CHAPTER III

MOMENT PROPERTIES OF GENERALIZED ORDER STATISTICS FROM EXPONENTIAL-WEIBULL LIFETIME DISTRIBUTION

3.1. Introduction

The concept of generalized order statistics (gos) was introduced by Kamps (1995). A variety of order models of random variables is contained in this concept, such as order statistics, upper record values, progressive Type II censoring order statistics, sequential order statistics and Pfeifer’s records. Several authors utilized the concept of gos in their studies. References may be made to Kamps and Gather (1997), Keseling (1999), Cramer and Kamps (2000), Ahsanullah (2000), Habibullah and Ahsanullah (2000), Raqab (2001), Kamps and Cramer (2001), Ahmad and Fawzy (2003), Beiniek and Syznal (2003), Al-Hussaini and Ahmad (2003), Cramer et al. (2004), Khan and Alzaid (2004), Jaheen (2005), Khan et al. (2006), Khan et al. (2010), Khan and Zia (2014) among others. Kamps (1998) investigated the importance of recurrence relations of order statistics in characterization. In this chapter, explicit expressions and some simple recurrence relations for the single and the product moments of the gos from exponential-Weibull lifetime distribution are obtained. Further, various deductions and particular cases are discussed. At the end, the characterization results based on conditional expectation and recurrence relations are presented and some computational works are also carried out.

A random variable \( X \) is said to have exponential-Weibull lifetime distribution (Cordeiro et al., 2014) if its pdf is of the form

\[
f(x) = (\alpha + \beta \theta x^{\theta-1})e^{-(\alpha x + \beta \theta)}, \quad x > 0, \, \alpha > 0, \, \beta > 0, \, \theta > 0
\]  

(3.1.1)

with the corresponding df

\[
F(x) = 1 - e^{-(\alpha x + \beta \theta)}, \quad x > 0, \, \alpha > 0, \, \beta > 0, \, \theta > 0.
\]  

(3.1.2)

It can be seen that

\[
f(x) = (\alpha + \beta \theta x^{\theta-1})F(x).
\]  

(3.1.3)

Part of the results of this chapter has appeared in Khan and Khan (2016)
The relation (3.1.3) will be used to derive explicit expressions and some recurrence relations for the moments of $gos$ from exponential-Weibull lifetime distribution.

We can obtain several special models from relation (3.1.1). The exponential and Weibull distributions are the special cases for $\theta = 1$ or $\theta = 1$, $\alpha = 0$ or $\beta = 0$ and $\alpha = 0$, respectively. The Rayleigh distribution arises when $\alpha = 0$ and $\theta = 2$. The two-parameter linear failure rate distribution is obtained when $\theta = 2$.

3.2. Relations for single moments

As given in Chapter I, the $pdf$ of the $r$-th $gos$, $X(r,n,m,k)$, is given by

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!} [\tilde{F}(x)]^{\gamma - 1} f(x) g_{r-1}^{m-1}(F(x)) .$$  \hspace{1cm} (3.2.1)

We shall first establish the existence of $E[X^j (r,n,m,k)]$. Using (3.2.1), we have when $m \neq -1$

$$E[X^j (r,n,m,k)] = \frac{C_{r-1}}{(r-1)!} \int_0^{\infty} x^j [\tilde{F}(x)]^{\gamma - 1} f(x) g_{r-1}^{m-1}(F(x)) \, dx .$$  \hspace{1cm} (3.2.2)

By using binomial expansion, (3.2.2) can be rewritten as

$$E[X^j (r,n,m,k)] = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \times \int_0^{\infty} x^j [\tilde{F}(x)]^{\gamma - u - 1} f(x) \, dx .$$  \hspace{1cm} (3.2.3)

Further, on using (3.1.3) in (3.2.3), we obtain

$$E[X^j (r,n,m,k)] = A \int_0^{\infty} x^j [\tilde{F}(x)]^{\gamma - u} (\alpha + \beta \theta x^{\theta^{-1}}) \, dx$$

$$= - \frac{A}{\gamma_{r-u}} \int_0^{\infty} x^j \left( \frac{d}{dx} (e^{-\gamma_{r-u}(\alpha + \beta \theta x^\theta)}) \right) \, dx ,$$

where

$$A = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} .$$
Integrating by parts now yields

\[ E[X^j (r, n, m, k)] = \frac{jA}{\gamma_{r-u}} \int_0^\infty x^{j-1} e^{-\gamma_{r-u}(\alpha x + \beta x^\theta)} \, dx. \]

On expanding \( e^{-\gamma_{r-u}\alpha x} \) in Taylor series, we get

\[ E[X^j (r, n, m, k)] = A^* \int_0^\infty x^{j+v-1} e^{-\gamma_{r-u}\beta x^\theta} \, dx, \quad (3.2.4) \]

where

\[ A^* = \frac{jA}{\gamma_{r-u}} \sum_{v=0}^\infty (-1)^v \frac{(\alpha \gamma_{r-u})^v}{v!}. \]

We have Gradshetyn and Ryzhik (2007, p-337)

\[ \int_0^\infty x^m e^{-\beta x^n} \, dx = \frac{\Gamma(m+1)/n}{n\beta^{(m+1)/n}}, \quad \beta, \ m, \ n > 0. \quad (3.2.5) \]

Now on substituting (3.2.5) in (3.2.4), we have

\[ E[X^j (r, n, m, k)] = \frac{jC_{r-1}}{(r-1)!\theta(m+1)} \sum_{v=0}^{r-1} \sum_{u=0}^{r-1} (-1)^v \frac{1}{u} \left( \begin{array}{c} r-1 \\ u \end{array} \right) \]

\[ \times \alpha^v (\gamma_{r-u})^{v-((j+v)/\theta)-1} \frac{1}{v!\beta^{(j+v)/\theta}} \Gamma\{(j+v)/\theta\}. \quad (3.2.6) \]

When \( m = -1 \) that

\[ E[X^j (r, n, m, k)] = \frac{0}{0} \quad \text{as} \quad \sum_{u=0}^{r-1} (-1)^u \left( \begin{array}{c} r-1 \\ u \end{array} \right) = 0. \]

Since (3.2.6) is of the form \( \frac{0}{0} \) at \( m = -1 \), therefore, we have

\[ E[X^j (r, n, m, k)] = B \sum_{u=0}^{r-1} (-1)^u \left( \begin{array}{c} r-1 \\ u \end{array} \right) \frac{(k + (n-r+u)(m+1))^{v-((j+v)/\theta)-1}}{(m+1)^{r-1}}, \quad (3.2.7) \]

where

\[ B = \frac{jC_{r-1}}{(r-1)!\theta} \sum_{v=0}^\infty (-1)^v \frac{\alpha^v \Gamma\{(j+v)/\theta\}}{v!\beta^{(j+v)/\theta}}. \]
Differentiating numerator and denominator of (3.2.7) \((r-1)\) times with respect to \(m\), we get

\[
E[X^j(r,n,m,k)] = B \sum_{u=0}^{r-1} (-1)^{u}(r-1)\left( \frac{(j+v)/\theta +1-v}{u} \right) \frac{[\Gamma((j+v)/\theta +1-v)\Gamma((j+v)/\theta -v)]}{\Gamma[(j+v)/\theta -v+1]} \frac{1}{(r-1)!}
\]

\[
\times \frac{[\{(j+v)/\theta +2-v\}...[\{(j+v)/\theta +r-1-v\}] (n-r+u)^{r-1}}{[k+(n-r+u)(m+1)]^{[(j+v)/\theta +r-v]}}.
\]

On applying L’Hospital rule, we have

\[
\lim_{m \to +1} E[X^j(r,n,m,k)] = B \left[ \frac{\{(j+v)/\theta +1-v\}...[\{(j+v)/\theta +r-1-v\}] (r-1)!}{k^{[(j+v)/\theta +r-v]}(r-1)!(j+v)/\theta +r-v} \right]
\]

\[
\times \sum_{u=0}^{r-1} (-1)^{u} \frac{(r-1)}{u} (r-n-u)^{r-1}.
\]

(3.2.8)

But for all integers \(n \geq 0\) and for all real numbers \(x\), we have Ruiz (1996)

\[
\sum_{i=0}^{n} (-1)^{i} \binom{n}{i} (x-i)^{n} = n!.
\]

(3.2.9)

Now substituting (3.2.9) in (3.2.8) and simplifying, we find that

\[
E[X^j(r,n,-1,k)] = E[(Y_r^{(k)})^j] = \frac{j}{(r-1)!\theta} \sum_{v=0}^{\infty} (-1)^{v}
\]

\[
\times \frac{\alpha^{v}\Gamma((j+v)/\theta)\Gamma((j+v)/\theta -v+1)}{v!k^{[(j+v)/\theta -v+1]}\beta^{(j+v)/\theta}}
\]

(3.2.10)

where \(Y_r^{(k)}\) denotes the \(k\) – th upper record values as defined in Chapter I.

**Remark 3.2.1.**

i) Putting \(\alpha = 0\) in (3.2.6), we get the explicit expression for single moments of \(gos\) from the Weibull distribution as given by Kamps (1995) pp-101.

ii) Putting \(\theta = 1\) in (3.2.6), the results for single moments of \(gos\) from exponential distribution with parameter \((\alpha + \beta)\) is deduced in the form

\[
E[X^j(r,n,m,k)] = C_{r-1} \frac{\sum_{u=0}^{r-1} (-1)^{u}(r-1)^{\alpha+\beta}}{(r-1)!(m+1)^{\alpha+\beta}}
\]

\[
\times \frac{\Gamma(j+1)}{(\gamma_{r-u})^{j+1}}.
\]
as obtained by Kamps (1995) pp-101 at $\alpha = 0$.

iii) Setting $\alpha = 0$ in (3.2.10), the result for single moments of $k$–th upper record values is deduced for the Weibull distribution, which verify the result obtained by Kamps (1995) pp-101.

iv) At $\theta = 1$ in (3.2.10), the result for single moments of $k$–th upper record values is deduced for the exponential distribution with parameter $(\alpha + \beta)$ in the form

$$E[X^j(r,n_1,k)] = E[(Y_r^{(k)})^j] = \frac{\Gamma(j + r)}{(r-1)! k^j (\alpha + \beta)^j},$$

as obtained by Kamps (1995) pp-101 at $\alpha = 0$.

**Special cases**

i) Putting $m = 0$ and $k = 1$ in (3.2.6), the exact expression for the single moments of order statistics from exponential-Weibull lifetime distribution can be obtained as

$$E(X^j(r,n_1,k)) = \frac{j C_{r,n}}{\theta} \sum_{v=0}^{\infty} \sum_{u=0}^{r-1} (-1)^{u+v} \binom{r-1}{u} \alpha^v (n-r+u+1)^{v-(j+v)/\theta-1} v! \beta^{(j+v)/\theta} \Gamma\{(j+v)/\theta\}, \quad (3.2.11)$$

where

$$C_{r,n} = \frac{n!}{(r-1)!(n-r)!}.$$

ii) Putting $k = 1$ in (3.2.10), we deduce the explicit formula for the single moments of upper records for exponential-Weibull lifetime distribution in the form

$$E[(Y_r^{(1)})^j] = E(X^j_{y(r)}) = \frac{j}{(r-1)! \theta} \sum_{v=0}^{\infty} (-1)^v \alpha^v \Gamma\{(j+v)/\theta\} \Gamma\{((j+v)/\theta) - v + r\} v! \beta^{(j+v)/\theta} \Gamma\{((j+v)/\theta) - v + 1\}. \quad (3.2.12)$$

Expressions (3.2.11) and (3.2.12) can be used to obtain the moments of order statistics and upper record values for arbitrary chosen values of $\alpha$, $\beta$, $\theta$ and various sample size $n = 1, 2, \ldots, 5$. Some numerical computations for the first four moments of order
statistics and upper record values from exponential-Weibull lifetime distribution are given in Table 3.1, 3.2, respectively.

### Table 3.1. First four moments of order statistics

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<th>$\beta = 1$</th>
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<td>$E(X^3)$</td>
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We can note that the relation \( \sum_{i=1}^{n} E(X^j_{en}) = nE(X^j) \) [David and Nagaraja (2003)] is satisfied here.

**Table 2.** First four moments of upper record values

<table>
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<tr>
<th>( n )</th>
<th>( \alpha = 1, \theta = 3 )</th>
<th>( \alpha = 1, \theta = 3 )</th>
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Now, we obtain the recurrence relations for single moments of exponential-Weibull lifetime distribution in the following theorem.

**Theorem 3.2.1.** For the distribution as given in (3.1.2) and \( n \in \mathbb{N} \), \( m \in \mathbb{R} \), \( 1 \leq r \leq n \), \( j = 0,1,2,... \)

\[
E[X^j(r,n,m,k)] = \frac{\alpha^r}{j+1} \left[ E[X^{j+1}(r,n,m,k)] - E[X^{j+1}(r-1,n,m,k)] \right] + \frac{\beta \theta \gamma_r}{j+\theta} \times \left[ E[X^{j+\theta}(r,n,m,k)] - E[X^{j+\theta}(r-1,n,m,k)] \right]. 
\]

(3.2.13)

**Proof.** From (3.2.1) and (3.1.3), we have

\[
E[X^j(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!} \left\{ \alpha \int_0^\infty x^j [\bar{F}(x)]^{\gamma_r} g_m^{r-1}(F(x))dx 
+ \beta \theta \int_0^\infty x^{j+\theta-1} [\bar{F}(x)]^{\gamma_r} g_m^{r-1}(F(x))dx \right\}. 
\]

(3.2.14)
Now (3.2.13) can be proved in view of Athar and Islam (2004)

\[
E[X^j(r,n,m,k)] - E[X^j(r-1,n,m,k)] = \frac{jC_{r-2}}{(r-1)!} \int_0^\infty x^{j-1} [\hat{F}(x)]^{\gamma r} g_{m}^{-1}(F(x)) \, dx.
\]

**Remark 3.2.2.**

i) Substituting \( m = 0, k = 1 \) in (3.2.13), we deduce the recurrence relation for single moments of order statistics from exponential-Weibull lifetime distribution in the form

\[
E(X_{r,n}^{j}) = \frac{\alpha(n-r+1)}{j+1} [E(X_{r,n}^{j+1}) - E(X_{r-1,n}^{j+1})] + \frac{\beta \theta(n-r+1)}{j+\theta} [E(X_{r,n}^{j+\theta}) - E(X_{r-1,n}^{j+\theta})].
\]

At \( \theta = 1, \alpha = 0 \), the result for single moments of order statistics is deduced for exponential distribution as given in Kamps (1995) pp-122.

ii) Putting \( m = -1 \) in (3.2.13), the result for single moments obtained by Khan et al. (2015) for upper \( k \)-th record values from exponential-Weibull lifetime distribution is deduced.

iii) Setting \( \alpha = 0 \) in (3.2.13), we get the recurrence relation for single moments of gos from the Weibull distribution, obtained by Khan et al. (2007) for \( j = j-\theta \).

iv) Assuming \( \beta = 0 \) and \( \alpha = 1 \) in (3.2.13), the result for single moments of gos is deduced for exponential distribution, established by Pawlas and Syznal (2001).

v) By putting \( \theta = 2 \) in (3.2.13), the result for single moments of gos obtained by Ahmad (2008) with \( \beta = v/2 \) for linear failure rate distribution is deduced.

vi) By putting \( \alpha = 0 \) and \( \theta = 2 \) in (3.2.13), the recurrence relation for single moments of gos is deduced for Rayleigh distribution in the form

\[
E(X^j(r,n,m,k)) = \frac{2\beta y}{j+2} \{E[X^{j+2}(r,n,m,k)] - E[X^{j+2}(r-1,n,m,k)]\}.
\]

### 3.3. Relations for product moments

The joint pdf of \( X(r,n,m,k) \) and \( X(s,n,m,k) \), \( 1 \leq r < s \leq n \), as defined in Chapter I, is
Moment properties of generalized order statistics from exponential-Weibull ...

\[
f_{X(r,m,k),X(s,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!} \left[ F(x) \right]^m f(x) g_{m-1}^{-1}(F(x)) \times \left[ h_m(F(y)) - h_m(F(x)) \right]^{s-r-1} \left[ F(y) \right]^{r-1} f(y), \quad x < y. \quad (3.3.1)
\]

**Theorem 3.3.1.** For the given exponential-Weibull distribution in (3.1.2) and \( n \in N, \ m \in \mathbb{R}, 1 \leq r < s \leq n \) and \( i, j \geq 0 \)

\[
E[X^i(r,m,k)X^j(s,m,k)] = \frac{\alpha \gamma s}{(j+1)} \left\{ E[X^i(r,m,k)X^{j+1}(s,m,k)] - E[X^i(r,m,k)X^{j+1}(s-1,m,k)] \right\} + \frac{\beta \gamma s}{(j+\theta)} \\
\times \left\{ E[X^i(r,m,k)X^{j+\theta}(s,m,k)] - E[X^i(r,m,k)X^{j+\theta}(s-1,m,k)] \right\}. \quad (3.3.2)
\]

**Proof.** From (3.1.1) and (3.1.3), we have

\[
E[X^i(r,m,k)X^j(s,m,k)] = \frac{\alpha \gamma}{(r-1)!(s-r-1)!} \int_0^\infty \int_0^\infty x^i y^j [F(x)]^m f(x) g_{m-1}^{-1}(F(x)) \\
\times \left[ h_m(F(y)) - h_m(F(x)) \right]^{s-r-1} \left[ F(y) \right]^{r-1} dy dx + \frac{\beta \gamma}{(r-1)!(s-r-1)!} \int_0^\infty \int_0^\infty x^i y^{j+\theta-1} \\
\times [F(x)]^m f(x) g_{m-1}^{-1}(F(x)) \left[ h_m(F(y)) - h_m(F(x)) \right]^{s-r-1} \left[ F(y) \right]^{r-1} dy dx. \quad (3.3.3)
\]

In view of Athar and Islam (2004), note that

\[
E[X^i(r,m,k)X^j(s,m,k)] - E[X^i(r,m,k)X^j(s-1,m,k)] \\
= \frac{j \gamma}{(r-1)!(s-r-1)!} \int_0^\infty \int_0^\infty x^i y^{j-1} [F(x)]^m f(x) g_{m-1}^{-1}(F(x)) \\
\times \left[ h_m(F(y)) - h_m(F(x)) \right]^{s-r-1} \left[ F(y) \right]^{r-1} dy dx. \quad (3.3.4)
\]

Substituting (3.3.4) in (3.3.3) and simplifying, we get the result given in (3.3.2).

At \( i = 0 \) in (3.3.2), the recurrence relation for product moments reduces to relation for single moments as obtained in (3.2.13).

**Remark 3.3.1.**

i) Putting \( m = 0 \) and \( k = 1 \) in (3.3.2), we obtain the recurrence relation for the product moments of order statistics from the exponential-Weibull lifetime distribution as
\[ E(X_i^j X_j^l) = \frac{\alpha(n-s+1)}{(j+1)} \{E(X_i^j X_j^{j+1}) - E(X_i^j X_{j+1}^{j+1})\} + \frac{\beta(n-s+1)}{(j+\theta)} \]
\[ \times [E(X_i^j X_j^{j+\theta}) - E(X_i^j X_{j+\theta}^{j+\theta})]. \]

ii) Setting \( m = -1 \) in Theorem 3.3.2 the relation for the product moments as obtained by Khan et al. (2015) for upper \( k \)-th record values from the exponential-Weibull lifetime distribution is deduced.

iii) Setting \( \alpha = 0 \) in (3.3.2), we get the recurrence relation for the product moments of \( gos \) from the Weibull distribution as obtained by Khan et al. (2007) for \( j = j - \theta \).

iv) Assuming \( \beta = 0 \) and \( \alpha = 1 \) in (3.3.2), the result for the product moments of \( gos \) is deduced for the exponential distribution, established by Pawlas and Syznal (2001).

v) By putting \( \theta = 2 \) in (3.3.2), the result for product moments of \( gos \) obtained by Ahmad (2008) with \( \beta = v/2 \) for linear failure rate distribution is deduced.

vi) By putting \( \alpha = 0 \) and \( \theta = 2 \) in (3.3.2), the recurrence relation for the product moments of \( gos \) is deduced for Rayleigh distribution in the form

\[ \begin{align*} 
E[X^l(r,n,m,k)X^j(s,n,m,k)] &= \frac{2\beta\gamma_s}{(j+2)} \{E[X^l(r,n,m,k)X^{j+2}(s,n,m,k)] \\
&\quad - E[X^l(r,n,m,k)X^{j+2}(s-1,n,m,k)] \}. 
\end{align*} \]

3.4. Characterizations

Let \( X(r,n,m,k), \ r = 1,2,\ldots,n \) be \( gos \), then the conditional \( pdf \) of \( X(s,n,m,k) \) given \( X(r,n,m,k) = x, \ 1 \leq r < s \leq n \), as defined in Chapter I,

\[ f_{X(s,n,m,k)|X(r,n,m,k)}(y \mid x) = \frac{C_{s-1}}{(s-r-1)!C_{r-1}} [F(x)]^{s-m+r+1} \]
\[ \times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{r-s-1} f(y), \quad x < y. \] (3.4.1)

**Theorem 3.4.1.** Let \( X \) be a non-negative random variable having an absolutely continuous \( df \) \( F(x) \) with \( F(0) = 0 \) and \( 0 \leq F(x) \leq 1 \) for all \( x > 0 \), then

\[ E[\xi[X(s,n,m,k) \mid X(l,n,m,k) = x] = e^{-(\alpha x + \beta x^\theta)} \prod_{j=1}^{s-l} \left( \frac{\gamma_{l+j}}{\gamma_{l+j} + 1} \right), \quad l = r, \ r+1 \] (3.4.2)

if and only if
\[ F(x) = e^{-(\alpha x + \beta x^\theta)}, \quad x > 0, \quad \alpha > 0, \quad \beta > 0, \quad \theta > 0, \]  

(3.4.3)

where

\[ \xi(y) = e^{-(\alpha y + \beta y^\theta)}. \]

**Proof.** We have from (3.4.1) for \( s > r + 1 \)

\[
E[\xi \{ X(s, n, m, k) \} | X(r, n, m, k) = x] = \frac{C_{s-1}}{(s - r - 1)!C_{r-1}(m + 1)^{s-r-1}} \times \int_x^\infty e^{-(\alpha y + \beta y^\theta)} \left( \frac{F(y)}{F(x)} \right)^{\gamma_r-1} \left[ 1 - \left( \frac{F(y)}{F(x)} \right)^{m+1} \right]^{s-r-1} \frac{f(y)}{F(x)} dy. \tag{3.4.4}
\]

By setting \( u = \frac{F(y)}{F(x)} = \frac{e^{-(\alpha y + \beta y^\theta)}}{e^{-(\alpha x + \beta x^\theta)}} \) from (3.1.2) in (3.4.4), we obtain

\[
E[\xi \{ X(s, n, m, k) \} | X(r, n, m, k) = x] = \frac{C_{s-1} e^{-(\alpha x + \beta x^\theta)}}{(s - r - 1)!C_{r-1}(m + 1)^{s-r-1}} \times \int_0^1 u^\gamma_r (1-u^{m+1})^{s-r-1} du. \tag{3.4.5}
\]

Again by setting \( t = u^{m+1} \) in (3.4.5), we get

\[
E[\xi \{ X(s, n, m, k) \} | X(r, n, m, k) = x] = \frac{C_{s-1} e^{-(\alpha x + \beta x^\theta)}}{(s - r - 1)!C_{r-1}(m + 1)^{s-r}} \times \int_0^1 t^{\gamma_r+1-1} (1-t)^{s-r-1} dt
\]

and hence the necessary part given in (3.4.2).

To prove the sufficient part, we have from (3.4.1) and (3.4.2)

\[
\frac{C_{s-1}}{(s - r - 1)!C_{r-1}(m + 1)^{s-r-1}} \int_x^\infty e^{-(\alpha y + \beta y^\theta)} [(F(x))^{m+1} - (F(y))^{m+1}]^{s-r-1} \times [F(y)]^{\gamma_r-1} f(y) dy = g_{s|r}(x)[F(x)]^{\gamma_r+1}, \tag{3.4.6}
\]

where
Differentiating (3.4.6) both sides with respect to $x$, we get

$$\frac{C_{s-1}^{m} f(x)}{(s-r-2)!C_{r-1}(m+1)^{s-r-2}} \int_{x}^{\infty} e^{-\alpha x + \beta x^\theta} \left[ (\frac{\gamma_{r+1}}{\gamma_{r} + 1})^{s-r+1} - \frac{\gamma_{r+1}}{\gamma_{r} + 1} \right] f(x)$$

$$\times \left[ \frac{\gamma_{r+1}}{\gamma_{r} + 1} \right] f(y) dy = g_{s|r}(x) \left[ \frac{\gamma_{r+1}}{\gamma_{r} + 1} \right] f(x)$$

or

$$-\gamma_{r+1} g_{s|r+1}(x) \left[ \frac{\gamma_{r+1}}{\gamma_{r} + 1} \right] f(x)$$

$$= g_{s|r}(x) \left[ \frac{\gamma_{r+1}}{\gamma_{r} + 1} \right] f(x) - \gamma_{r+1} g_{s|r}(x) \left[ \frac{\gamma_{r+1}}{\gamma_{r} + 1} \right] f(x),$$

where

$$g_{s|r}(x) = -(\alpha + \beta x^\theta) e^{-\alpha x + \beta x^\theta} \prod_{j=1}^{s-r} \left( \frac{\gamma_{r+j}}{\gamma_{r} + 1} \right)$$

$$g_{s|r+1}(x) = e^{-\alpha x + \beta x^\theta} \left( \frac{\gamma_{r+1} + 1}{\gamma_{r+1}} \right) \prod_{j=1}^{s-r} \left( \frac{\gamma_{r+j}}{\gamma_{r+j} + 1} \right).$$

Therefore,

$$\frac{f(x)}{F(x)} = -\frac{g_{s|r}(x)}{\gamma_{r+1} \left[ g_{s|r+1}(x) - g_{s|r}(x) \right]} = \alpha + \beta x^\theta x^{\theta-1}$$

which proves that

$$F(x) = 1 - e^{-\alpha x + \beta x^\theta}, \quad x > 0, \; \alpha > 0, \; \beta > 0, \; \theta > 0.$$

**Remark 3.4.1.**

i) As $m \to -1$ in (3.4.2), we get the characterization results from the exponential-Weibull distribution based on $k$-th upper record values.

ii) Setting $m = 0, \; k = 1$ in (3.4.2), we obtain the characterization results of the exponential-Weibull lifetime distribution based on order statistics.
Following theorem contains characterization of this distribution by a recurrence relation for the single moments of \(gos\).

**Theorem 3.4.2.** Fix a positive integer \(k\) and let \(j\) be a non-negative integer. A necessary and sufficient condition for a random variable \(X\) to be distributed with pdf given by (3.1.1) is that

\[
E(X^j (r, n, m, k)) = \frac{\alpha r}{j + 1} \{E[X^{j+1} (r, n, m, k)] - E[X^{j+1} (r-1, n, m, k)]\} \\
+ \frac{\beta \theta r}{j + \theta} \{E[X^{j+\theta} (r, n, m, k)] - E[X^{j+\theta} (r-1, n, m, k)]\}.
\] (3.4.7)

**Proof.** The necessary part follows from (3.2.13). On the other hand if the relation in (3.4.7) is satisfied, then on using (3.2.1), we have

\[
\frac{C_{r-1}}{(r-1)!} \int_0^\infty x^j [F(x)]^{\gamma r-1} f(x) g_m^{r-1}(F(x)) \, dx = \frac{\alpha C_{r-1}}{(r-1)!(j+1)} \int_0^\infty x^{j+1}[F(x)]^\gamma \, dx \\
\times \int_0^\infty f(x) g_m^{r-2}(F(x)) \left(\frac{\gamma_r g_m(F(x))}{F(x)} - (r-1)[F(x)]^m\right) \, dx + \frac{\beta \theta C_{r-1}}{(r-1)!(j+\theta)} \int_0^\infty x^{j+\theta}[F(x)]^\gamma f(x) g_m^{r-2}(F(x)) \left(\frac{\gamma_r g_m(F(x))}{F(x)} - (r-1)[F(x)]^m\right) \, dx.
\]

Let

\[
h(x) = -(F(x))^{\gamma r} g_m^{r-1}(F(x)).
\] (3.4.8)

Differentiating both the sides in (3.4.8), we get

\[
h'(x) = [F(x)]^{\gamma r} f(x) g_m^{r-2}(F(x)) \left(\frac{\gamma_r g_m(F(x))}{F(x)} - (r-1)[F(x)]^m\right).
\]

Thus,

\[
\frac{C_{r-1}}{(r-1)!} \int_0^\infty x^j [F(x)]^{\gamma r-1} f(x) g_m^{r-1}(F(x)) \, dx
\]

\[
= \frac{\alpha C_{r-1}}{(r-1)!(j+1)} \int_0^\infty x^{j+1} h'(x) \, dx + \frac{\beta \theta C_{r-1}}{(r-1)!(j+\theta)} \int_0^\infty x^{j+\theta} h'(x) \, dx.
\] (3.4.9)

Integrating right hand side in (3.4.9) by parts and using the value of \(h(x)\) from (3.4.8), we find that

\[
\frac{C_{r-1}}{(r-1)!} \int_0^\infty x^j [F(x)]^{\gamma r-1} g_m^{r-1}(F(x)) \{\alpha F(x) - \beta \theta x^{\theta-1} F(x) - f(x)\} \, dx = 0.
\] (3.4.10)
Applying the extension of Müntz-Szász Theorem, (see for example Hwang and Lin, 1984), to (3.4.10), we get

\[ f(x) = (\alpha + \beta \theta x^{\theta-1}) \bar{F}(x) \]

which proves the result.

**Remark 3.4.2.** Theorem 3.4.2 can be used to characterize the exponential, Weibull, linear failure rate and Rayleigh distributions by setting parameters.