This chapter emphasizes on some motivation of the results described in this monograph. These motivations help to study to many interesting research problems in the field of our study. As we know, though the growth of the general theory on bifurcations and chaos seem to be old, the real breakthrough for the development of these areas came through the most surprising discovery of the physicist Feigenbaum. Feigenbaum’s work gave a link between chaos and phase transitions, and enticed a generation of physicists to study of dynamics. Since then, this field is not only confined in a small area in mathematical sciences with an independent life, but also serves as a unifying thread interlacing many other branches of mathematics and science such as Weather Forecasting, Laser Technology, Population Dynamics, Functional analysis, Measure theory, Fluid mechanics, Quantum mechanics, Physics, Chemistry, Biosciences, Engineering, Medical science etc. and thus, it suggests a very wide scope of doing research. Some problems are posed as follows: [2, 4, 52, 65, 70, 72, 77]

Open Problems:

Problem 1

In chapter - 2, we have determined different bifurcation values leading to chaos with a one dimensional model by applying numerical methods. Now, a question arises: can we develop rigorous analytical methods to establish the
existence of different bifurcation values and the corresponding periodic points?
Moreover, if we consider a polynomial model of degree 'n':

\[ f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n : \]

can we determine the ranges of the coefficients \( a_i \)s for which this model exhibits Feigenbaum tree of bifurcation points leading to chaos? Can we also analytically determine some formulae for controlling chaos?

**Problem 2**

Fractal is an object in every branch of science and various fractal dimensions have far reaching applications in real world problems. So, in order to study various fractal dimensions effectively and smoothly, it is urgent need of the hour for some analytical formulations for determining these dimensions very accurately. In our chapter -6, we have evaluated only a few dimensions. Can we also determine the other dimensions with the help of some suitable mathematical mechanisms both in one-dimensional and higher dimensional population models?

**Problem 3**:

In chapter - 5, we have calculated bifurcation values and the corresponding periodic points of a two-dimensional population model by using numerical methods; Can we do it with the aid of analytical explanation? Here, for each value of b, we have bifurcation value of p and thus P is a continuous of b. Now the questions is : how smooth is this curve?

**Problem 4**:

In the models discussed in chapter - 2 and chapter - 5, can we determine periodic orbits of periods in the Sarkovskii orders [52]:

\[ 3 \rightarrow 5 \rightarrow 7 \rightarrow \cdots \rightarrow 2 \times 3 \rightarrow 2 \times 5 \rightarrow 2 \times 7 \rightarrow \cdots \rightarrow 2^2 \times 3 \rightarrow 2^2 \times 5 \rightarrow 2^2 \times 7 \rightarrow \cdots \]
Problem 5:

One important method of characterizing an attractor makes use of a probability distribution function. This notion becomes particularly important as the number of state space dimensions increases. For a larger number of state space dimensions, we have more and more geometric possibilities for attractors. For higher dimensional state spaces, we need more abstract and less geometric methods of characterizing the attractor/strange attractor. Various kinds of probability distributions are useful in this case. Now we can ask: what is the probability that a given trajectory point of the dynamical system falls within some particular region of a state space?

Problem 6:

The concept of entropy is developed from the point of view of statistical mechanics. From this perspective, the most fundamental definition of entropy is given by counting the number of “accessible” states for the system under consideration. The idea is that statistical mechanics is concerned with relating macroscopic (large-scale) properties of a system, such as pressure, volume occupied, and temperature for a gas, to the microscopic description in terms of positions and velocities of the atoms or molecules that make up that gas. In almost all cases, there is a vast number of microscopic states that correspond to the same macroscopic state. One can assume that an isolated system of atoms and molecules in thermal equilibrium visits equally all of these microstates compatible with this set of properties. Then we define the entropy $S$ of the system as

$$S = k \ln N$$
where \(N\) is the number of microstates compatible with the assumed macroscopic conditions: \(k\) is called Boltzmann’s constant and determines the units in which entropy is measured (in the SI system, \(k = 1.38 \times 10^{23}\) Joule / K).

We let a single trajectory run for a long time to map out an attractor. We then cover the attractor region of state space with cells. Next we start a trajectory in one of the cells and label that cell \(b(o)\). At a time \(\tau\) later, the trajectory point will be in cell \(b(1)\). At \(t = 2\tau\), the trajectory is in cell \(b(2)\), and so on, up to time \(t = N\tau\), thereby recording a particular sequence of cell labels: \(b(0), b(1), b(2), \ldots, b(N)\).

We then start off a second trajectory from the same initial cell. Because the exact initial conditions are slightly different, however, we generally get a different sequence of cell labels for the second sequence. We repeat this process many times, thereby generating a large number of sequences. Next, we calculate the relative number of times a particular sequence of \(N\) cell labels occurs. Let us call that relative number \(p(i)\) for the with sequence. We then define the entropy \(S_N\) to be

\[
S_N = \langle \sum_i p(i) \ln p(i) \rangle,
\]

where the sum is taken over all sequences of \(N\) cell labels that start with \(b(0)\). The brackets \(\langle \rangle\) mean that we average the sum over all starting cells on the attractor.

Finally, we define the K- entropy to be the average rate of increase of the entropy with respect to sequence length:

\[
K = \lim_{N \to \infty} \frac{1}{N} (S_N - S_0)
\]

Kolmogorov- Sinai Entropy can be described in the similar way [71].
Now we can pose a question: can we apply effectively this kind of Entropy to quantify chaos in our concerned models?

**Problem: 7**

In statistical mechanics, the primary computational tool for a system in thermal equilibrium is the so-called “partition function” which is generated by summing a Boltzmann-type factor over all the energy states of the system (in case of discrete systems): 

\[ Z = \sum g_n e^{-\beta E_n} \]

where \( g_n \) is the degeneracy factor for the \( n \)th energy, that is, \( g_n \) specifies how many states have the energy \( E_n \). As usual, \( \beta \) is proportional to the reciprocal of the absolute (Kelvin) temperature for the system. The crucial point is that all the thermodynamic properties of the system can be computed from the partition function.

Now the question is: can we use this partition function to compute bifurcation points, corresponding periodic points and to quantify chaos in chaotic models? How to measure chaos in this case?

**Problem: 8**

The family of Visibility algorithm was developed by Lacasa et.al in 2008 and Laque et.al in 2009, which represents a transformation from time series analysis to a network system opening the possibility of a bridge between nonlinear dynamics and graph theory. Let \( G = \{ x_0, x_1, x_2, \ldots, x_n \} \) be a set of ordered points. Then the Horizontal visibility graph (HVG) consists of the set of points \( G \) and the set of edges \( E = \{ (x_i, x_j) \mid x_i > x_k \text{ whenever } i < k < j \} \). Similarly another visibility graph
called natural visibility algorithm consists of \( G = \{ x_0, x_1, x_2, \ldots, x_n \} \), the set of ordered points and the set of edges \( E = \{ (x_i, x_j) | j \leq k < j \} \). It is a culture in the literature to transform the various universal properties of the time series data of the non-linear dynamical system to network analysis and vice versa.

An analysis of the dynamical properties of 1-dimensional discrete model and its effect in the presence of another 1-dimensional discrete map can be made. For example, the logistic map \( f(x) = ax(1-x) \) shows a period doubling scenario to chaos as the controlling parameter “a” is increased. The phenomenon can be explained by taking the composition \( fof, f0f0f0f, \ldots \) and so on, whose fixed points and their stability is checked. Now instead of taking \( fof \) if it is made as fog where g is another particular map say another logistic map whose control parameter is say b, then how the dynamical behavior is affected as compared to the single logistic map when the parameter value a is increased controlling the other value b. In particular if \( b = a \) then it will completely merge as the logistic map. An analysis may also be done of the properties of the network suitably defined as above in the chaotic region. Now the questions may be raised as follows:

Can we make some intrinsic relation between period-doubling bifurcation and the family of Visibility algorithm in graph theory? Can we establish any relation between Fractal dimensions and graph theory? These are some challenging problems in Dynamical systems and Graph theory! We are quite hopeful that discussion of the period doubling behavior of 2-dimensional discrete model may be done. Also a discussion may be done of the analysis of the network of the periodic points at different parameters, for example the HVG graph,
whose vertices are the periodic points say \( x_1, x_2, ..., x_n \), where \( x_i \in R^2 \) for all \( i \). A suitable partial order relation is so defined that the HVG or NVG graph may be constructed. Then, how some universal properties of the chaotic map are seen in the network that is observable. Further the defined procedure may be adapted in 3-dimensional discrete map and if possible the same may be generalized.

**Problem: 9**

Can we also do that the dynamical behavior of continuous dynamical system may be analyzed with the help of Graph theory?