order matrix difference equations. In Chapter 3 to Chapter 6, we studied $\Psi$-bounded, $\Psi$-stable and $\Psi$-asymptotic stable solutions of linear and nonlinear matrix difference equations. The controllability and observability properties of fuzzy matrix difference control systems were studied in Chapter 7 and Chapter 8. The technique of Kronecker product of matrices was used as a tool to study the matrix difference equations.

A brief sketch of the work done in each chapter is mentioned here under.

The importance of present study and literature review about the proposed work was systematically presented in Chapter 1. In Chapter 2, the notation has been introduced in this thesis and presented some properties of Kronecker product of matrices [35] and also presented some standard results on difference equations [1, 31], fuzzy sets and systems [45, 70] etc., which are useful for later discussion.

The existence of $\Psi$-bounded solutions for system of linear differential equations and matrix differential equations were studied by many authors [18, 20, 22, 60, 63]. Recently, this concept was extended to linear difference and matrix difference equations [21, 24, 37, 80]. In Chapter 3, we continue to study the existence of $\Psi$-bounded solutions of first order non-homogeneous matrix difference equations. First, we establish existence and uniqueness of solutions of initial value problems associated with nonhomo-
geneous matrix difference equation (1.1.4), where $F(n)$ is a $m \times m$ real valued matrix function on $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, by converting the problem into a Kronecker product initial value problem. We obtain necessary and sufficient conditions for the existence of at least one $\Psi$-bounded solution for the matrix difference equation (1.1.4), under the assumption that $F(n)$ is a $\Psi$-summable matrix function or $\Psi$-bounded matrix function on $\mathbb{Z}^+$. Further, results relating to the asymptotic behaviour of the $\Psi$-bounded solutions of this equation on $\mathbb{Z}^+$ were proved. The introduction of the matrix function $\Psi$ permits to obtain a mixed asymptotic behaviour of the components of the solutions. The results obtained in this chapter are illustrated with suitable examples. This chapter includes some of the results of Han and Hong [37], Diamandescu [24] as a particular case when $B(n) = 1$, $X$ and $F$ are column vectors.

The existence and uniqueness of $\Psi$-bounded solutions for nonlinear difference and matrix difference equations are not studied so far. Chapter 4 is devoted to establish sufficient conditions for the existence and uniqueness of $\Psi$-bounded solutions for first order nonlinear matrix difference equations on $\mathbb{Z}$. First, we developed sufficient conditions for the existence and uniqueness of $\Psi$-bounded solutions for the nonlinear difference equation (1.1.3) on $\mathbb{Z}$, using Banach contraction principle. Further, we obtain sufficient conditions for the existence and uniqueness of $\Psi$-bounded solutions for nonlinear matrix difference equation

$$X(n + 1) = A(n)X(n)B(n) + F(n, X(n))$$  \hspace{1cm} (1.2.1)
on \( Z \), using the technique of Kronecker product of matrices.

The concepts of \( \Psi \)-(uniform)stability and \( \Psi \)-asymptotic stability for nonlinear matrix difference equations are studied in Chapter 5 and Chapter 6. The purpose of these chapters are to obtain sufficient conditions for the \( \Psi \)-(uniform)stability and \( \Psi \)-asymptotic stability of trivial solution of nonlinear matrix difference equation (1.2.1). First, we establish sufficient conditions for the \( \Psi \)-(uniform)stability and \( \Psi \)-asymptotic stability of trivial solution of (1.1.3) as a perturbed equation of (1.1.2), where \( A(n) \), \( B(n) \) are \( m \times m \) matrix-valued functions and \( f(n, x(n)) \) is a vector-valued function of order \( m \) on \( \mathbb{N} = \{0, 1, 2, \ldots \} \). Also, we develop new difference equations corresponding to (1.1.3), (1.1.2) which are (uniformly)stable and asymptotic stable on \( \mathbb{N} \), provided (1.1.3), (1.1.2) are \( \Psi \)-(uniformly)stable and \( \Psi \)-asymptotically stable on \( \mathbb{N} \). Here, we investigate conditions on the fundamental matrix \( Y(n) \) of the linear equation (1.1.3) and on the function \( f(n, x) \) under which the trivial solution of (1.1.3) is \( \Psi \)-(uniformly)stable on \( \mathbb{N} \). Further, we extend these results to linear matrix difference equations (1.1.1) and nonlinear matrix difference equations (1.1.3), using the technique of Kronecker product of matrices. We investigate conditions on the fundamental matrices \( Y(n), Z(n) \) for the linear equations

\[
X(n + 1) = A(n)X(n), \tag{1.2.2}
\]

\[
X(n + 1) = B^T(n)X(n) \tag{1.2.3}
\]

and on the nonlinear matrix function \( F(n, X) \) under which the trivial solu-
tion of (1.2.1) is $\Psi$-(uniformly)stable and $\Psi$-asymptotic stable on $\mathbb{N}$. The results obtained in these chapters are illustrated with suitable examples and extends some of the results of differential and matrix differential equations [17, 20, 60, 64] to vector and matrix difference equations.

Control theory is an interdisciplinary area of research, where many mathematical concepts and methods work together to produce an impressive body of applied mathematics. Controllability and observability are the two basic concepts that arise in the control of dynamical systems [5, 68]. Recently, many authors [26, 29, 59, 61, 62, 65] studied controllability and observability of matrix Lyapunov systems, fuzzy dynamical systems and fuzzy dynamical matrix Lyapunov systems. In Chapters 7 and 8, we provide a way to incorporate difference equations with fuzzy sets to form a new fuzzy logic system called fuzzy difference control system which can be regarded as a new approach to intelligent control.

Chapter 7 is devoted to study the controllability of first order fuzzy matrix difference control systems modelled by

$$X(n+1) = A(n)X(n)B(n) + F(n)U(n), \quad X(0) = X_0,$$

(1.2.4)

$$Y(n) = C(n)X(n) + D(n)U(n),$$

(1.2.5)

where $U(n)$ is a $m \times m$ fuzzy input matrix called fuzzy control and $Y(n)$ is a $m \times m$ fuzzy output matrix. Here $A(n), B(n)$ are nonsingular matrices, $F(n), C(n)$ and $D(n)$ are matrices of order $m \times m$, whose elements are functions of $n$ on $J = [0, L] \cap \mathbb{N}$, $L \in \mathbb{N} = \{0, 1, 2, \ldots \}$. First, we generate
a deterministic control system

\[ x(n + 1) = A(n)x(n) + B(n)u(n), \ x(0) = x_0, \quad (1.2.6) \]

\[ y(n) = C(n)x(n) + D(n)u(n). \quad (1.2.7) \]

with fuzzy inputs and outputs called a fuzzy difference control system. Here, the fuzzy input is taken as a fuzzy set defined on \( \mathbb{R}^n \). Further, we obtain sufficient conditions for the controllability of the fuzzy difference control system. These results are extended to matrix difference control systems using Kronecker product of matrices. The results of this chapter are highlighted with suitable examples. This chapter extend some of the results of Ding and Kandel [26], M.S.N.Murty et al. [65] to fuzzy difference control systems.

In Chapter 8, we continue to study the first order fuzzy matrix difference control system \((1.2.4), (1.2.5)\) and discuss the concept of observability. First, we formulated a fuzzy difference control system and fuzzy matrix difference control systems and also obtain their solution sets. Next, we introduced the notion of likely observability and provide sufficient conditions for fuzzy control systems to be likely observable. The main results of this chapter are illustrated with suitable examples. This chapter extends some of the results of Ding, Maa and Kandel [29], Murty and Suresh Kumar [62] to fuzzy difference control systems.
Section 1.3.

The following numbering system is used in this thesis. [1], [2], . . . , [86] stand for references given at the end of the eighth chapter. This thesis is divided into eight chapters. In each chapter Definitions, Results, Theorems, Lemmas etc., are numbered in the decimal notation. The first integer indicates the number of the chapter, the second integer the number of the section and the third one indicates the relative order of the result etc., in that section. All these are marked on the right side of the results.

In addition to these, certain equations and expressions are numbered on the right side of the page as (a.b.c), where ‘c’ denotes the relative order of the equation in b\textsuperscript{th} section of the Chapter ‘a’.