CHAPTER VI

GRAVITATIONAL POTENTIALS OF SOME CHARGED FLUID DISTRIBUTIONS:

1. INTRODUCTION:

The hurdles in understanding the implications of Einstein field equations

\[ R_{ab} - \frac{1}{2} R g_{ab} = - K T_{ab}, \]

are the non-linearity of the field equations, the non-integrability of tensors over finite regions of space-time, the invariance of field equations under arbitrary continuous co-ordinate transformations, besides the minuteness of the magnitude of physical quantities amenable for experimental detection in the laboratory. A way out of these difficulties is to construct exact solutions under ideal situations to serve as a substratum for more complex and realistic situations. For instance, the so called 'crucial tests' confirming the superiority of Einstein's theory of gravitation over Newton's theory are based upon the determination of the exact form of the gravitational potentials in a suitably restricted empty space-time.

Ehlers and Kundt (1962) have given a comprehensive...

Gravitational potentials for electrovac universes, discovered by Einstein and Rosen, Rosen, Curzon, Bonnor,
Weber and Wheeler, Marder, Das, Misra and Radhakrishna have been surveyed by Radhakrishna (1965). The determination of the metric tensor characterizing homogeneous non-rotating cosmological model by Misner (1969), homogeneous cosmological models by Matzner (1970, 1971), Matzner and Chitre (1971), field of shells and disks by Morgan and Morgan (1970), null fluid by Morgan (1972), Vaidya (1973), magnetofluids by Date (1972), Shaha (1972), inhomogeneous spherically symmetric model by Dodson (1972), are a tribute to the rich content of Einstein's geometrodynamical relations.

Reissner-Nordstrom (Vide Misner, Thorne, Wheeler, 1973) have determined the gravitational field of an electron, while Shah and Vaidya (1968) have found the field of a charged particle embedded in homogeneous universe. Exact solutions of Penney's field equations corresponding to a charged fluid, have been obtained by Bohra and Mehra (1971). An exquisite method has been evolved by Synge (1971), for expressing the gravitational potentials as functionals of stress-tensor. His treatment has been extended by Hogan (1972) to charged bodies. The gravitational potentials of a photon rocket by Kinnersley (1969) and that of a moving charged particle emitting null fluid and null current by Bonnor and Vaidya (1972) are exciting investigations.
All these investigations justify Lichnerowicz's (1962) claim that the primary concern in the general theory of relativity is to find out gravitational potentials of various self-gravitating distributions of matter in motion. In this chapter we determine the potentials for self-gravitating Charged Fluid distributions, under certain restrictions. In Section 2, after establishing the non-existence of non-static Charged Fluid spheres, the potentials of a static Charged Fluid sphere are presented and Kuchowicz's potentials for hydrodynamical perfect fluid can be recovered as a particular case of the present investigation. Section 3 contains the potentials of a class of cosmological constant Charged Fluid spheres, derived by using the characteristic system for a spherically symmetric space-time developed by Takeno (1966). The last section deals with the potentials of 'radiating' Charged Fluid and a charged 'curvature' fluid in cylindrically symmetric space-times.

2. ON THE NON-EXISTENCE OF NON-STATIC CHARGED FLUID SPHERES

Theorem I: The gravitational potentials of self-gravitating Charged Fluid sphere are independent of the time co-ordinate.
Proof: The gravitational potentials of a spherically symmetric space-time satisfy (Takeno, 1966),

\[ \xi \ell g_{\alpha \beta} = 0, \quad (A) \]

where \( a, s \) are given in spherical polar co-ordinates as

\[ a\xi = (0, \sin \phi, \cot \theta \cos \phi, 0), \quad (2.2) \]

\[ a\eta = (0, -\cos \phi, \cot \theta \sin \phi, 0), \quad (2.3) \]

\[ a\zeta = (0, 0, -1, 0). \quad (2.4) \]

The equations (2.1) to (2.4) imply

\[ g_{11} = A(\rho, t), \quad g_{22} = B(\rho, t), \]

\[ g_{33} = g_{22} \sin^2 \phi, \quad g_{44} = C(\rho, t), \]

\[ g_{14} = D(\rho, t). \quad (2.5) \]

Further by a co-ordinate transformation (Tolman, 1934), one obtains a simpler set as follows:

\[ g_{11} = e^\gamma, \quad g_{22} = \rho^2, \]

\[ g_{33} = \rho^2 \sin^2 \phi, \quad g_{44} = e^\phi, \quad (2.6) \]

where \( \gamma \) and \( \phi \) are functions of \( \rho \) and \( t \).
We assume co-moving coordinate system, i.e.,

\[ u^a = (0, 0, 0, u^4). \quad \ldots \quad (2.7) \]

Then, for metric in (2.6), by virtue of Einstein equation (I-3.20), viz.,

\[ R_{ab} - \frac{1}{2} R g_{ab} = - K T_{ab}, \quad \ldots \quad (2.8) \]

for the Charged Fluid, i.e., when

\[ T_{ab} = (\gamma + p) u^a u^b - p g_{ab} + E_{ab}, \quad \ldots \quad (2.9) \]

where

\[ E_{ab} = \frac{1}{4} g_{ab} F_{mn} F^{mn} F_{am} F^b, \quad \ldots \quad (2.10) \]

we get

\[ T^4_1 = - KE^4_1 = \frac{1}{c} \gamma \dot{\gamma}. \quad \ldots \quad (2.11) \]

Dot denotes differentiation with respect to time.

Also for the spherically symmetric space-times we have (Takeno, 1966)

\[ \frac{\partial F_{ab}}{\partial \xi} = 0 \]

\[ \ldots \quad (2.12) \]

which implies that there exist only two non-vanishing components of \( F_{ab} \) viz.,

\[ F_{14} = - F_{41}, \quad F_{23} = - F_{32} \ldots \quad (2.13) \]

Now it follows that

\[ E^4_1 = 0. \quad \ldots \quad (2.14) \]
Thus (2.7) implies

\[ i = 0, \text{i.e., } \gamma = \gamma(\kappa), \quad \ldots \]  

(2.15)

The choice

\[ i = 0 \quad \text{i.e.} \quad \psi = \psi(\kappa), \quad \ldots \]  

(2.16)

is possible by using transformation

\[ t = \int e^{\psi/2} dt. \]

This completes the proof of the theorem.

**Remark 1):** Birkhoff's theorem states that "Every spherically symmetric solution of field equations of general relativity \( R_{ab} = 0 \), may be reduced, by a co-ordinate transformation to the Schwarzschild solution, viz.,

\[ ds^2 = - (1 - \frac{2m}{\kappa})^{-1} d\kappa^2 - \kappa^2 (d\varphi^2 + \sin^2 \varphi \, d\varphi^2) + (1 - \frac{2m}{\kappa}) \, dt^2. \]

This theorem is the extension of Birkhoff's theorem to the Charged Fluid.

**Remark ii):** Birkhoff's theorem was extended by Bonnor (1961) to field equation \( R_{ab} = \wedge g_{ab} \), by Das (1960) and Hoffman (1962) to electromagnetic fields by Shaha (1974) to definite magnetofluid schemes.

**Remark iii):** Birkhoff's theorem is obeyed by a plane-symmetric space-time (Taub, 1972).
Gravitational Potentials for Some Static Charged Fluids:

We consider the field equations for the Charged Fluid in co-moving co-ordinate system, for the metric

\[ ds^2 = -e^{-\gamma} \, dt^2 + \frac{1}{\kappa^2} \left( \frac{\gamma' + \frac{\gamma'}{\kappa} \gamma''}{\kappa^2} \right) - \frac{1}{\kappa^2} \left( d\phi^2 + \sin^2 \phi \, d\theta^2 \right) + \psi \, dt^2, \quad (2.17) \]

where \( \gamma \) and \( \psi \) are functions of \( \kappa \) only. For this metric flow vector is

\[ u^a = (0, 0, 0, e^{-\gamma}), \]

and the surviving components of stress-tensor (2.9) are

\[ K^1_1 = -e^{-\gamma} \left( \frac{\gamma'}{\kappa} + \frac{1}{\kappa^2} \right) + \frac{1}{\kappa^2} = K \left( -p + E^1_1 \right), \quad (2.18) \]

\[ K^1_2 = -e^{-\gamma} \left( \frac{\gamma''}{2} - \frac{\gamma' \gamma''}{4} + \frac{\gamma'}{4} + \frac{\gamma'}{2 \kappa} \right) = K \left( -p + E^2_2 \right) = K^3_3 = K \left( -p + E^3_3 \right), \quad (2.19) \]

\[ K^1_4 = e^{-\gamma} \left( \frac{\gamma'}{\kappa} - \frac{1}{\kappa^2} \right) + \frac{1}{\kappa^2} = K \left( \mu + E^4_4 \right), \quad (2.20) \]

Prime denotes differentiation with respect to \( \kappa \). On integrating (2.20) we get

\[ e^{-\gamma} = 1 - \frac{2m}{\kappa} - \frac{K}{\kappa} \int \left( \mu + E^4_4 \right) \kappa \, d\kappa, \quad \ldots \quad (2.21) \]

where \( m \) is a constant of integration.

If we assume that the electromagnetic field is spherically symmetric as in (2.12) then we have by (2.13) the surviving components of stress-tensor \( E^b_a \) are

\[ E^1_1 = -E^2_2 = -E^3_3 = E^4_4. \quad \ldots \quad (2.22) \]
Subtracting equation (2.19) from (2.18) we get
\[ e^{-\gamma} \left( \frac{\psi''}{2} - \frac{\psi'^2}{4} + \frac{\psi'''}{4} - \frac{\psi'}{2} - \frac{1}{4} \psi - \frac{1}{2} \right) = -2K \frac{e^2}{2}, \]
\[ e^{-\gamma} \left( \frac{\psi''}{2} - \frac{\psi'^2}{4} + \frac{\psi'''}{4} - \frac{\psi'}{2} - \frac{3}{2} \frac{\psi'}{2} \right) = -K (\mu - E_1) \] (2.23)
by virtue of (2.20), (2.22).

The substitution
\[ \frac{\psi}{2} = y, \quad v = e^{-\gamma}, \]
\[ \frac{d}{dx} + \int v^{1/2} \, dx, \quad ... \] (2.24)
reduces the equation (2.23) to
\[ \frac{d^2y}{dz^2} + \left[ \frac{3}{4z} + \frac{K}{4z} (\mu - E_1) \right] y = 0, \quad ... \] (2.25)

Now we observe that the Maxwell's equations
\[ F_{ab} = \mu^a \] (2.26)
are satisfied, in the co-moving co-ordinate system (2.7) by the following choice of \( F_{ab} \),
\[ F_{41} = -\varepsilon_0 \frac{n}{\varepsilon} \exp \left( \frac{1}{2} (\gamma + \psi) \right), \quad ... \] (2.27)
\[ F_{23} = \frac{\varepsilon_0}{\varepsilon} \frac{n+4}{\varepsilon} \sin \theta = 0 \] (2.28)
with
\[ \theta = (n + 2) \varepsilon_0 \frac{\mu^{n-1}}{\varepsilon}, \quad (2.29) \]
where \( \varepsilon_0, \varepsilon, n \) are constants.

We readily have
\[ F_{41} = \varepsilon_0 \frac{n}{\varepsilon} \exp \left( \frac{1}{2} (\gamma + \psi) \right), \] (2.30)
The equations (2.10), (2.22), (2.27), (2.28), (2.30), (2.31) yield

\[ E_1^1 = \frac{H_0}{\kappa} \quad \text{by (2.32)}, \quad \ldots \]  \quad (2.34)

and

\[ e^{-\gamma} = 1 - K (A_0 - H_0), \quad \text{by (2.21)}, \]

\[ = B_0, \quad \text{say}. \quad \ldots \]  \quad (2.35a)

Here \( A_0, H_0 \) are arbitrary constants. Integration of equation (2.23) yields

\[ e^\psi = \left( \frac{1}{p} \right)^{1+1} + \frac{2}{p} \left( \frac{1-1}{2} \right)^2, \quad \ldots \]  \quad (2.35b)

where

\[ B_0 \kappa^2 = \frac{B_0 - K (A_0 - H_0)}{\kappa}, \quad \ldots \]  \quad (2.35c)

and \( \frac{1}{p}, \frac{1}{\kappa} \) are arbitrary constants of integration.

Remark 1: When \( H_0 = 0 \), solution (2.35) reduces to Kuchowicz solution (1966, solution III.1).
Case 11: The assumptions

\[ \mu = A_0, \text{ (constant), } n = 1, m = 0, \] (2.36)

yield the following gravitational potentials

\[ e^{-\gamma} = 1 - \frac{A_0 K \kappa^2}{3} - \frac{H_0 K \kappa^4}{5}, \ldots \] (2.37a)

\[ \psi = (P \sinh z) + (P \cosh z)^2 \] (2.37b)

where

\[ l^2 = \frac{11}{20} H_0 K, \] (2.37c)

\[ \sin z \left( \frac{Q_0}{5} \right)^{1/2} = \left( \frac{2}{5} + \frac{5}{6} \frac{A_0}{H_0} \right) \left( \frac{5}{H_0 K} + \frac{25A_0^2}{36H_0^2} \right)^{1/2} \] (2.37d)

and \( P, Q \) are arbitrary constants of integration.

3. GRAVITATIONAL POTENTIALS OF A COSMOLOGICAL CONSTANT CHARGED FLUID SPHERE

Characteristic System: A spherically symmetric space-time is a 4-dimensional Riemannian space having the following properties (Takeno, 1966)

\[ R_{abcd} = \frac{1}{\kappa} A_a A_b B_c B_d + \frac{2}{\kappa} \varepsilon_{[a \mid c \mid b \mid d]} A_d + \frac{2}{\kappa} \varepsilon_{[a \mid c \mid b \mid d]} B_c A_d + \frac{2}{\kappa} \varepsilon_{[a \mid c \mid b \mid d]} B_c B_d + \frac{2}{\kappa} \varepsilon_{[a \mid c \mid b \mid d]} A_d B_c, \] (3.1)

where

\[ Q_{ab} = A_{(a \mid b)}, \] (3.2)

and vectors \( A_a, B_a \) satisfy

\[ A_a A^a = -B_a B^a = -1, \] (3.3)

\[ A_a B^a = 0, \]
The scalars $X, X, X, X, Y, Y, Z, Z$ are determined by the equations (3.1) to (3.5).

II) One of the scalars $X, X, X, X$ and $R = R_{ab} g^{ab}$ is such that its gradient vector is a linear combination of $A_a$ and $B_a$.

III) $X, Z, \bar{Z}$ satisfy

$$\frac{1}{4} X - 2 (Z^2 - \bar{Z}^2) \neq 0, \quad \ldots (3.6)$$

The equation (3.1) yields

$$R_{ab} = - \frac{1}{4} (X + X + 6 X) A_{ab} + \frac{1}{4} (X + 2 X) A_a A_b - \frac{1}{4} (X + 2 X) B_a B_b - \frac{5}{2} X Q_{ab}, \quad \ldots (3.7)$$

$$R = - \frac{1}{2} (X + 3 X + 3 X + 12 X). \quad (3.8)$$

Integrability conditions of (3.4) and (3.5) are

$$Y, a = -(A^b Y, b) A_a + (B^b Y, b) B_a, \quad \ldots (3.9)$$
$$\bar{Y}, a = -(A^b \bar{Y}, b) A_a + (B^b \bar{Y}, b) B_a, \quad \ldots (3.10)$$
$$Z, a = -(A^b Z, b) A_a + (B^b Z, b) B_a, \quad \ldots (3.11)$$
$$\bar{Z}, a = -(A^b \bar{Z}, b) A_a + (B^b \bar{Z}, b) B_a, \quad \ldots (3.12)$$
Integrability conditions of (3.1) are\textit{Bianchi Identities} which are equivalent to the following equations,

\begin{align}
\frac{1}{4} \left( \frac{1}{x} + \frac{2}{x^2} + \frac{3}{x^3} + \frac{4}{x^4} \right) &= y^2 - y^2 + \frac{B^a y_3}{2} - A^a y_1, \quad (3.13) \\
\frac{1}{4} \left( \frac{2}{x} + \frac{4}{x^2} \right) &= y_z + z^2 + A^a z_1, \quad \ldots \quad (3.14) \\
\frac{1}{4} \left( \frac{3}{x} + \frac{4}{x^2} \right) &= \bar{y} z - z^2 - B^a z_2, \quad \ldots \quad (3.15) \\
\frac{1}{8} x^5 &= B^a z_3 - z (y - z), \quad \ldots \quad (3.16) \\
&= A^a z_3 + z (y + z), \quad \ldots \quad (3.16a)
\end{align}

Equations (3.9) to (3.21) constitute a set of differential equations subject to the condition (3.6) to be solved.

Finally for the metric
\[ ds^2 = -Ad \zeta^2 - B(d\phi^2 + \sin^2 \phi d\phi^2) + Ct^2, \quad (3.22) \]
where \( A, B, C \) are function of \( \zeta, t \), we have
Equation of state

\[ \rho + p = 0 \tag{3.29} \]

characterizes the cosmological constant Charged Fluid,

(McIntosh, 1970). Now in comoving co-ordinate system,

which implies in characteristic system

\[ \frac{5}{X} = 0, \quad \ldots \quad \ldots \tag{3.30} \]

we have for the metric (3.22), the non-vanishing components

dependent on coordinates 1, 2, 3, 4.

\[ R_1^1 = - \frac{1}{4} (X + 3X + 3X + 6X) = - \frac{K}{2} (p - \mu + 2 E_4^1), \tag{3.31} \]

\[ R_2^2 = R_3^3 = - (X + \frac{3}{2}X + 3X + 6X) = - \frac{K}{2} (p - \mu + 2 E_3^1), \tag{3.32} \]

\[ R_4^4 = - \frac{1}{4} (\frac{1}{X} + \frac{2}{2}X + 3\frac{3}{2}X + 6X) = - \frac{K}{2} (\mu + 3p + 2 E_4^4), \tag{3.33} \]
For the assumption that electromagnetic field is spherically symmetric, we have (Equation 2.22)

\[ E_1^1 = - E_2^2 = - E_3^3 = E_4^4. \]  

(3.34)

Equation of state (3.29) implies by (3.31), (3.33) that

\[ 2 \frac{x}{\gamma} = x. \]  

(3.35)

In view of the theorem I, of Section 2, we seek static solution, accordingly,

\[ \gamma = \bar{z} = 0, \text{ by (3.25), (3.28)}. \]  

(3.36)

We further assume

\[ B = r^2. \]  

(3.37)

Now using equations (3.23), (3.24), (3.36), (3.37) we obtain from (3.19), (3.20) respectively

\[ \frac{2}{\gamma} = \frac{3}{x} = \frac{1}{\gamma} + \frac{1}{\gamma} \int \frac{1}{x} \, d\tau, \]  

(3.38)

\[ \frac{4}{x} = \frac{2}{\gamma} + \int \frac{2}{x} \, d\tau, \]  

(3.39)

where \( P, P \) are constants of integration. Further on equating (3.14) (3.15), we get by virtue of (3.23), (3.24), (3.36), (3.37) that

\[ \frac{C_1'}{C} = - \frac{A'}{A}, \]  

(3.40)

i.e.,

\[ C = \frac{1}{A}, \text{ on integrating (3.40)}. \]  

(3.41)

by taking constant of integration equal to unity. Finally for the metric (3.22), equation (3.6) becomes (Takeno, 1968)

\[ \frac{4}{x} - 2 (z^2 - \bar{z}^2) = - \frac{2}{B}, \]  

(3.42)
which yields by virtue of (3.27), (3.37),

\[ \frac{4}{X} \gamma^2 = 2 (\gamma^{-1} - 1), \ldots \ldots \] (3.43)

i.e., \( A = (1 + \frac{4}{X} \gamma^2)^{-1} \) \ldots \ldots (3.43a)

Equation (3.13) serves as a consistency condition which is to be satisfied. Thus we have by (3.41), (3.43) the line element

\[ ds^2 = - (1 + \frac{X}{2} r^2)^{-1} d\gamma^2 = \gamma^2 (d\gamma^2 + \sin^2 \phi d\phi^2) + \] 
\[ + (1 + \frac{4}{X} \gamma^2) dt^2. \ldots \ldots \] (3.44)

This gives a class of gravitational potentials of a cosmological constant Charged Fluid sphere.

**Particular Cases:**

**Case I:** Let \( X = - \frac{4b}{\gamma^2} \), (b = constant) \ldots \ldots (3.45)

then \( X = \frac{2a}{\gamma} + \frac{4b}{\gamma^2} \) by (3.38), (p=2a) (3.46)

and \( \frac{4}{X} = - \frac{2a}{\gamma} + \frac{2b}{\gamma^2} \) by (3.39), (q = 0). (3.47)

The resultant metric (3.44) yields the gravitational potentials as

\[ g_{11} = -(1 - a \gamma - b)^{-1}, \quad g_{22} \sin^2 \phi = \gamma^2 \sin^2 \phi, \]
\[ g_{44} = 1 - a \gamma - b. \ldots \ldots \] (3.48)

**Remark 1:** We observe the following properties, corresponding to these gravitational potentials
i) \( p = \frac{1}{2K} \left( \frac{b^2}{\kappa^2} + \frac{2a}{\kappa} \right) \), by using (3.31), (3.32), (3.33).

ii) \( E_1^l = - p - \frac{a}{2K\kappa} \), by using (3.31), (3.32), (3.33).

iii) \( F_{14} = \frac{a^2}{2Ke\kappa^2} \left( 1 - a\kappa - b \right)^{1/2} \),

\[ F_{23} = 0, \]

\[ S = \frac{a^2}{4\kappa^2} - \frac{a^2}{\kappa^2 (a\kappa + b)} \],

by using Maxwell equation, viz., \( (E^a_b \sqrt{-g})_a = GE_{bm}u^m \).

iv) Solution is characterized by

\[ \frac{2}{X} \frac{3}{X} \frac{5}{X} = \frac{1}{X + 2} \frac{2}{X + 2} \frac{4}{X} = 0. \]

v) It is of Class \( S_a \) in Takeno's sense.

vi) Eigen values of \( R_{ab} \) are

\( \left( X_1, X_1, X_2, X_2 \right) \),

where

\[ X_1 = - \frac{1}{2} \left( X + \frac{3}{X} \right) = \frac{2a}{\kappa} + \frac{b}{\kappa^2}, \]

\[ X_2 = \frac{1}{4} \left( X + \frac{4}{X} + \frac{6}{X} \right) = \frac{a}{\kappa}. \]

vii) Lorentz force is

\[ GE_a = \left( \frac{a}{2K\kappa^2}, 0, 0, 0 \right). \]

viii) Gravitational tidal force \( R_{abcd}u^b u^d = B_{ac} \) has only two surviving components, viz.,

\[ B_2^2 = B_3^3 = \frac{1}{4} \left( X + \frac{2}{X} \right) = - \frac{a}{2\kappa}. \]
ix) When \( a = b = 0 \), solution reduces to that of flat space-time.

**Case II:** Let \( \frac{1}{r} = - \frac{24m}{\kappa^3} + \frac{24H_0K}{\kappa^3} \),

where \( m \) and \( H_0 \) are constants.

Then \( \frac{2}{3} \left( \frac{3}{2} \right) ^{-1} = \frac{12m}{\kappa^3} + \frac{8H_0K}{\kappa^4} \), by (3.38), \( \left( \frac{1}{p} = 0 \right) \),

and \( \frac{4}{3} \left( \frac{3}{2} \right) ^{-1} = - \frac{4m}{\kappa^3} + \frac{2KH_0}{\kappa^4} - \frac{2KA_0}{3} \), by (3.39) \( \left( \frac{1}{p} = - \frac{2KA_0}{3} \right) \),

where \( A_0 \) and \( H_0 \) are constants.

Then space-time is characterized by the line element

\[
ds^2 = - A^{-1} d\zeta^2 - \kappa^2 (d\theta^2 + \sin^2 \theta d\varphi^2) + A dt^2,
\]

where

\[
A = 1 - \frac{2m}{\kappa} - \frac{A_0^2}{3} + \frac{H_0K}{\kappa^2}.
\]

**Remark:** We observe the following properties:

1) \( \mu = - \rho = A_0 = \) constant.

ii) \( F_{14}^1 = \frac{H_0}{\kappa^4} \), \( \left( \theta_0 = \text{constant} \right) \),

\[
F_{23} = \beta_0 \kappa^2 \sin \theta \left( \frac{\beta_0}{\kappa^2} = \text{constant} \right) \neq 0.
\]

\( H_0 = \frac{1}{2} \left( \theta_0^2 + \frac{\beta_0^2}{\kappa^2} \right) \),

\( G = 0 \).

iii) When \( A_0 = 0 \), we recover Das solution (1960) and obviously external Schwarzschild solution results when \( A_0 = H_0 = 0 \).
4. GRAVITATIONAL POTENTIALS OF A CHARGED FLUID WITH RADIATION:

We consider the Einstein-Rosen metric, viz.,

\[ ds^2 = e^{\gamma-\psi} (dt^2 - dr^2) - \frac{\kappa^2}{r^2} e^{2\psi}d\theta^2 - e^{\psi}dz^2, \]  

(4.1)

where \( \gamma, \psi \) are functions of \( r, t \). The non-vanishing components of stress-tensor \( T^a_b \) for (4.1) are,

\[ - KT_1^1 = KT_4^1 = \frac{1}{2} e^{\psi-\gamma} \left( - \frac{\psi'^2}{2} + \frac{\psi}{r} \right), \] ...

(4.2)

\[ - KT_2^2 = - \frac{1}{2} e^{\psi-\gamma} \left( - \gamma'' + \frac{\gamma}{r} - \frac{\psi'^2}{2} + \frac{\psi}{r} \right), \] ...

(4.3)

\[ - KT_3^3 = - \frac{1}{2} e^{\psi-\gamma} \left( - \gamma'' + \frac{\gamma}{r} - \frac{\psi'^2}{2} + 2\gamma'' - 2\psi + \frac{2\psi}{r} \right), \]

(4.4)

\[ - KT_4^1 = KT_4^4 = - \frac{1}{2} e^{\psi-\gamma} \left( \psi' - \frac{\psi}{r} \right). \] ...

(4.5)

**Theorem II:** In cylindrically symmetric space-times, the electromagnetic field of a Charged Fluid distribution reduces to an electromagnetic wrench when

\[ u^a = (0, 0, 0, 0, u^4). \] ...

(4.6)

**Proof:** The relations \( T_1^2 = T_2^3 = T_4^4 = T_3^3 = 0 \), by virtue of (4.1) and (4.6) yield two cases (Radhakrishna, 1963),

1) \( F_{14} = F_{23} = 0 \), \( F_{12}F_{13} - F_{24}F_{34} = 0 \),

(4.7)

11) \( F_{12} = F_{13} = F_{24} = F_{34} = 0 \), ...

(4.8)
When $F_{14} = 0$, the Maxwell equations $F_{ab}^{\gamma \beta} = \delta u^a$, imply $\delta = 0$, for $u^a \neq 0$. Thus this case corresponds to an
electrovac universe, i.e., Case (i) is incompatible with
a Charged Fluid. Case (ii) corresponds to an electromagentic wrench as $F_{14}$ and $F_{23}$ are the only non-vanishing components.

**Solution of Maxwell equations**: For the Charged Fluid
we have to consider only Case (ii) i.e., (4.8). The
following choice of $F_{14}$ and $F_{23}$ satisfies the Maxwell
equations, in comoving co-ordinate system

\[ F_{14} = e^{-\phi} \frac{d}{dt}, \quad \ldots \]  
\[ F_{23} = \beta_0 = \text{constant}, \quad \ldots \]  

where $f(t)$ is arbitrary function and the charge density $\delta$
is given by

\[(\gamma f)' = \delta \frac{\rho}{c} \exp \frac{1}{2} (\gamma - \phi). \]

Equations (4.9), (4.10) yield

\[ E_1 = - E_2 = - E_3 = E_4 = \frac{1}{2} \beta_0^2 (\beta^2 + \beta_0^2). \]

**Set of Equations for Integration**: Equations (4.2) to
(4.5), in comoving co-ordinate system for the Charged Fluid
can be reduced to the following set of equations:

\[ \psi'' - \ddot{\psi} + \frac{\beta^2}{\gamma} = 0, \quad \ldots \]  
\[ \gamma \psi'' - \gamma = 0. \quad \ldots \]
Equations (4.15), (4.16), (4.17) yield
\[ p - \mu = \frac{2E^1}{\lambda}, \quad \ldots \quad (4.18) \]
Equation of continuity is written as
\[ \frac{\partial}{\partial \tau} \left[ \left( \mu + P \right) u^a \right] + a_{,a} u^a p, a = 0. \quad \ldots \quad (4.19) \]

**Remark:** Non-existence of Zeldovich Charged Fluid (\( \mu = p \)) is established by (4.18), since \( \mu = p \Rightarrow E^a_b = 0 \).

**An Exact Solution:** As the simultaneous equations for integration are non-linear, we make certain simplifying assumptions. Let
\[ \psi (\kappa, t) = f_1(\kappa) + f_2(t). \quad \ldots \quad (4.20) \]
Integrating (4.13) with the help of (4.20) we obtain
\[ \psi = \frac{1}{4} \kappa^2 + 1 \log m \kappa + \frac{h}{2} t^2 + nt, \quad (4.21) \]
where \( h, l, m, n \) are constants of integration. Choosing
\[ l = 1, \quad h = 0, \quad \ldots \quad (4.22) \]
we obtain, on integrating (4.14)
\[ \gamma = nt + f_3(\kappa). \quad \ldots \quad (4.23) \]
Further equations (4.18), (4.19) imply that \( \mu + P \) is a function of \( \kappa \) only. Now we assume
\[ 2(\mu + P) = \exp. (\psi - \gamma + 2F(\kappa)), \quad (4.24) \]
where \( F ( r ) \) is a function of \( r \). Then integrating equation (4.17) with the help of (4.21), (4.23), (4.29) we get

\[
\gamma = nt + \log b \sqrt{r} + \frac{n^2 r^2}{4} + \frac{K}{2} \int r e^{2F} d\lambda,
\]

where \( b \) is constant of integration. The substitution

\[
\Omega ( r ) = \int r e^{2F} d\lambda,
\]

gives the following gravitational potentials:

\[
\begin{align*}
\sigma_{11} &= \frac{1}{m} \exp \left( \frac{n^2 r^2}{4} + \frac{K}{2} \Omega \right) = \sigma_{44}, \\
\sigma_{22} &= -\frac{r}{m} \exp (-nt), \\
\sigma_{33} &= -m r \exp (nt),
\end{align*}
\]

Remark: For the set of gravitational potentials (4.27) we observe that

1) \( p = \frac{1}{4} \frac{m \frac{r^2}{m^2}}{b} (1 + F', r) \exp \left( -\frac{n^2 r^2}{4} - \frac{K}{2} + 2F \right) \).

2) \( \mu = \frac{m \frac{r^2}{m^2}}{4b} (1 - F', r) \exp \left( -\frac{n^2 r^2}{4} - \frac{K}{2} + 2F \right) \).

3) \( F_{14} = \frac{m \frac{r^2}{m^2}}{b} \left( \frac{m}{4} F', \frac{r^2}{2} \exp \left( -\frac{n^2 r^2}{4} - \frac{K}{2} + 2F \right) - \right) \frac{1}{2} \times \exp \left( -\frac{n^2 r^2}{4} - \frac{K}{2} \right), \)

\[
F_{23} = \beta_0,
\]

\[
E_i = \frac{K m \frac{r^2}{m^2}}{4b} F' \exp \left( -\frac{n^2 r^2}{4} - \frac{K}{2} + 2F \right),
\]

... (4.32)
Particular Cases:

Case-A: A Disordered Radiation Field: The assumption

\[ 2F = \log \frac{P}{\epsilon}, \quad (P = \text{constant}), \quad \ldots \]  

(4.33)
yields

\[ \Omega = \frac{P}{\epsilon}, \quad \text{by (4.26)}, \quad \ldots \]  

(4.34)
and

\[ \mu = 3p, \quad \text{by (4.28) (4.29)}, \]  

(4.35)

which is the equation of state, characterizing radiation field (Tolman, 1934). Thus gravitational potentials for a Charged Fluid filled with disordered radiation are, (by 4.27)

\[ -g_{11} = \frac{b}{m} \frac{\lambda}{\epsilon} \frac{1}{2} \exp \left( \frac{n^2 \lambda^2}{4} + \frac{K \lambda}{2} \right) = g_{44}, \quad \ldots \]  

(4.36a)

\[ g_{22} = -\frac{\lambda}{m} \exp \left( -\mu t \right), \quad \ldots \]  

(4.36b)

\[ g_{33} = -m \lambda \exp \left( \mu t \right). \]

Case-B: A Curvature Fluid: The assumption

\[ F = -2 \log P \epsilon, \quad (P = \text{constant}), \quad \ldots \]  

(4.37)

entails

\[ \Omega = -\frac{1}{2} \frac{1}{\epsilon p^4 \lambda^2}, \quad \ldots \]  

(4.38)
and

\[ \mu + 3p = 0, \quad \ldots \]  

(4.39)

which characterizes a curvature fluid (McIntosh, 1972).

We thus have the gravitational potentials for the
charged curvature fluid as

\[- \varepsilon_{11} = \frac{b}{m} \frac{1}{2} \exp\left(\frac{n^2 \chi^2}{4} - \frac{K^4 \chi^{-2}}{4}\right) = \varepsilon_{44}\] (4.40a)

\[\varepsilon_{22} = - \frac{\chi}{m} \exp\left(\chi - nt\right), \quad \ldots \] (4.40b)

\[\varepsilon_{33} = - m \exp\left(nt\right), \quad \ldots \] (4.40c)
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