Chapter 2
Cardinality, ideals and separation axioms

2.1 Introduction

Since Levine introduced the concept of semi-open sets in his article [14], there are a number of generalizations for separation axioms in terms of generalized open sets and generalized closed sets through closure operator and interior operator. One may realize these operators as addition and deletion of sets in the ideal of nowhere dense subsets (see:[15, 16]). So, a number of generalizations for separation axioms in terms of members of an ideal in a topological space are also available in literature.

The first aim of this chapter is to develop a new method of introducing separation axioms; called pseudo axioms. This is achieved by replacing single elements by singleton sets; when singleton sets are closed.

The second aim is to develop a new method of introducing cardinal numbers in separation axioms. This is achieved by replacing separating open sets by separating intersections of open sets; when intersections depend on cardinalities.

The third aim is to explain extension of these methods in different directions.
This is achieved by means of applying ideal concepts in topological spaces.

2.2 Pseudo-axioms

Definition 2.2.1 A topological space \((X, \tau)\) is said to be pseudo-Hausdorff, if for any two distinct points \(x\) and \(y\) in \(X\) such that \(\{x\}\) and \(\{y\}\) are closed, there are two disjoint open sets \(U\) and \(V\) such that \(x \in U\) and \(y \in V\).

Example 2.2.2 Let \(X = \{1, 2, 3\}\) and \(\tau = \{\phi, X, \{3\}\}\) then \((X, \tau)\) is not Hausdorff, but it is pseudo-Hausdorff.

Definition 2.2.3 A topological space \((X, \tau)\) is said to be pseudo-regular, if for any given closed subset \(A\) of \(X\) and a point \(x\) not in \(A\) such that \(\{x\}\) is closed, there are disjoint open subsets \(U\) and \(V\) of \(X\) such that \(A \subseteq U\) and \(x \in V\).

Example 2.2.4 The topological space \((X, \tau)\) given in Example 2.2.2 is a pseudo-regular space, whereas it is not a regular space.

Example 2.2.5 Let \(X = \{1, 2, 3, 4, 5\}\) and \(\tau = \{\phi, X, \{3, 4, 5\}, \{3, 4\}, \{1, 2, 3, 4\}\}\). Then \((X, \tau)\) is pseudo-Hausdorff, but it is not pseudo-regular (For, consider the closed sets \(\{1, 2\}\) and \(\{5\}\)).

Example 2.2.6 Let \(X = \{1, 2, 3, 4, 5\}\) and \(\tau_X = \{\phi, X, \{3, 4, 5\}, \{1, 2, 3\}, \{3\}\}\). Then \((X, \tau_X)\) is pseudo-Hausdorff and pseudo-regular.

Let \(Y = \{2, 3, 4\}\).

Let \(\tau_Y = \{\phi, Y, \{3, 4\}, \{2, 3\}, \{3\}\}\) be the subspace topology on \(Y\) derived from \(\tau_X\).

Then \((Y, \tau_Y)\) is not pseudo-Hausdorff and not pseudo-regular.
Thus a subspace of a pseudo-Hausdorff (pseudo-regular) space need not be pseudo-Hausdorff (pseudo-regular). However, we have the following result.

**Proposition 2.2.7** A closed subspace of a pseudo-Hausdorff (pseudo-regular) space is pseudo-Hausdorff (pseudo-regular).

**Proof:** Let \((X, \tau)\) be a pseudo-Hausdorff space, and let \(A\) be a given closed subset of \(X\). Let \(\tau_A = \{A \cap U : U \in \tau\}\) be the subspace topology on \(A\).

Let \(x, y \in A\) be distinct points such that \(\{x\}\) and \(\{y\}\) are \(\tau_A\)-closed subsets of \(A\). Then \(\{x\}\) and \(\{y\}\) are \(\tau\)-closed subsets of \(X\), and \(x \neq y\).

Then there are disjoint \(\tau\)-open subsets \(U\) and \(V\) of \(X\) such that \(x \in U\) and \(y \in V\). Then \(U \cap A \in \tau_A\), \(V \cap A \in \tau_A\), \(x \in U \cap A\), \(y \in V \cap A\) and \((U \cap A) \cap (V \cap A) = \emptyset\).

This proves that \((A, \tau_A)\) is also pseudo-Hausdorff.

A similar proof is applicable for pseudo-regularity. ■

**Proposition 2.2.8** A product of pseudo-Hausdorff (pseudo-regular) spaces is pseudo-Hausdorff (pseudo-regular).

**Proof:** Let \(((X_i, \tau_i))_{i \in I}\) be a given family of topological spaces.

Let \(X = \prod_{i \in I} X_i\). Let \(\tau\) be the product topology on \(X\) induced by \((\tau_i)_{i \in I}\).

To each \(i \in I\), let \(P_i : X \to X_i\) denote the projection.

Suppose each \((X_i, \tau_i)\) is pseudo-Hausdorff.

Let \(x = (x_i)_{i \in I}\) and \(y = (y_i)_{i \in I}\) be two distinct points in \(X\) such that \(\{x\}\) and \(\{y\}\) are \(\tau\)-closed. Then \(x_j \neq y_j\) for some \(j\).

Since \(\{x_j\}\) and \(\{y_j\}\) are \(\tau_j\)-closed, there are disjoint \(\tau_j\)-open subsets \(U_j\) and \(V_j\) of \(X_j\) such that \(x_j \in U_j\) and \(y_j \in V_j\).
Then $P_j^{-1}(U_j)$ and $P_j^{-1}(V_j)$ are two disjoint $\tau$-open subsets of $X$ containing $x$ and $y$ respectively. Thus $(X, \tau)$ is pseudo-Hausdorff.

Suppose each $(X_i, \tau_i)$ is pseudo-regular.

Let $A$ be a non-empty $\tau$-closed subset of $X$.

Let $x = (x_i)_{i \in I}$ be a point in $X$ such that $x \notin A$ and $\{x\}$ is $\tau$-closed.

Then there are $j_1, j_2, ... j_n \in I$ such that $x_{j_k} \in U_{j_k}$ for some $U_{j_k} \in \tau_{j_k}$, $k = 1, 2, 3, ... n$ and such that $A \subseteq X \setminus \bigcap_{k=1}^{n} P_{j_k}^{-1}(U_{j_k})$.

For each $k$, there are disjoint $\tau_{j_k}$ open sets $V_{j_k}$ and $W_{j_k}$ such that $x_{j_k} \in V_{j_k}$ and $X_{j_k} \setminus U_{j_k} \subseteq W_{j_k}$.

Thus

$$V = \bigcap_{k=1}^{n} P_{j_k}^{-1}(V_{j_k}) \text{ and } W = \bigcup_{k=1}^{n} P_{j_k}^{-1}(W_{j_k})$$

are disjoint open subsets of $X$ such that $x \in V$ and $A \subseteq W$.

This proves that $(X, \tau)$ is also pseudo-regular. $\blacksquare$

All these concepts and results can be extended to ideal set up.

**Definition 2.2.9** Let $(X, \tau)$ be a topological space with an ideal $\mathcal{I}$.

(a) $(X, \tau)$ is said to be $\mathcal{I}$-pseudo Hausdorff if for any two distinct points $x$ and $y$ in $X$ such that $\{x\}$ and $\{y\}$ are closed, there are two $\tau$-open sets $U$ and $V$ such that $x \in U$, $y \in V$ and $U \cap V \in \mathcal{I}$

(b) $(X, \tau)$ is said to be $\mathcal{I}$-pseudo regular if for any given non-empty closed subset $A$ of $X$ and a point $x \notin A$ such that $\{x\}$ is closed, there are two $\tau$-open sets $U$ and $V$ such that $x \in U$, $A \subseteq V$ and $U \cap V \in \mathcal{I}$. 

11
If \((X, \tau)\) is a topological space with an ideal \(\mathcal{I}\), let us consider a given subset \(Y\) of \(X\) with the ideal \(\{Y \cap A : A \in \mathcal{I}\}\) when \(Y\) is considered as a topological subspace. If this convention is followed then we have the following result.

**Proposition 2.2.10** A closed subspace \(Y\) of an \(\mathcal{I}\)-pseudo Hausdorff (\(\mathcal{I}\)-pseudo regular) space \(X\) is \(\mathcal{I}\)-pseudo Hausdorff (\(\mathcal{I}\)-pseudo regular) when \(\mathcal{I} = \{Y \cap A : A \in \mathcal{I}\}\).

**Proposition 2.2.11** Let \(((X_i, \tau_i, \mathcal{I}_i))_{i \in I}\) be a collection such that each \((X_i, \tau_i)\) is \(\mathcal{I}_i\)-pseudo Hausdorff (\(\mathcal{I}_i\)-pseudo regular). Let \(X = \prod_{i \in I} X_i\). Let \(\tau\) be the product topology on \(X\) induced by \((\tau_i)_{i \in I}\). Let \(\mathcal{I}\) be any ideal on \(X\) such that \(\mathcal{I}\) contains all members of the form \(P_i^{-1}(A_i)\), where \(i \in I\), \(A_i \in \mathcal{I}_i\), and \(P_i : X \to X_i\) is the projection. Then \((X, \tau)\) is \(\mathcal{I}\)-pseudo Hausdorff (\(\mathcal{I}\)-pseudo regular).

The proofs for propositions 2.2.10 and 2.2.11 follow from the classical arguments applied in the proofs of propositions 2.2.7 and 2.2.8. Note that all sets of the form \(P_i^{-1}(A_i)\), with \(i \in I\) and \(A_i \in \mathcal{I}_i\), form an ideal in \(X\).

### 2.3 Cardinality and separation axioms

Let \(\aleph\) denote a cardinal number.

**Definition 2.3.1** A subset \(U\) of a non-empty set \(X\) is said to be an intersection (or union) of \(\aleph\)-number of subsets \((A_i)_{i \in I}\) of \(X\), if the cardinality of \(I\) is less than or equal to \(\aleph\), and if \(U = \bigcap_{i \in I} A_i\) (or \(U = \bigcup_{i \in I} A_i\)). In particular, a subset \(A\) of a topological space is a \(G_\delta\) (or \(F_\sigma\)) set if and only if it is an intersection (or union) of \(\aleph_0\)-number of open (or closed) subsets of \(X\).
Definition 2.3.2 A topological space \((X, \tau)\) is said to be \(\aleph\)-pseudo regular if for a given non-empty closed subset \(A\) of \(X\) and for a given point \(x \notin A\) such that \(\{x\}\) is closed, there are disjoint subsets \(U\) and \(V\) of \(X\) such that \(x \in U\), \(A \subseteq V\), \(U\) is an intersection of \(\aleph\)-number of \(\tau\)-open subsets of \(X\), and \(V\) is a \(\tau\)-open subset of \(X\).

Proposition 2.3.3 Every Hausdorff Lindeloff space is \(\aleph_0\)-pseudo regular.

Proof: Let \(A\) be a given non-empty closed subset of a given Hausdorff Lindeloff space \((X, \tau)\). Let \(y\) be a point in \(X\) such that \(y \notin A\).

To each \(x \in A\), there are disjoint open sets \(U_x\) and \(V_x\) such that \(x \in U_x\) and \(y \in V_x\).

Since \(X\) is Lindeloff, there is a sequence \(x_1, x_2, \ldots\) in \(A\) such that \(A \subseteq \bigcup_{i=1}^{\infty} U_{x_i}\). Write

\[
X_y = \bigcup_{i=1}^{\infty} U_{x_i} \quad \text{and} \quad W_y = \bigcap_{i=1}^{\infty} V_{x_i}.
\]

Then \(X_y\) is open and \(W_y\) is a \(G_\delta\) set such that \(X_y \cap W_y = \emptyset\), \(A \subseteq X_y\) and \(y \in W_y\).

This proves that \((X, \tau)\) is \(\aleph_0\)-pseudo regular.

Remark 2.3.4 If \((X, \tau)\) is a Hausdorff space such that every open covering of \(X\) has a subcovering consisting of \(\aleph\)-number of members, then for a given non-empty closed subset \(A\) of \(X\) and for any point \(x \notin A\) (but in \(X\)), there are disjoint subsets \(U\) and \(V\) of \(X\) such that \(A \subseteq U\), \(U\) is open, \(x \in V\), and \(V\) is an intersection of \(\aleph\)-number of open subsets of \(X\).

Proof: It follows from the arguments applied in the proof of the previous proposition 2.3.3.

Example 2.3.5 Let \(X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}\).

Let \(\tau = \{\emptyset, X\} \cup \{\{0, \frac{1}{n}, \frac{1}{n+1}, \ldots\}, \{\frac{1}{n}, \frac{1}{n+1}, \ldots\} : n = 3, 4, 5, \ldots\} \cup \{\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}\}.\)
Then $(X, \tau)$ is not pseudo-regular in view of the closed sets \(\{1, \frac{1}{2}\}\) and \(\{0\}\) without disjoint open sets containing them separately.

However $(X, \tau)$ is $\aleph_0$-pseudo regular.

So, $\aleph$-pseudo regularity need not imply pseudo regularity. However the converse is always true:

pseudo regularity implies $\aleph$-pseudo regularity

**Proposition 2.3.6** Every closed subspace of an $\aleph$-pseudo regular space is $\aleph$-pseudo regular.

**Proposition 2.3.7** A product of $\aleph$-pseudo regular spaces is an $\aleph$-pseudo regular space.

More precisely we can have the following proposition.

**Proposition 2.3.8** Let $((X_i, \tau_i))_{i \in I}$ be a given collection of topological spaces. To each $i \in I$, suppose $(X_i, \tau_i)$ is $\aleph_i$-pseudo regular, for some cardinal number $\aleph_i$. If $\aleph = \sup_{i \in I} \aleph_i$, then the product of $((X_i, \tau_i))_{i \in I}$ is $\aleph$-pseudo regular.

For a proof of proposition 2.3.6, we refer to the proof of proposition 2.2.7. For proofs of the propositions 2.3.7 and 2.3.8 we refer to the arguments given in the proof of proposition 2.2.8. We shall not present proofs for subsequent results, because earlier arguments are applicable to them.

Let us recall that a topological space $(X, \tau)$ with an ideal $\mathcal{I}$ is said to be $\mathcal{I}$-modulo Lindelöf, if for every open covering $\mathcal{U}$ of $X$, there is a countable sub-collection $\{A_1, A_2, \ldots\}$ of $\mathcal{U}$ such that $X \setminus \bigcup_{i=1}^{\infty} A_i \in \mathcal{I}$.
Definition 2.3.9 Let \((X, \tau)\) be a topological space with an ideal \(\mathcal{I}\) of subsets of \(X\). Let \((X, \tau)\) is said to be \(\mathcal{I}\)-modulo \(\aleph\)-pseudo regular if for a given non-empty closed subset \(A\) of \(X\) and for a given point \(x \notin A\) such that \(\{x\}\) is closed, there are disjoint subsets \(U\) and \(V\) of \(X\) such that \(x \in U\), \(V \setminus A \in \mathcal{I}\), \(U\) is an intersection of \(\aleph\)-number of \(\tau\)-open subsets of \(X\), and \(V\) is a \(\tau\)-open subset of \(X\).

Proposition 2.3.10 Every Hausdorff \(\mathcal{I}\)-modulo Lindeloff space is \(\mathcal{I}\)-modulo \(\aleph_0\)-pseudo regular.

We do have more:

Remark 2.3.11 Suppose \((X, \tau)\) is a Hausdorff space with an ideal \(\mathcal{I}\). Suppose every open covering \(\mathcal{U}\) of \(X\) has a subcollection \(\mathcal{V}\) consisting of \(\aleph\)-number of members such that \(X \setminus \bigcup \{B : B \in \mathcal{V}\} \in \mathcal{I}\). Then for a given non-empty closed subset \(A\) of \(X\) and for any point \(x \notin A\) (but in \(X\) ), there are disjoint subsets \(U\) and \(V\) of \(X\) such that \(x \in V\), \(V\) is an intersection of \(\aleph\)-number of open subsets of \(X\), \(U\) is open and \(U \setminus A \in \mathcal{I}\).

Proposition 2.3.12 A closed subspace \(Y\) of an \(\mathcal{I}\)-modulo \(\aleph\)-pseudo regular space \(X\) is \(\mathcal{I}\)-modulo \(\aleph\)-pseudo regular when \(\mathcal{I} = \{Y \cap A : A \in \mathcal{I}\}\).

Proposition 2.3.13 Let \((\langle X_i, \tau_i, \mathcal{I}_i \rangle)_{i \in I}\) be a collection such that each \((X_i, \tau_i)\) is \(\mathcal{I}_i\)-modulo \(\aleph_i\)-pseudo regular, where \(\aleph_i\)-is a given cardinal number. Let \(\aleph = \sup_{i \in I} \aleph_i\).

Let \(X = \prod_{i \in I} X_i\). Let \(\tau\) be the product topology on \(X\) induced by \((\tau_i)_{i \in I}\). Let \(\mathcal{I}\) be any ideal on \(X\) such that \(\mathcal{I}\)-contains all members of the form \(P_i^{-1}(A_i)\), where \(i \in I\), \(A_i \in \mathcal{I}_i\), and \(P_i : X \to X_i\) is the projection. Then \((X, \tau)\) is \(\mathcal{I}\)-modulo \(\aleph\)-pseudo regular.
Remark 2.3.14 The notation $\aleph$ is used to denote an infinite cardinal number. However, if $\aleph$ is considered as a finite cardinal number, then we remove $\aleph$ in definition 2.3.9 (and in definition 2.3.2). In this case we can remove $\aleph$ from proposition 2.3.12 (and proposition 2.3.6). If each $\aleph_i$ is considered as a finite cardinal number in proposition 2.3.13, then we can consider them as 1 and hence we can take $\aleph = 1$, and so we can remove $\aleph_i$ and $\aleph$ from proposition 2.3.13 (and from proposition 2.3.8).

Definition 2.3.15 Let $(X, \tau)$ be a topological space with an ideal $\mathcal{I}$ of subsets of $X$. $(X, \tau)$ is said to be $\mathcal{I}$-$\aleph$-pseudo regular if for a given non-empty closed subset $A$ of $X$ and for a given point $x \notin A$ such that $\{x\}$ is closed, there are subsets $U$ and $V$ of $X$ such that $x \in U$, $A \subseteq U$, $U$ is an intersection of $\aleph$-number of $\tau$-open subsets of $X$, $V$ is a $\tau$-open subset of $X$, and $U \cap V \in \mathcal{I}$.

Let us recall that a topological space $(X, \tau)$ with an ideal $\mathcal{I}$ is said to be $\mathcal{I}$-modulo Hausdorff, if for any two distinct points $x, y \in X$ there are two open subsets $U$ and $V$ of $X$ such that $x \in U$, $y \in V$ and $U \cap V \in \mathcal{I}$.

Then we have the following facts:

Proposition 2.3.16 Every $\mathcal{I}$-modulo Hausdorff space, which is also Lindeloff, is $\mathcal{I}$-$\aleph_0$-pseudo regular, provided $\mathcal{I}$ is a $\sigma$-ideal (that is, $\mathcal{I}$ is closed under countable unions).

Remark 2.3.17 Suppose $(X, \tau)$ is an $\mathcal{I}$-modulo Hausdorff space with respect to an ideal $\mathcal{I}$ in $X$. Suppose that the ideal is closed under unions of $\aleph$-number of members of $\mathcal{I}$. Suppose every open covering $\mathcal{U}$ of $X$ has a subcover $\mathcal{V}$ consisting
of $ℵ$-number of members. Then for a given non-empty closed subset $A$ of $X$ and for any point $x \notin A$ (but in $X$), there are subsets $U$ and $V$ of $X$ such that $x \in V$, $V$ is an intersection of $ℵ$-number of open subsets of $X$, $U$ is open and $U \cap V \in \mathcal{I}$.

**Proposition 2.3.18** Every closed subspace $Y$ of an $\mathcal{I}$-$ℵ$-pseudo regular space $X$ is $\mathcal{J}$-$ℵ$-pseudo regular when $\mathcal{J} = \{Y \cap A : A \in \mathcal{I}\}$.

**Proposition 2.3.19** Let $((X_i, \tau_i, \mathcal{I}_i))_{i \in I}$ be a collection such that each $(X_i, \tau_i)$ is $\mathcal{I}_i$-$ℵ_i$-pseudo regular, where $ℵ_i$ is a given cardinal number. Let $ℵ = \sup_{i \in I} ℵ_i$. Let $X = \prod_{i \in I} X_i$. Define $τ, X$ and $\mathcal{I}$ as in proposition 2.3.13. Then $(X, τ)$ is $\mathcal{I}$-$ℵ$-pseudo regular.

**Remark 2.3.20** If $ℵ$ in definition 2.3.15 and proposition 2.3.18 and if $ℵ_i$ in proposition 2.3.19 are finite cardinal numbers, then $ℵ$ and $ℵ_i$ can be removed from definition 2.3.15, proposition 2.3.18 and proposition 2.3.19.

### 2.4 Concluding remark

The second section provides a method to introduce pseudo axioms corresponding to known separation axioms which involve closed singleton sets for single elements. For example, this method gives an idea to define a pseudo axiom for completely regular spaces. One may replace open sets by generalized open sets in extending these definitions. Then these axioms may be extended to topological spaces with ideals and to bitopological spaces. The third section provides a method to replace separating open sets in the axioms by separating intersections of open sets; when the number of members in the intersections may depend on some fixed cardinalities. In
the definition of \(\aleph\)-pseudo regularity, closed set is contained in a finite intersection of open sets and closed singleton set is contained in an intersection of \(\aleph\)-number of open sets. Thus different entities may be separated by intersections by different numbers of open sets, when new separation axioms are introduced. Then axioms may be extended to topological spaces with ideals and to bitopological spaces, and they may be extended by replacing open sets by generalized open sets. All these generalizations may be done when their applications are required.