Chapter 3

Glue Graphs of Thorn Graphs
3.1 Introduction.

In this chapter we consider only finite, undirected graphs without loops and multiple edges. Let $V(G)$ and $E(G)$ be the vertex and edge sets of a graph $G$ respectively. In order to define $d(u,v)$ for all pairs $u,v \in V(G)$, the graph $G$ must be connected. For two vertices $u$ and $v$ in a graph $G$, the distance $d(u,v)$ from $u$ to $v$ is the length of a shortest $u,v$ path in $G$. A $u,v$ path of length $d(u,v)$ is called a $u−v$ geodesic [3].

For a vertex $v$ in a connected graph $G$, the eccentricity $e(v)$ of $v$ is the distance to a vertex farthest from $v$. The minimum eccentricity among the vertices of $G$ is its radius and the maximum eccentricity is the diameter, which are denoted by $r(G)$ and $diam(G)$ respectively. The vertex $v$ is a central vertex if $e(v) = r(G)$ and $\{\xi_r\}$ is the set of vertices having minimum eccentricity. A graph is self centered if every vertex is in the center. For a vertex $v$, each vertex at distance $e(v)$ from
\( v \) is an eccentric vertex for \( v \) [3]. The average eccentricity of a graph \( G \) is mean eccentricity of a vertex in \( G \), denoted by \( \text{avec}(G) \). The status \( s(v) \) of a vertex \( v \) in \( G \) is the sum of the distances from \( v \) to each other vertex in \( G \). The median \( M(G) \) of a graph \( G \) is the set of vertices with minimum status. The total status \( \sigma(G) \) is the sum of all the status values [3].

The concept of thorn graphs proposed recently by Gutman is extended to the broader concept of generalized thorny graphs. The Thorn graph \( G^* \) for any graph \( G \) that can be obtained from a parent connected graph \( G \) by attaching \( p_i \geq 0 \) new vertices of degree one to each vertex \( i \). The later preserve unchanged certain parts of the parent graph while applying the procedure of thorn graph generation to the rest of the graphs. Thorn stars and rods are typical examples of such generalized thorn graphs. In this chapter, the basic properties like, radius, diameter, total status, average eccentricity and centrality properties of glue graphs of thorn graph \( G_g^* \) are studied.
Definition 3.1. A $t$-thorny ring has a simple cycle as the parent and $(t - 2)$ thorns at each cycle vertex.

Definition 3.2. Thorn stars are graphs obtained from $K_{t}$ arm star by attaching $(t - 1)$ terminal vertices to each of the star arms.

Definition 3.3. Proper graphs are special case of thorny graphs with uniform vertex degree $t$ of all non-terminal vertices and terminal vertices of degree one. We will call such graphs $t$-thorn graphs. The $t$-thorn graphs are of importance for polymer theory, especially for dendrimers.

Definition 3.4. A thorn rod is a graph which includes a linear chain of $p$ vertices and degree-$t$ terminal vertices at each of the two rod ends.

Definition 3.5. A caterpillar is a thorn tree $T^{*}(a,b)$ whose nonterminal vertices $b \geq 1$ are of the same degree $a > 2$, and whose parent graph is the $b$-site chain graph, $P_{b}$ [1].

In the following figure $3(a)$ and $3(b)$, a thorny ring $G^{*}$
3.2 Results

Observation 3.1. For any thorny ring and thorn stars, average eccentricity is given by, 
\[ \text{avec}(G^*) = \left\lfloor \frac{2r(G^*) + 1}{2} \right\rfloor \]
and 
\[ \text{avec}(G^*_g) = 2. \]

In graph theory, chromatic number of a graph plays an important role. Some of the important applications of chromatic number are - Schedules (conferences and events), Programs, Time
table (trains) and Distribution of items. The chromatic number of a graph $G$ and it’s glue graph $G_g$ are studied and compared. By assigning colors to the vertices of glue graph, overlapping of lines in telecommunication networks can be prevented. We have obtained chromatic number of a thorny ring and it’s glue graph.

**Observation 3.2.** Suppose, $G^*$ is a thorny ring and $G^*_g$ is it’s glue graph then,

$$
\chi(G^*) = \begin{cases} 
2, & \text{if } n \text{ is even;} \\
3, & \text{otherwise.}
\end{cases}
$$

$$
\chi(G^*_g) = \begin{cases} 
\frac{n}{2}, & \text{if } n \text{ is even;} \\
\frac{n}{2} + 1, & \text{otherwise.}
\end{cases}
$$

We have obtained some basic properties of glue graphs which will be helpful in our further results.
Centrality Properties for Glue graphs of Thorn Graphs

**Theorem 3.1.** Glue graph of thorn star is self-centered.

**Proof.** For thorn stars, \( r(G) = 2 \) and \( d(G) = 2r \). Hence by the Proposition 2.4, which states, the glue graph \( G_g \) is self-centered if and only if \( r(G) \leq 2 \). Hence the proof. 

**Theorem 3.2.** The glue graph of a thorny ring is self-centered with eccentricity two.

**Proof.** Consider a thorny ring \( G^* \) with interior vertices \( v_1, v_2, \cdots, v_n \) and terminal vertices \( u_1, u_2, \cdots, u_n \).

By the definition of glue graph, in \( G^*_g \) all \( v_i's \) are adjacent to each other. Similarly all \( u_i's \) are adjacent to each other. \( v_i \) is adjacent to at most one \( u_i \) and vice-versa.

In \( G^*_g \), \( e(v_i) = d(v_i, u_n \setminus u_n \text{ is not adjacent to } v_i) \)
which is equal to \( d(v_i, v_n) + d(v_n, u_n) = 2 \rightarrow (1) \).

Consider, \( e(u_n) = d(u_n, v_i \setminus v_i \text{ is not adjacent to } u_n) \)
which is equal to \( d(u_n, v_n) + d(v_n, v_i) = 2 \rightarrow (2) \).

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From (1) and (2) \( e(v_n) = e(u_n) = 2 \). Which implies \( G_g^* \) is self-centered with eccentricity two.

**Corollary 3.1.** Adding a pendant vertex to each vertex of thorny ring \( G^* \) and constructing its glue graph, its centrality property lost and \( r(G_g^*) = 2 \) and \( diam(G_g^*) = 3 \).

**Proof.** In \( G_g^* \), \( V(G_g^*) = v_n \cup u_n \cup w_n \), where \( v_n \) are interior vertices, \( u_n \) are middle vertices and \( w_n \) are terminal vertices.

\[
e(v_n) = \frac{d(v_n, w_n)}{w_n} \text{ is not adjacent to } u_i.
\]

\[
e(v_n) = \{d(v_n, v_i) + d(v_i, u_n) + d(u_n, w_n)\} \text{ such that } u_n \text{ is adjacent to } v_n \text{ and } w_n \text{ is adjacent to } u_n \rightarrow (1).
\]

\[
e(u_n) = d(u_n, w_n) \text{ or } d(u_n, v_n), \text{ such that } u_n \text{ is not adjacent to } v_i \text{ which is equal to}
\]

\[
\{d(u_n, u_i) + d(u_i, w_n)\} \text{ or } d(u_n, v_i) + d(v_i, v_n) \rightarrow (2).
\]

From (1) and (2) \( e(u_n) < e(v_n) \). Which implies \( G_g^* \) is not self-centered.

**Proposition 3.1.** The total status of glue graph of a cycle \( C_n \)
is given by \( \sigma(G_g) = n[n - e(G_g)] \), where \( n \) is the order of \( G_g \).

**Theorem 3.3.** The total status of glue graph of thorny ring is given by \( \sigma(G_g^*) = n[\frac{3n}{2} - e(G_g)] \).

**Proof.** Suppose \( G_g^* \) be a glue graph of a thorny ring \( G \) with order \( n \) as shown in Figure 3(b). Let \( v_1, v_2, \ldots, v_n \) and \( u_1, u_2, \ldots, u_n \) are interior and terminal vertices of thorny ring.

\( V(G_g^*) = v_n \cup u_n \). In \( G_g^* \), number of interior vertices are equal to the number of terminal vertices which is equal to half of the vertex set of glue graph. Therefore, total status of \( G_g^* \) is given by,

\[
\sigma(G_g^*) = s(v_n) + s(u_n),
\]

where \( s(v) \) denotes status of a vertex \( v \) in the graph \( G_g^* \).

In \( (G_g^*) \), \( |v_n| = |u_n| = \frac{n}{2} \).

Let us find the status of a vertex \( v_1 \).

\[
s(v_1) = d(v_1, v_i) + d(v_1, u_1/u_1 \text{ is adjacent to } v_1) + d(v_1, u_i/u_i \text{ is not adjacent } v_1).
\]
\[ s(v_1) = \frac{n}{2} - 1 + 1 + 2\left(\frac{n}{2} - 1\right) \]
\[ s(v_1) = (\frac{3n}{2} - 2) \]

Similarly, \( s(u_1) = (\frac{3n}{2} - 2) \)

Now, \( s(v_n) = s(u_n) = \frac{n}{2} [\frac{3n}{2} - 2] \)

Therefore \( \sigma(G^*_g) = s(v_n) + s(u_n) \)
\[ \sigma(G^*_g) = n[\frac{3n}{2} - 2] \]
\[ \sigma(G^*_g) = n[\frac{3n}{2} - e(G^*_g)] . \]

Since for \( G^*_g \), eccentricity is equal to two.

\[ \blacksquare \]

**Note.** Except thorny ring all other thorn graphs are acyclic.

**Theorem 3.4.** For any thorn graph \( G^* \) and it’s glue graph \( (G^*_g) \), \( r(G) \leq r(G^*_g) \leq r(G^*) \).

**Proof.** Let us consider an acyclic graph \( G \) with vertices \( v_1, v_2, \ldots, v_n \).

Let \( G^* \) be the thorn graph of \( G \). Let \( (G^*_g) \) be the glue graph of \( G^* \). Consider, \( r(G) \leq r(G^*) \).

By the definition of thorn graphs, \( |G| < |G^*| \).

\[ r(G) = e(v_r), \quad v_r \in \left\{ \xi_r \right\} . \]
\[ d(v_r, v_d). \]

\[ = d(v_r, v_{r+1}) + d(v_{r+1}, v_{r+2}) + \cdots + d(v_{r+n}, v_d) = r \rightarrow (1). \]

\[ r(G^*) = e(v_r), (v_r) \in \{\xi_r\}. \]

\[ = d(v_r, v_{r+1}) + d(v_{r+1}, v_{r+2}) + \cdots + d(v_{r+n}, v_d) + d(v_d, u_d) \]

\[ = r + 1 \rightarrow (2). \]

From (1) and (2) \( r(G) \leq r(G^*) \rightarrow (i). \)

\( r(G_g^*) \) is obtained by connecting equi - eccentric vertices of \( r(G^*) \)
and the resulting graph contains a cycle. Hence, \( r(G_g^*) \) is more
connected than thorn graphs which decreases the radius.

Therefore, \( r(G_g^*) \leq r(G^*) \rightarrow (ii). \)

We know that \( r(T_g) \leq r(T) \), where \( T \) and \( T_g \) are the tree and
its glue graph respectively.

By Observation 2.1., \( r(G) \leq r(G^*) \rightarrow (iii) \)

From (i), (ii) and (iii) the result follows.
REFERENCES


