Chapter 1

Introduction
1.1 INTRODUCTION

Graph theory is the major area of Discrete Mathematics, and during recent decades, graph theory has developed into a major area of mathematics. The present thesis entitled Some Topics on Graph Theory and its Applications focuses mainly on distance theory, domination and applications of Glue graphs.

1. BRIEF HISTORY OF GRAPH THEORY

Graph theory was once considered an unimportant branch of topology (some even called it the “slum of topology”), graph theory has long since, justified its existence through many important contributions to a wide range of fields. Rich in interesting problems and applications, it is one of the most studied and fastest growing areas within discrete mathematics and computer
science. Countless problems involving a collection of discrete objects can be phrased and solved by graph-theoretic methods. Many such methods have become standard and are considered a natural part of every curriculum in discrete mathematics and computer science [33].

Graph theory may be said to have begun in the year 1736 paper by Leonhard Euler devoted to the Konigsberg bridge problem [54]. More than one century after Euler’s paper on the bridges of Konigsberg and Johann Benedict Listing introduced topology. Arthur Cayley was led by the study of particular analytic forms arising from differential calculus to study a particular class of graphs, the trees. The study had many implications in theoretical chemistry. The involved techniques mainly concerned the enumeration of graphs having particular properties. Enumerative graph theory then rose from the results of Cayley and the fundamental results published by George Polya between 1935 and 1937 and the generalization of these by Nicolaas Govert
de Bruijn in 1959. The fusion of the ideas coming from mathematics with those coming from chemistry is at the origin of a part of the standard terminology of graph theory.

In particular, the term "graph" was introduced by James Joseph Sylvester [56] in a paper published in 1878 in Nature, where he draws analogy between "quantic invariants" and "co-invariants" of algebra and molecular diagrams. One of the most famous and stimulating problems in graph theory is the four color problem.

Graph theory is an exploration of proof techniques in discrete mathematics, and its results have applications in many areas of computing, social and natural sciences. Graph theory is widely applied in almost in every line of thought, right from Sociology, Psychology, Anthropology, Architecture, Biology, Chemistry, Physics, Economics, Computer science, Communication network, Linguistics, Discrete optimization problems, Combinatorial problems, Classical algebraic problems are few to quote
There are many aspects one has to cover, to study graph theory, namely, Colorability, Hamiltonicity, Convexity, Symmetry, Domination, Ramsey Theory, Random graphs, Distance concepts etc. Some graph theoretic parameters like vertex - connectivity, edge - connectivity, chromatic number, domination number, independence number, geodetic number, radius, diameter etc are helpful in understanding the structure of graphs in particular.

1.2. A BRIEF HISTORY OF DISTANCE THEORY IN GRAPHS

One concept that pervades all of graph theory is that of distance. Distance is used in isomorphism testing, graph operations, hamiltonian problems, extremal problems on connectivity
and convexity in graphs. Distance is the basis of many concepts of symmetry in graphs. The important application of facility location on networks is based on various types of graphical centrality, all of which are defined using distance. Many graph algorithms depend on the idea of finding collection of long paths within a graph or network [22].

More often distance is used in facility location problems like locating post office, power station, banks or public distribution systems in a region. We try to minimize distance from each utility location to the service station, so that emergency service stations must be accessible easily. In graph theoretic words the problem reduces to find the center of the underlying graphs. There are some classic results on centers due to Jordan [14], Zelinka [23], Proskurowski [13], Laskar and Schier [11], Hedetniemi and Hedetniemi [16], Fred Buckley [2] and some more.

There are different situations when we try to locate some facility centers like post office, power stations, or banks, we min-
imize the total distance traveled by people within that region. Thus, comes the concepts of ”median” in graphs. This was introduced by Harary [5], as the set of vertices with minimum sum of distances from the considered vertex to all other vertices of the graph.

The facility location problem had still some queries, as where to construct a road, or to build a set of service stations, or some general service facilities like shopping centers, food outlets, etc. All these cases deal with generalized centers, like path center, n-center, n-median, cutting center, path centrix, and several others. These are studied by several graph-theorists like Slater [28][29][30], Cockayne, Hedetniemi and Hedetneiemi [6], Harary and Ostrand [10], Reid [19], Hage and Harary [16], Buckley and Lewinter [5], Freeman [8], Reid and Gu [21], Holbert [18].

Two fundamental distance related concepts in graph theory are radius and diameter. Prisner [20] has dealt sharper inequalities concerning the radius and diameter of a graph.
Akiyama, Ando, and Avis defined the eccentric graph $G_e$ of $G$ to be the graph with $V(G_e) = V(G)$, where two nodes $u$ and $v$ of $V(G_e)$ are adjacent if one of them is an eccentric node of the other [1].

In this thesis, we study the some distance parameters of glue graphs. By applying glue graph definition to non-trivial graphs, we can reduce the distance between the vertices. These graphs may be useful in locating emergency facilities. As these graphs are more connected, can be used in construction of roads, bridges and also communication networks.

1.3. A BRIEF HISTORY OF DOMINATION THEORY IN GRAPHS

Domination is a rapidly developing area of research in graph theory. The concept of domination has existed for a long
time. Although the mathematical study of dominating sets in graphs began around 1960, the subject has historical roots dating back to 1862 when de Jaensich [27] studied the problem of determining the minimum number of queens which are necessary to cover (or dominate) an n x n chessboard. As reported by W.W.Rouse Ball in 1892 [51], chess enthusiasts in the late 1800’s studied, among others, the basic types of problems like

i) covering

ii) independent covering

iii) independence.

These three types of problems were studied in detail by Yaglom and Yaglom around 1964 [59].

In 1958, domination was formalized as a theoretical area in graph theory by C. Berge [4]. He referred to the domination number as the coefficient of external stability and denoted it as $\beta(G)$. In 1962, Ore [46] was the first to use the term ”domination” for undirected graphs and denoted the domination number
by $\gamma(G)$ and also introduced the concepts of minimal and minimum dominating sets of vertices in a graph.

In 1977 Cockayane and Hedetniemi [24] published a survey of the few results known at that time about dominating sets in graphs. In this survey paper, Cockayne and Hedetniemi were the first to use the notation $\gamma(G)$ for the domination number of a graph, which subsequently became the accepted notation. This survey paper seems to have set in motion the modern study of domination in graph.

In our work we have studied domination number and maximal edge dominating set for Glue graphs.

1.4. OUTLINE OF THE PRESENT WORK

Nowadays distance of a graph is very much applied in facility location problems. Suppose that each vertex in a graph
represents a site where customers are located, and we can choose one or more sites at which to locate facilities to serve these customers optimally. Measures of optimality typically involve centrality measures such as choosing centers, medians, or centroid. Like distance in graphs, domination theory plays an important role in solving real world problems such as, School bus routing, Communication networks, Radio stations, Controlling epidemics, Land surveying etc.

This thesis is organized into six chapters. Chapter I is introductory in nature, chapters II to V concentrate on the study of the properties of glue graphs and chapter VI is on cycle-complete Graphs. The chapters are as follows:

Chapter 1. Introduction.

Chapter 2. Some Properties of Glue Graphs.

Chapter 3. Glue graphs of Thorn graphs.

Chapter 4. Domination on Glue graphs.

Chapter 5.

Chapter 6.
Chapter 5. Maximal Edge Domination in Glue graphs.

Chapter 6. Cycle-complete graphs

A brief summary of each chapter is as follows:

Chapter 1, is introductory in nature.

In Chapter 2, we have studied the glue graphs.

The glue graph is defined as, let $G = (V, E)$ be a graph.

The glue graph of $G$ is the graph denoted by $G_g$, with vertex set $V(G_g) = V(G)$ and $(u, v)$ is an edge if and only if $e_G(u) = e_G(v)$, where $e(v_i)$ is the eccentricity of any vertex of a graph $G$. In this chapter, glue graphs of some class of graphs and their distance invariants like, status, radius, diameter, eccentricity and centrality properties are discussed. Also glue graphs of path, cycle and star are widely studied. We have deduced the relation between parameters of a graph and its glue graph.

In Chapter 3, the glue graphs of Thorn graphs are studied.
The eccentricity of a graph $G$ is the farthest distance between two vertices $u$ and $v$ of $G$. The vertices which have the same eccentricity are called equi-eccentric vertices.

The glue graph $G_g$ of $G$ is a graph with $V(G_g) = V(G)$ and $u, v \in V(G)$ are adjacent in $V(G_g)$ if and only if $e(u) = e(v)$. The thorn graph $G^*$ for any graph $G$ that can be obtained from a parent connected graph $G$ by attaching $p_i \geq 0$ new vertices of degree one to each vertex $i$.

In this chapter, the basic properties like, radius, diameter, total status, average eccentricity and centrality properties of glue graphs of thorn graphs $G^*_g$ are studied.

In Chapter 4, domination parameters are studied for some class of graphs and glue graphs. For any graph $G$, the equi-eccentric point set graph $G_{ee}$ is defined on the same set of vertices by joining two vertices in $G_{ee}$ if and only if they correspond to two vertices of $G$ with equal eccentricities. The glue graph $G_g$ is defined on the same set of
vertices by joining two vertices in $G_g$ if and only if they correspond to two adjacent vertices of $G$ or two adjacent vertices of $G_{ee}$.

A set $D \subseteq V$ is a dominating set of $G$ if each vertex in $V - D$ is adjacent to some vertex in $D$. The domination number $\gamma(G)$ is the smallest cardinality of a dominating set. A dominating set is said to be minimal, if no proper subset of $D$ is a dominating set of $G$. In this chapter, some results on domination number and the upper bound for the domination number of the glue graph in terms of order and maximum degree are obtained. We have compared the domination number and connected domination number of a graph $G$ with glue graph $G_g$.

In Chapter 5, the maximal edge domination of glue graphs is studied. A set $S$ of edges is a edge dominating set if every edge not in $S$ is adjacent to some edge of $S$. The edge domination number $\gamma'(G)$ is the minimum cardinality
of a minimum edge dominating set of $G$. This concept was introduced by S.R.Jayaram [41]. An edge dominating set $S$ of a graph $G$ is a maximal edge dominating set of $G$ if $E - S$ is not an edge dominating set of $G$. The maximal edge domination number $\gamma'_m(G)$ of $G$ is the minimum cardinality of a maximal edge dominating set. In this chapter, we study the maximal edge dominating sets and obtain bounds for the maximal edge domination number.

In Chapter 6, cycle-complete graph and its properties are studied. The distance is the basic definition of several graph parameters including center, diameter, radius, total status of the graph, status of a vertex etc. These invariants are examined for cycle-complete graph $C_nK_m$. We obtained the general solution to find the status of these graphs of any order. These graphs are self-centered. The construction of such graphs is described and properties are studied. Also we have calculated degree of each vertex, distribution of
vertices by considering their respective status.

We conclude the work by giving conclusion, list of symbols and suitable references.

**BASIC TERMINOLOGY AND DEFINITIONS**

The following terminologies are used in the present thesis.

A graph $G$ consists of a pair $(V(G), E(G))$, where $V(G)$ is the non-empty finite set whose elements are called *vertices* and $E(G)$ is a set of unordered pairs of distinct elements of $V(G)$. The elements of $E(G)$ are called *edges* of the graph $G$. We consider the graphs in which there is no direction for edges, hence *undirected*; graphs in which not more than one edge joins two vertices, hence *simple*; also the graphs which do not contain edges joining the same vertex, that is not containing *loops*. So we consider, simple, undirected graphs without loops.
If $uv$ is an edge of $G$, then $u$ and $v$ are said to be *adjacent* in $G$. An edge with identical ends is called a *loop* and an edge with distinct ends a *link*. A graph with $p$ vertices and $q$ edges is called a *(p,q)-graph*. The (1,0) graph is called *trivial graph* and all other graphs *nontrivial*. The cardinality of the vertex set of a graph $G$ is called the *order* of $G$ while the number of edges is its *size*.

A *walk* of a graph $G$ is an alternating sequence of vertices, $v_i$ and edges, $e_j$ as $v_0, e_1, v_1, e_2, \cdots, v_{n-1}, e_n, v_n$ such that every $e_i = v_{i-1}v_i$ is an edge of $G$, $1 \leq i \leq n$. A walk begins and ends with vertices immediately preceded and followed by edges. A walk is *closed* if $v_0 = v_n$ and is *open* otherwise. A walk is a *trial* if all edges are distinct and a path if all vertices (so also edges) are distinct. A *path* with $n$ vertices is denoted by $P_n$. A closed walk is a *cycle* if its $n$ vertices are distinct and $n \geq 3$.

A *circuit* in a graph $G$ is a closed trial of length three or
more. Hence a circuit begins and ends at the same vertex but repeats no edges. A circuit can be described by choosing any of its vertices as the beginning (and ending) vertex provided with vertices are listed in the same cyclic order. In a circuit, vertices can be repeated, in addition to the first and last. A circuit that repeats no vertex, except for the first and last, is a cycle. A \( k \)-cycle is a cycle of length \( k \). The length of a cycle or a path is the number of occurrences of edges in it. The cycle of odd length is called an \textbf{odd cycle}; while a cycle of even length is called an \textbf{even cycle}. For \( n \geq 4 \), the \textbf{wheel} \( W_n \) is defined to be the graph \( K_1 + C_{n-1} \), where \( C_n \) is the cycle with \( n \) vertices.

The \textbf{degree} of a vertex \( v \) is the number of edges of \( G \) incident with \( v \), and is denoted by \( deg_G(v) \) or simply \( deg(v) \). The \textbf{edge degree} of an edge \( x = uv \) of a graph \( G \) is the sum of the degrees of \( u \) and \( v \). The minimum degree among all the vertices of \( G \) is denoted by \( \delta(G) \) and maximum degree is denoted by \( \Delta(G) \). A vertex of degree zero in \( G \) is called an \textbf{iso-}
lated vertex. A vertex of degree one in \( G \) is called a **pendant** vertex. A pendant edge is an edge incident to a pendant vertex. If \( \delta(G) = \Delta(G) = r \), then all the vertices have same degree and \( G \) is said to be a **regular graph** of regularity \( r \).

So if \( G \) is a graph of order \( n \) and \( v \) is any vertex of \( G \), then

\[
0 \leq \delta(G) \leq \text{deg}(v) \leq \Delta(G) \leq n - 1.
\]

A complete graph \( K_p \) has every pair of its vertices adjacent. Thus \( K_p \) has \( \frac{p(p-1)}{2} \) edges and is regular graph of degree \( p - 1 \).

A graph is said to be **connected** if every pair of its vertices are joined by a path. A graph which is not connected is said to be **disconnected**. A graph whose edge set is empty is called a **totally disconnected** graph. A graph \( H \) is called a **subgraph** of a graph \( G \), written \( H \subseteq G \), if \( V(H) \) is subset of \( V(G) \) and \( E(H) \subseteq E(G) \). If \( H \subseteq G \) and either \( V(H) \) is a **proper subset** of \( V(G) \) or \( E(H) \) is a proper subset of \( E(G) \), then \( H \) is a **proper subgraph** of \( G \). If \( H \) is a subgraph of a graph \( G \) and \( V(H) = V(G) \), then we say that \( H \) is a **span-**
**Induced subgraph** of $G$. A subgraph $F$ of a graph $G$ is called an
**induced subgraph** of $G$ if whenever $u$ and $v$ are vertices of $F$
and $uv$ is an edge of $G$, then $uv$ is an edge of $F$ as well. If $S$ is
a nonempty set of vertices of a graph $G$, then the subgraph of $G$
induced by $S$ is the induced subgraph with vertex set $S$. This
induced subgraph is denoted by $\langle S \rangle$. For a nonempty set $X$ of
edges, the subgraph $\langle X \rangle$ induced by $X$ has edge set $X$ and
consists of all vertices that are incident with at least one edge in $X$. This subgraph is called an **edge induced subgraph** of $G$.

A connected subgraph of $G$ that is not a proper subgraph
of any other connected subgraph of $G$ is a **component** of $G$.
A graph $G$ is then connected if and only if it has exactly one
component. A **cut point** of a graph $G$ is a point whose re-
moval increases the number of components. A **nonseparable
graph** is connected, nontrivial and has no cut points. A **block**
of a graph $G$ is a **maximal nonseparable subgraph**. If $G$ is
nonseparable, then $G$ itself is often called a **block**.
The **complement** $\overline{G}$ of a graph $G$ is that graph whose vertex set is $V(G)$, such that for each pair $(u, v)$ of vertices of $G$, $uv$ is an edge of $\overline{G}$ if and only if $uv$ is not an edge of $G$. A graph $G$ is a **bipartite graph** if $V(G)$ can be partitioned into two subsets $V_1$ and $V_2$, called **partite sets**, such that every edge of $G$ joins a vertex of $V_1$ to a vertex of $V_2$. If $G$ contains every edge joining $V_1$ and $V_2$ then $G$ is a **complete bipartite graph**. A complete bipartite graph with $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$. A graph $G$ is a **$k$-partite graph** if $V(G)$ can be partitioned into $k$-subsets $V_1, V_2, \ldots, V_k$ such that $uv$ is an edge of $G$ if $u$ and $v$ belong to different partite sets. A **star** is a complete bipartite graph $K_{1,n}$. A graph with cycles is called **cyclic graph**, otherwise **acyclic**. A **tree** is a connected acyclic graph.

Two graphs $G$ and $H$ are **isomorphic** written as $G \cong H$ if there exists a one-to-one correspondence between their vertex sets which preserves adjacency. $G$ is said to be a **self-**
complementary graph if \( G \cong \overline{G} \). An isomorphism of a graph \( G \) on-to itself is called an automorphism of \( G \). A set \( S \subseteq V \) of vertices which covers all the edges of a graph \( G \) is called a vertex cover of \( G \). An edge cover of \( G \) is a set of edges that covers the vertices of \( G \). A set of edges in a graph is independent if no two edges in the set are adjacent. The vertex connectivity \( k(G) \) (edge connectivity \( \lambda(G) \)) of a graph \( G \) is the minimum number of vertices (edges) whose removal results in a disconnected or trivial graph. The neighborhood of a vertex \( u \) in \( V \) is the \( N(u) \) consisting of all vertices \( v \) which are adjacent with \( u \). The closed neighborhood is \( N[u] = N(u) \cup \{u\} \). The smallest number of vertices in any vertex cover for \( G \) is called its vertex covering number and is denoted by \( \alpha_0 \). Similarly, \( \alpha_1 \) is the smallest number of edges in any edge cover of \( G \) and is called its edge covering number. The largest number of vertices in independent set is called the vertex independent number of \( G \) and is denoted by \( \beta_0 \). An independent set of edges of \( G \)
has no two of its edges adjacent and the maximum cardinality of such a set is the edge independence number $\beta_1$.

A cycle complete graph is obtained by taking disjoint copies of the cycle of even length and a complete graph and joining three consecutive vertices of the cycle to all vertices in the complete graph.

In this paragraph, distance and its related parameters are defined. For a connected graph $G$, distance $d(u, v)$ between two vertices $u$ and $v$ is the minimum of the lengths of the $u - v$ paths of $G$. Under this distance function, the set $V(G)$ is a metric space, that is the following properties hold.

1. $d(u, v) \geq 0$, for all pairs $u, v$ of vertices of $G$ and $d(u, v) = 0$, if and only if $u = v$.

2. Symmetric property: $d(u, v) = d(v, u)$ for all pairs $u, v$ of vertices of $G$.

3. Triangle inequality: $d(u, v) + d(v, w) \geq d(u, w)$ for all triples
of vertices of $G$.

If $G$ is disconnected, $d(u, v) = \infty$. The **eccentricity** $e(v)$ of a vertex $v$ of a connected graph $G$ is the number $\max_{u \in V(G)} \{d(u, v)\}$. That is, $e(v)$ is the distance between the vertex $v$ and a vertex farthest from $v$. The **radius**, $rad(G)$ is the minimum eccentricity among the vertices of $G$, while the **diameter** $diam(G)$ of $G$ is the maximum eccentricity. Consequently, $diam(G)$ is the greatest distance between any two vertices of $G$. A vertex $v$ is called a **central vertex** if $e(v) = rad(G)$, and the **center** $C(G)$ of a graph $G$, is the subgraph induced by all central vertices of $G$. A vertex $v$ is called **peripheral vertex**, if $e(v) = diam(G)$, and the periphery is the set of all peripheral vertices. For a vertex $v$, each vertex at distance $e(v)$ from $v$ is an **eccentric vertex** for $v$. A graph is said to be **self-centered** if every vertex is in the center.

A **diameter-preserving** spanning tree of a graph $G$, is
a spanning tree $T$ for which $\text{diam}(T) = \text{diam}(G)$. Similarly, a \textbf{radius-preserving} spanning tree of a graph $G$ is a spanning tree $T$ for which $\text{rad}(T) = \text{rad}(G)$. A graph $G$ is said to be \textbf{unique eccentric node} graph if every vertex of $G$ has exactly one eccentric vertex. For a graph $G$ distance matrix $D(G) = d_{ij}$ of $G$, such that $d_{ij} = d(v_i, v_j)$ the distance between $v_i$ and $v_j$ where the vertex set $V(G)$ is labeled as $\{v_1, v_2, \cdots, v_n\}$. The \textbf{status} $s(v)$ of a vertex $v$ in $G$ is the sum of the distances from $v$ to each other vertex in $G$. The \textbf{median} $M(G)$ of a graph $G$ is the set of vertices with minimum status. The \textbf{total status} of a graph $G$, denoted by $\sigma(G)$ is the sum of the status of all the vertices $v_i \in G$. A graph $G$ is called \textbf{radius-minimal} if $r(G - e) > r(G)$ for every edge $e$ in $G$.

For any graph $G$ the \textbf{equi-eccentric point set graph} $G_{ee}$ is defined on the same set of vertices by joining two vertices in $G_{ee}$ if and only if they correspond to two vertices of $G$ with equal eccentricities. The \textbf{glue graph} of $G$ denoted by $G_g$, is a
graph with vertex set $V(G_g) = V(G)$ and $uv$ is an edge if and only if $e_G(u) = e_G(v)$ where $e(G)$ is the eccentricity of a graph $G$.

A set $D$ of vertices in a graph $G$ is a **dominating set** if every vertex in $V - D$ is adjacent to some vertex in $D$. The **domination number** $\gamma(G)$ of $G$ is the minimum cardinality of a dominating set of $G$. A minimum dominating set of a graph $G$ is called a $\gamma$- set of $G$. A **connected dominating set** is a dominating set in which $\langle D \rangle$ is connected. A dominating set $D$ is called a **maximal dominating set** if $\langle V - D \rangle$ is not a dominating set.

**Note:** We have given all possible definitions and terminologies require to understand the results discussed in this thesis. Some specific definitions required are defined in the respective chapters. For other terms not defined, one may refer to [31],[22] and [34].