CHAPTER IV

MOMENT PROPERTIES OF GENERALIZED ORDER STATISTICS FROM ADDITIVE WEIBULL DISTRIBUTION

4.1. Introduction

In this chapter, we have discussed and established simple exact expressions and some recurrence relations satisfied by single and product moments of generalized order statistics from additive Weibull distribution. These relations are deduced for moments of order statistics and record values. Further, conditional moments and recurrence relation for single moments of the generalized order statistics are used to characterize this distribution and some computational works are also carried out.

A random variable $X$ is said to have additive Weibull distribution (Lemonte et al., 2014) if its pdf is of the form

$$f(x) = (\alpha \beta x^{\beta-1} + \theta \delta x^{\delta-1}) e^{-(\alpha x^{\beta} + \theta x^{\delta})},$$

$$x > 0, \; \alpha > 0, \; \beta > 0 \; \text{and} \; \delta, \; \beta > 0 \quad (4.1.1)$$

and the corresponding df is

$$F(x) = 1 - e^{-(\alpha x^{\beta} + \theta x^{\delta})}, \; x > 0, \; \alpha > 0, \; \beta > 0 \; \text{and} \; \delta, \; \beta > 0 . \quad (4.1.2)$$

It is easy to see that

$$f(x) = (\alpha \beta x^{\beta-1} + \theta \delta x^{\delta-1}) F(x) . \quad (4.1.3)$$

The relation (4.1.3) will be utilized to derive exact expressions and some recurrence relations for the moments of gos from additive Weibull distribution.

The concept of generalized order statistics is as introduced in Chapter I.

4.2. Relations for single moments

As given in Chapter I, the pdf of the $r$-th gos, $X(r, n, m, k)$, is given by

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!} \left[ F(x) \right]^{r-1} f(x) g_{m}^{-1}(F(x)) . \quad (4.2.1)$$

Part of the results of this chapter has appeared in Khan et al. (2016)
We shall first establish the exact expression for $E[X^j (r, n, m, k)]$. Using (4.2.1), we have when $m \neq -1$

$$E[X^j (r, n, m, k)] = \frac{C_{r-1}}{(r-1)!} \int_0^\infty x^j [F(x)]^{\gamma r^{-1}} f(x) g_m^{r-1} (F(x)) \, dx .$$

(4.2.2)

On expanding $g_m^{r-1} (F(x)) = \left( \frac{1}{m+1} \left( 1 - (F(x))^{m+1} \right) \right)^{r-1}$ binomially in (4.2.2), we get

$$E[X^j (r, n, m, k)] = \int_0^\infty x^j [F(x)]^{\gamma r^{-u-1}} f(x) \, dx,$$

(4.2.3)

where

$$A = \frac{C_{r-1}}{(r-1)! (m+1)^{r-1}} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} .$$

On using (4.1.3) in (4.2.3), we obtain

$$E[X^j (r, n, m, k)] = A \int_0^\infty x^j (\alpha \beta x^{\beta-1} + \theta \delta x^{\delta-1}) [F(x)]^{\gamma r^{-u}} \, dx$$

$$= - \frac{A}{\gamma r^{-u}} \int_0^\infty x^j \left( \frac{d (e^{-\gamma r^{-u} (\alpha \beta x^{\beta} + \theta \delta x^{\delta}))}{dx} \right) \, dx .$$

Integrating by parts now yields

$$E[X^j (r, n, m, k)] = j A \int_0^\infty x^{j-1} e^{-\gamma r^{-u} (\alpha x^{\beta} + \theta x^{\delta})} \, dx .$$

Further, on using the Taylor series expansion

$$e^{-\gamma r^{-u} \theta} x^\delta = \sum_{v=0}^\infty (-1)^v \frac{(\gamma r^{-u} \theta)^v}{v!} x^{\nu \delta}$$

and simplifying the resulting expression, we get

$$E[X^j (r, n, m, k)] = A^* \int_0^\infty x^{j+\nu \delta-1} e^{-\gamma r^{-u} \alpha \beta} \, dx ,$$

(4.2.4)

where

$$A^* = \frac{j A}{\gamma r^{-u}} \sum_{v=0}^\infty (-1)^v \frac{(\gamma r^{-u} \theta)^v}{v!} .$$
We have Gradshetyn and Ryzhik (2007, p-337)

\[
\int_0^\infty x^m e^{-\beta x^n} \, dx = \frac{\Gamma(m+1)/n}{n^\beta(m+1)/n}, \quad \beta, m, n > 0. \tag{4.2.5}
\]

By substituting (4.2.5) in (4.2.4) and simplifying the resulting expression, we find that

\[
E[X^j (r,n,m,k)] = \frac{jC_{r-1}}{(r-1)! \beta (m+1)^{r-1}} \sum_{v=0}^{r-1} \sum_{u=0}^{\infty} (-1)^{u+v} 
\times \left( r-1 \right) \theta^\nu \left( y_{r-u} \right)^{-\nu \left( j + v \delta \right) / \beta} \Gamma \left( \nu \left( j + v \delta \right) / \beta \right).
\tag{4.2.6}
\]

When \( m = -1 \), we have

\[
E[X^j (r,n,m,k)] = 0 \quad \text{as} \quad \sum_{u=0}^{r-1} (-1)^u \left( \frac{r-1}{u} \right) = 0.
\]

Since (4.2.6) is of the form \( \frac{0}{0} \) at \( m = -1 \), therefore, we have

\[
E[X^j (r,n,m,k)] = B \sum_{u=0}^{r-1} (-1)^u \left( \frac{r-1}{u} \right) \frac{\left[ k + (n-r+u)(m+1) \right]^{-\nu \left( j + v \delta \right) / \beta}}{(m+1)^{r-1}},
\]

where

\[
B = \frac{jC_{r-1}}{(r-1)! \beta} \sum_{v=0}^{\infty} (-1)^v \theta^\nu \left( y_{r-u} \right)^{-\nu \left( j + v \delta \right) / \beta} \Gamma \left( \nu \left( j + v \delta \right) / \beta \right). \tag{4.2.7}
\]

Differentiating numerator and denominator of (4.2.7) \( r-1 \) times with respect to \( m \), we get

\[
E[X^j (r,n,m,k)] = B \sum_{u=0}^{r-1} (-1)^{u+(r-1)} \left( \frac{r-1}{u} \right) \frac{\left[ \frac{((j + v \delta) / \beta) + 1 - v}{(r-1)!} \right] \times \left[ \frac{((j + v \delta) / \beta) + 2 - v}{(r-1)!} \right] \ldots \left[ \frac{((j + v \delta) / \beta) + r-1 - v}{(n-r+u) (m+1)^{r-1}} \right]}{\left[ k + (n-r+u)(m+1) \right]^{(j + v \delta) / \beta} + r-1 - v}.
\]

On applying L’ Hospital rule, we have

\[
\lim_{m \to 1} E[X^j (r,n,m,k)] = B \left[ \frac{(j + v \delta) / \beta + 1 - v}{(r-1)! k^{(j + v \delta) / \beta + r-1 - v}} \times \sum_{u=0}^{r-1} (-1)^u \left( \frac{r-1}{u} \right) (r-n-u)^{r-1}. \tag{4.2.8}
\]
But for all integers $n \geq 0$ and for all real numbers $x$, we have Ruiz (1996)
\[
\sum_{i=0}^{n} \binom{n}{i} (-1)^i (x - i)^n = n!.
\] (4.2.9)

Now substituting (4.2.9) in (4.2.8) and simplifying, we find that
\[
E[X^j (r, n-1, k)] = E[(Y_r^{(k)})^j] = \frac{j}{(r-1)! \beta} \sum_{v=0}^{\infty} (-1)^v \\
\times \frac{\theta^v \Gamma\{((j + v\delta) / \beta) - v + r\} \Gamma\{((j + v\delta) / \beta) - v + 1\}}{v! \alpha^{(j+v\delta)/\beta} \Gamma\{((j + v\delta) / \beta) - v\}}.
\] (4.2.10)

where $Y_r^{(k)}$ denotes the $k$–th upper record values as defined in Chapter I.

Special cases

i) Putting $m=0$ and $k=1$ in (4.2.6), the exact expression for the single moments of order statistics from additive Weibull distribution can be obtained as
\[
E(X^j_{r,n}) = \frac{jC_{rn}}{(r-1)! \beta (m+1)^r} \sum_{u=0}^{\infty} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \\
\times \frac{(n-r+u+1)^{v-(j+v\delta)/\beta-1}}{v! \alpha^{(j+v\delta)/\beta} \Gamma\{((j + v\delta) / \beta) - v\}}.
\] (4.2.11)

where $C_{rn} = \frac{n!}{(r-1)! (n-r)!}$.

ii) Putting $k=1$ in (4.2.10), we deduce the explicit formula for the single moments of upper records for additive Weibull distribution in the form
\[
E[(Y_r^{(1)})^j] = E(X^j_{U(r)}) = \frac{j}{(r-1)! \beta} \sum_{v=0}^{\infty} (-1)^v \\
\times \frac{\theta^v \Gamma\{((j + v\delta) / \beta) - v + r\} \Gamma\{((j + v\delta) / \beta) - v + 1\}}{v! \alpha^{(j+v\delta)/\beta} \Gamma\{((j + v\delta) / \beta) - v\}}.
\] (4.2.12)

Expressions (4.2.11) and (4.2.12) can be used to obtain the moments of order statistics and upper record values for arbitrary chosen values of $\alpha$, $\beta$, $\theta$, $\delta$ and various sample size $n = 1, 2, \ldots, 5$. Numerical computations for the first four moments of order statistics and upper record values from additive Weibull distribution are given in Table 4.1, 4.2, respectively.
Table 4.1. First four moments of order statistics

<table>
<thead>
<tr>
<th>$n$</th>
<th>$r$</th>
<th>$\alpha = 1, \beta = 1, \theta = 1$</th>
<th>$\alpha = 2, \beta = 1, \theta = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$E(X)$</td>
<td>$E(X^2)$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.54564</td>
<td>0.45435</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.32783</td>
<td>0.17216</td>
</tr>
<tr>
<td>2</td>
<td>0.76344</td>
<td>0.73655</td>
<td>0.83862</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.23902</td>
<td>0.09430</td>
</tr>
<tr>
<td>2</td>
<td>0.50546</td>
<td>0.32787</td>
<td>0.25270</td>
</tr>
<tr>
<td>3</td>
<td>0.89243</td>
<td>0.94089</td>
<td>1.13158</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0.18946</td>
<td>0.06053</td>
</tr>
<tr>
<td>2</td>
<td>0.38771</td>
<td>0.19562</td>
<td>0.11819</td>
</tr>
<tr>
<td>3</td>
<td>0.62320</td>
<td>0.46012</td>
<td>0.38721</td>
</tr>
<tr>
<td>4</td>
<td>0.98217</td>
<td>1.10115</td>
<td>1.37970</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0.15748</td>
<td>0.04251</td>
</tr>
<tr>
<td>2</td>
<td>0.31738</td>
<td>0.13261</td>
<td>0.06680</td>
</tr>
<tr>
<td>3</td>
<td>0.49319</td>
<td>0.29013</td>
<td>0.19526</td>
</tr>
<tr>
<td>4</td>
<td>0.70988</td>
<td>0.57345</td>
<td>0.51518</td>
</tr>
<tr>
<td>5</td>
<td>1.05025</td>
<td>1.23308</td>
<td>1.59584</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$n$</th>
<th>$r$</th>
<th>$\alpha = 1, \beta = 2, \theta = 2$</th>
<th>$\alpha = 2, \beta = 2, \theta = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$E(X)$</td>
<td>$E(X^2)$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.57912</td>
<td>0.29764</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.44000</td>
<td>0.16540</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.71824</td>
<td>0.42987</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0.37341</td>
<td>0.11643</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0.79078</td>
<td>0.51314</td>
</tr>
</tbody>
</table>

We can note that the fact $\sum_{i=1}^{n} E(X^j_{En}) = nE(X)^j$ [David and Nagaraja (2003)] is satisfied here, using Table 4.1.
Table 4.2. First four moments of upper record values

<table>
<thead>
<tr>
<th>n</th>
<th>$\alpha = 1$, $\beta = 1$, $\theta = 1$</th>
<th>$\alpha = 2$, $\beta = 1$, $\theta = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\delta = 2$</td>
<td>$\delta = 2$</td>
</tr>
<tr>
<td></td>
<td>$E(X)$</td>
<td>$E(X^2)$</td>
</tr>
<tr>
<td>1</td>
<td>0.54564</td>
<td>0.45435</td>
</tr>
<tr>
<td>2</td>
<td>0.93205</td>
<td>1.06794</td>
</tr>
<tr>
<td>3</td>
<td>1.24176</td>
<td>1.75823</td>
</tr>
<tr>
<td>4</td>
<td>1.50624</td>
<td>2.49375</td>
</tr>
</tbody>
</table>

Remark 4.2.1.

i) Putting $\delta = 1$ or $\beta = 1$ in (4.2.6), we get the explicit expression for single moments of $gos$ from the exponential-Weibull lifetime distribution, established by Khan and Khan (2016).

ii) Putting $\alpha = 0$ or $\theta = 0$ in (4.2.6), we get the explicit formula for single moments of $gos$ from the Weibull distribution, established by Kamps (1995) pp-101.

iii) Putting $\beta = 1$ and $\delta = 1$ in (4.2.6), the results for single moments of $gos$ from exponential distribution with parameter $(\alpha + \theta)$, established by Khan and Khan (2016).

iv) Setting $\beta = 1$, $\theta = 0$ or $\alpha = 0$, $\delta = 1$ in (4.2.6), we get the explicit expression for single moments of $gos$ from the exponential distribution, established by Kamps (1995) pp-101.

v) Setting $\delta = 1$ or $\beta = 1$ in (4.2.10), the result for single moments of $k$–th upper record values is deduced for the exponential-Weibull lifetime distribution, which verify the result obtained by Khan and Khan (2016).
vi) Setting \( \alpha = 0 \) or \( \theta = 0 \) in (4.2.10), the result for single moments of \( k \)–th upper record values is deduced for the Weibull distribution, which verify the result obtained by Kamps (1995) pp-101.

vii) At \( \beta = 1 \) and \( \delta = 1 \) in (4.2.10), the result for single moments of \( k \)–th upper record values is deduced for the exponential distribution with parameter \((\alpha + \theta)\), as given by Khan and Khan (2016).

viii) Setting \( \beta = 1, \theta = 0 \) or \( \alpha = 0, \delta = 1 \) in (4.2.10), the results for single moments of \( k \)–th upper record values is deduced for the exponential distribution, as given by Kamps (1995) pp-101.

Now we obtain the recurrence relations for single moments of additive Weibull distribution in the following theorem.

**Theorem 4.2.1.** For the distribution as given in (4.1.2) and \( n \in N, m \in \mathbb{R}, 1 \leq r \leq n, j = 0,1,2,\ldots \)

\[
E(X^j(r,n,m,k)) = \frac{\alpha \beta^{\gamma_r}}{j + \beta} \{E[X^{j+\beta}(r,n,m,k)] - E[X^{j+\beta}(r-1,n,m,k)]\} \\
+ \frac{\delta^{\gamma_r}}{j + \delta} \{E[X^{j+\delta}(r,n,m,k)] - E[X^{j+\delta}(r-1,n,m,k)]\}. \quad (4.2.13)
\]

**Proof.** From (4.2.1) and (4.1.3), we have

\[
E[X^j(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!} \left\{ \alpha \beta \int_0^\infty x^{j+\beta-1} \left( \frac{\bar{F}(x)}{g_m} \right)^{\gamma_r} g_m^{r-1}(F(x))dx \right. \\
+ \delta \int_0^\infty x^{j+\delta-1} \left( \frac{\bar{F}(x)}{g_m} \right)^{\gamma_r} g_m^{r-1}(F(x))dx \right\}.
\]

Now (4.2.13) can be seen by noting that in view of Athar and Islam (2004).

\[
E[X^j(r,n,m,k)] - E[X^j(r-1,n,m,k)] = \frac{jC_{r-2}}{(r-1)!} \int_0^\infty x^{j-1} \left( \frac{\bar{F}(x)}{g_m} \right)^{\gamma_r} g_m^{r-1}(F(x))dx.
\]

**Remark 4.2.2.**

i) Substituting \( m = 0, k = 1 \) in (4.2.13), we deduce the recurrence relation for single moments of order statistics from additive Weibull distribution in the form

\[
E(X_{r_{jn}}^j) = \frac{\alpha \beta (n-r+1)}{j + \beta} \{E[X_{r_{jn}}^{j+\beta}] - E[X_{r-l_{jn}}^{j+\beta}]\} + \frac{\delta (n-r+1)}{j + \delta} \\
\times \{E[X_{r_{jn}}^{j+\delta}] - E[X_{r-l_{jn}}^{j+\delta}]\}.
\]
At \( \beta = 1 \) and \( \delta = 1 \), the result for single moments of order statistics is deduced for exponential distribution with parameter \((\alpha + \theta)\) as given in Kamps (1995) pp-122.

ii) Putting \( m = -1 \) and \( k = 1 \) in (4.2.13), the result for single moments of upper record values from additive Weibull distribution is deduced as

\[
E(Y_r^{(3)} j) = E(X_{U(r)}^j) = \frac{\alpha \beta}{j + \beta} \{E(X_{U(r)}^{j+\beta}) - E(X_{U(r-1)}^{j+\beta})\} \\
+ \frac{\theta \delta}{j + \delta} \{E(X_{U(r)}^{j+\delta}) - E(X_{U(r-1)}^{j+\delta})\}.
\]

iii) Setting \( \alpha = 0 \) or \( \theta = 0 \) in (4.2.13), we get the recurrence relation for single moments of gos from the Weibull distribution, obtained by Khan et al. (2007) for \( j = j - \delta \) or \( j = j - \beta \), respectively.

iv) Assuming \( \alpha = 1, \theta = 0, \beta = 1 \) or \( \alpha = 0, \theta = 1, \beta = 1 \) in (4.2.13), the result for single moments of gos is deduced for exponential distribution, established by Pawlas and Syznal (2001).

v) By putting \( \delta = 2 \) and \( \beta = 1 \) in (4.2.13), the result for single moments of gos obtained by Ahmad (2008) with \( \theta = \nu / 2 \) for linear failure rate distribution is deduced.

vi) By putting \( \alpha = 0 \) and \( \delta = 2 \) in (4.2.13), the recurrence relation for single moments of gos is deduced for Rayleigh distribution, established by Khan and Khan (2016).

vii) Setting \( \beta = 1 \) or \( \delta = 1 \) and \( m = -1 \) in (4.2.13), the result for single moments of Khan et al. (2015) for upper \( k \) record values from exponential-Weibull lifetime distribution is deduced.

4.3. Relations for product moments

The joint pdf of \( X(r,n,m,k) \) and \( X(s,n,m,k) \), \( 1 \leq r < s \leq n \), as discussed in Chapter I, is

\[
f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = C_{s-1} \frac{(s-r)!}{(r-1)!(s-r-1)!} \left[ \frac{\bar{F}(x)}{m} \right]^m f(x) g_m^{r-1}(F(x)) \\
	imes [h_m(F(y)) - h_m(F(x))]^{s-r-1} \left[ \frac{\bar{F}(y)}{m} \right]^{y^{-1}} f(y), \quad x < y. \tag{4.3.1}
\]
Theorem 4.3.1. For the given distribution as in (4.1.2) and \( n \in \mathbb{N}, m \in \mathbb{R}, 1 \leq r < s \leq n \) and \( i, j \geq 0 \)

\[
E[X^i(r, n, m, k)X^j(s, n, m, k)] = \frac{\alpha \beta \gamma_s}{(j + \beta)} [E[X^i(r, n, m, k)X^{j+\beta}(s, n, m, k)]
\]

\[-E[X^i(r, n, m, k)X^{j+\beta}(s - 1, n, m, k)] + \frac{\theta \delta \gamma_s}{(j + \delta)}
\]

\[
\times \{E[X^i(r, n, m, k)X^{j+\delta}(s, n, m, k)] - E[X^i(r, n, m, k)X^{j+\delta}(s - 1, n, m, k)]\}.
\] (4.3.2)

Proof. From (4.3.1) and (4.1.3), we have

\[
E[X^i(r, n, m, k)X^j(s, n, m, k)] = \frac{\alpha \beta C_{s-1}}{(r - 1)!(s - r - 1)!} \int_{-\infty}^{\infty} x^{i+j+\beta-1} [F(x)]^m f(x)
\]

\[
\times g_m^{-r}(F(x))[h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{y_s} \, dy \, dx
\]

\[
+ \frac{\theta \delta C_{s-1}}{(r - 1)!(s - r - 1)!} \int_{-\infty}^{\infty} x^{i+j+\delta-1} [F(x)]^m f(x)
\]

\[
\times g_m^{-r}(F(x))[h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{y_s} \, dy \, dx.
\] (4.3.3)

In view of Athar and Islam (2004), we have

\[
E[X^i(r, n, m, k)X^j(s, n, m, k)] - E[X^i(r, n, m, k)X^j(s - 1, n, m, k)]
\]

\[
= \frac{j C_{s-2}}{(r - 1)!(s - r - 1)!} \int_{-\infty}^{\infty} x^{i+j-1} [F(x)]^m f(x) g_m^{-r-1}(F(x))
\]

\[
\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{y_s} \, dy \, dx.
\] (4.3.4)

Substituting (4.3.4) in (4.3.3) and simplifying, we get the result given in (4.3.2).

At \( i = 0 \) in (4.3.2), the recurrence relation for product moments reduces to relation for single moments as obtained in (4.2.13).

Remark 4.3.1.

i) Putting \( m = 0 \) and \( k = 1 \) in (4.3.2), we obtain the recurrence relation for the product moments of order statistics from the additive Weibull distribution as

\[
E(X^i_{rn}X^j_{sn}) = \frac{\alpha \beta (n - s + 1)}{j + \beta} \{E(X^i_{rn}X^{j+\beta}_{sn}) - E(X^i_{rn}X^{j+\beta}_{s-ln})\}
\]

\[
+ \frac{\theta \delta (n - s + 1)}{j + \delta} \{E(X^i_{rn}X^{j+\delta}_{sn}) - E(X^i_{rn}X^{j+\delta}_{s-ln})\}.
\]
ii) Setting \( m = -1 \) and \( k \geq 1 \) in Theorem 4.3.2 the relation for the product moments of upper \( k \) record values from the additive Weibull distribution is deduced.

iii) Setting \( \alpha = 0 \) or \( \theta = 0 \) in (4.3.2), we get the recurrence relation for the product moments of \( gos \) from the Weibull distribution as obtained by Khan et al. (2007) for \( j = j - \delta \) or \( j = j - \beta \) respectively.

iv) Assuming \( \alpha = 0, \beta = 1, \theta = 0 \) or \( \alpha = 0, \theta = 1, \delta = 1 \) in (4.3.2), the result for the product moments of \( gos \) is deduced for the exponential distribution, established by Pawlas and Syznal (2001).

v) By putting \( \delta = 2 \) and \( \beta = 1 \) in (4.3.2), the result for product moments of \( gos \) obtained by Ahmad (2008) with \( \theta = \nu/2 \) for linear failure rate distribution is deduced.

vi) By putting \( \alpha = 0 \) and \( \delta = 2 \) in (4.3.2), the recurrence relation for the product moments of \( gos \) is deduced for the Rayleigh distribution, established by Khan and Khan (2016).

vii) Setting \( \beta = 1 \) or \( \delta = 1, m = -1 \) and \( k \geq 1 \) in Theorem 4.3.2 the relation for the product moments in Khan et al. (2015) for upper \( k \) record values from the exponential-Weibull lifetime distribution is deduced.

### 4.4. Characterizations

Let \( X(r, n, m, k), r = 1, 2, \ldots, n \) be \( gos \), then the conditional pdf of \( X(s, n, m, k) \) given \( X(r, n, m, k) = x, 1 \leq r < s \leq n \), as defined in Chapter I, is

\[
 f_{X(s, n, m, k)| X(r, n, m, k)}(y | x) = \frac{C_{s-1}}{(s-r-1)!C_{r-1}}[\bar{F}(x)]^{m-\gamma_{r+1}}
 \times[h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_{s-1}}f(y), \quad x < y. \tag{4.4.1}
\]

**Theorem 4.4.1.** Let \( X \) be a non-negative random variable having an absolutely continuous df \( F(x) \) with \( F(0) = 0 \) and \( 0 \leq F(x) \leq 1 \) for all \( x > 0 \), then

\[
 E[\xi(X(s, n, m, k)) | X(l, n, m, k) = x] = e^{-(\alpha x^{\theta} + \theta x^{\delta})} \prod_{j=1}^{s-1} \left( \frac{\gamma_{l+j}}{\gamma_{l+j} + 1} \right),
 l = r, r + 1 \tag{4.4.2}
\]
if and only if

\[
\bar{F}(x) = e^{-(\alpha x^\beta + \theta x^\delta)}, \quad x > 0, \quad \alpha > 0, \quad \theta > 0 \text{ and } \delta, \beta > 0 \tag{4.4.3}
\]

where

\[
\hat{\xi}(y) = e^{-(\alpha y^\beta + \theta y^\delta)}.
\]

**Proof.** From (4.4.1) for \( s > r + 1 \), we have

\[
E[\xi\{ X(s, n, m, k) \} | X(r, n, m, k) = x] = \frac{C_{s-1}}{(s-r)!C_{r-1}(m+1)^{s-r-1}}
\]

\[
\times \int_x^\infty e^{-(\alpha y^\beta + \theta y^\delta)} \left( \frac{\bar{F}(y)}{\bar{F}(x)} \right)^{\gamma_s-1} \left\{ 1 - \left( \frac{\bar{F}(y)}{\bar{F}(x)} \right)^{m+1} \right\}^{s-r-1} f(y) \,dy. \tag{4.4.4}
\]

By setting \( u = \frac{\bar{F}(y)}{\bar{F}(x)} = \frac{e^{-(\alpha y^\beta + \theta y^\delta)}}{e^{-(\alpha x^\beta + \theta x^\delta)}} \) from (4.1.2) in (4.4.4), we find that

\[
E[\xi\{ X(s, n, m, k) \} | X(r, n, m, k) = x] = \frac{C_{s-1} e^{-(\alpha x^\beta + \theta x^\delta)}}{(s-r)!C_{r-1}(m+1)^{s-r-1}}
\]

\[
\times \int_0^1 u^{\gamma_s-1} (1 - u^{m+1})^{s-r-1} du. \tag{4.4.5}
\]

Again by setting \( t = u^{m+1} \) in (4.4.5), we get

\[
E[\xi\{ X(s, n, m, k) \} | X(r, n, m, k) = x] = \frac{C_{s-1} e^{-(\alpha x^\beta + \theta x^\delta)}}{(s-r)!C_{r-1}(m+1)^{s-r}}
\]

\[
\times \int_0^{t^{\gamma_s-1}} t^{\gamma_s-1} (1 - t)^{s-r-1} dt
\]

and hence the necessary part.

To prove the sufficient part, we have from (4.4.1) and (4.4.2)

\[
\frac{C_{s-1}}{(s-r)!C_{r-1}(m+1)^{s-r-1}} \int_x^\infty e^{-(\alpha y^\beta + \theta y^\delta)} \left[ (\bar{F}(x))^{m+1} - (\bar{F}(y))^{m+1} \right]^{s-r-1}
\]

\[
\times [\bar{F}(y)]^{\gamma_s-1} f(y) \,dy = g_{s|r}(x) [\bar{F}(x)]^{\gamma_s+1}, \tag{4.4.6}
\]
where

\[ g_{s|r}(x) = e^{-(\alpha x^\beta + \theta x^\delta)} \prod_{j=1}^{s-r} \left( \frac{\gamma_{r+j}}{\gamma_{r+j} + 1} \right). \]

Differentiating (4.4.6) both sides with respect to \( x \), we get

\[ \frac{C_{s-1}[\bar{F}(x)]^m f(x)}{(s - r - 2)!C_{r-1}(m + 1)^{s-r-2}} \int_x^\infty e^{-(\alpha y^\beta + \theta y^\delta)}((\bar{F}(x))^{m+1} - (\bar{F}(y))^{m+1})^{s-r-2} \times [\bar{F}(y)]^{\gamma_{r+1} - 1} f(y) dy = g'_{s|r}(x)[\bar{F}(x)]^{\gamma_{r+1} - 1} - \gamma_{r+1} g_{s|r}(x)[\bar{F}(x)]^{\gamma_{r+1} - 1} f(x) \]

or

\[-\gamma_{r+1} g_{s|r+1}(x)[\bar{F}(x)]^{\gamma_{r+1} + m} f(x) = g'_{s|r}(x)[\bar{F}(x)]^{\gamma_{r+1} - 1} - \gamma_{r+1} g_{s|r}(x)[\bar{F}(x)]^{\gamma_{r+1} - 1} f(x). \]

where

\[ g'_{s|r}(x) = -(\alpha \beta x^{\beta - 1} + \theta \delta x^{\delta - 1})e^{-(\alpha x^\beta + \theta x^\delta)} \prod_{j=1}^{s-r} \left( \frac{\gamma_{r+j}}{\gamma_{r+j} + 1} \right) \]

\[ g_{s|r+1}(x) = e^{-(\alpha x^\beta + \theta x^\delta)} \left( \frac{\gamma_{r+1} + 1}{\gamma_{r+1}} \right) \prod_{j=1}^{s-r} \left( \frac{\gamma_{r+j}}{\gamma_{r+j} + 1} \right). \]

Therefore,

\[ \frac{f(x)}{\bar{F}(x)} = -\frac{g'_{s|r}(x)}{\gamma_{r+1}[g_{s|r+1}(x) - g_{s|r}(x)]} = (\alpha \beta x^{\beta - 1} + \theta \delta x^{\delta - 1}) \]

which proves that

\[ F(x) = 1 - e^{-(\alpha x^\beta + \theta x^\delta)}, \quad x > 0, \quad \alpha > 0, \quad \theta > 0 \text{ and } \delta, \beta > 0. \]

**Remark 4.4.1.**

i) As \( m \to -1 \) in (4.4.2), we get the characterization results from the additive Weibull distribution based on \( k \)th upper record values.

ii) Setting \( m = 0, \ k = 1 \) in (4.4.2), we obtain the characterization results of the additive Weibull distribution based on order statistics.
Following theorem contains characterization of this distribution by a recurrence relation for the single moments of $gos$.

**Theorem 4.4.2.** Fix a positive integer $k$ and let $j$ be a non-negative integer. A necessary and sufficient condition for a random variable $X$ to be distributed with pdf given by (4.1.1.) is that

$$E(X^j (r, n, m, k)) = \frac{\alpha \beta r}{j + \beta} \{E[X^{j+\beta} (r, n, m, k)] - E[X^{j+\beta} (r-1, n, m, k)]\}$$

$$+ \frac{\partial \delta}{j + \delta} \{E[X^{j+\delta} (r, n, m, k)] - E[X^{j+\delta} (r-1, n, m, k)]\}. \tag{4.4.7}$$

**Proof.** The necessary part follows immediately from equation (4.2.13). On the other hand if the relation in (4.4.7) is satisfied, then on using (4.2.1), we have

$$\frac{C_{r-1}}{(r-1)!} \int_0^\infty x^j \{F(x)\}^{r-1} f(x) g_m^{-1}(F(x)) \, dx = \frac{\alpha \beta C_{r-1}}{(r-1)!}(j + \beta)$$

$$\times \int_0^\infty x^{j+\beta-1} \{F(x)\}^{r-2} f(x) g_m^{-2}(F(x)) \left\{\frac{\gamma_r g_m(F(x))}{F(x)} - (r-1)[F(x)]^m\right\} \, dx$$

$$+ \frac{\delta \delta}{(r-1)!}(j + \delta) \int_0^\infty x^{j+\delta-1} \{F(x)\}^{r-2} f(x) g_m^{-2}(F(x)) \left\{\frac{\gamma_r g_m(F(x))}{F(x)} - (r-1)[F(x)]^m\right\} \, dx.$$

Let

$$h(x) = -[\{F(x)\}^{r-2} g_m^{-1}(F(x))]. \tag{4.4.8}$$

Differentiating both the sides of (4.4.8), we get

$$h'(x) = [\{F(x)\}^{r-2} f(x) g_m^{-2}(F(x)) \left\{\frac{\gamma_r g_m(F(x))}{F(x)} - (r-1)[F(x)]^m\right\}].$$

Thus,

$$\frac{C_{r-1}}{(r-1)!} \int_0^\infty x^j \{F(x)\}^{r-1} f(x) g_m^{-1}(F(x)) \, dx = \frac{\alpha \beta C_{r-1}}{(r-1)!}(j + \beta)$$

$$\times \int_0^\infty x^{j+\beta-1} h'(x) \, dx + \frac{\delta \delta}{(r-1)!}(j + \delta) \int_0^\infty x^{j+\delta-1} h'(x) \, dx. \tag{4.4.9}$$
Integrating right hand side in (4.4.9) by parts and using the value of \( h(x) \) from (4.4.8), we find that

\[
\frac{C_{r-1}}{(r-1)!} \int_0^\infty x^j (\bar{F}(x))^{\gamma r-1} g_m^{-1}(F(x)) \times \{f(x) - (\alpha \beta x^{\beta-1} + \theta \delta x^{\delta-1})\bar{F}(x)\} \, dx = 0. \tag{4.4.10}
\]

Applying the extension of Müntz-Szász Theorem, (see for example Hwang and Lin, 1984), to (4.4.10), we get

\[ f(x) = (\alpha \beta x^{\beta-1} + \theta \delta x^{\delta-1})\bar{F}(x) \]

which proves that \( f(x) \) has the form as in (4.1.3).

**Remark 4.4.2.** Theorem 4.4.2 can be used to characterize the exponential-Weibull, Weibull, exponential, linear failure rate and Rayleigh distributions by setting different values of parameters.