Chapter 1

Preliminaries

1.1 Introduction

In many recent literature [17, 21, 51, 57, 61, 62, 65, 75], researchers have discussed fuzzy automaton algebraically. It seems that the main motto behind them was to discuss algebraic properties of fuzzy automata in relation to their structure (i.e. fuzzy transition function) [21, 52, 68, 65, 99]. Structure preserving fuzzy transition functions - isomorphisms in general and homomorphisms in particular - plays crucial role for equivalent and reduction of a fuzzy automaton [51, 52, 58, 61, 62, 65, 75]. Three different approaches of the study of (fuzzy) automata motivated the present work of the thesis. The first approach of study is the kind of generalization of fuzzy automata over finite group by Das [21] and few followers [1, 17, 57]. The second approach is the study of various researchers in classical automata by the use of properties of crisp semigroups and groups [6, 7, 8, 11, 27, 28, 29, 31, 52, 74, 97, 100]. The third approach is the study of fuzzy automata by the topology induced by successor as a Kuratowski operator by Tiwari and Srivastava [90] and many following researchers [11, 75].

In first three sections we have recalled concepts in fuzzy group theory, automata theory and fuzzy automata theory. We have also introduced new concepts
for fuzzy automata theory in the last section.

1.2 Concepts in fuzzy group theory

In this section we recall few concepts from fuzzy group theory, ranging over 8-10 research papers, which we feel as basis building blocks for the rest of this thesis.

Definition 1.2.1. [64, 108] A fuzzy set of set \( X \) is a function from \( X \) into \([0, 1]\). The set of all fuzzy sets of \( X \) is called the fuzzy power set of \( X \) and is denoted by \( \mathcal{FP}(X) \).

Definition 1.2.2. [64] Let \( \alpha \in \mathcal{FP}(X) \). Then the set \( \{ \alpha(x) | x \in X \} \) is called the image of \( \alpha \) and is denoted by \( \alpha(X) \) or \( \text{Im}(\alpha) \). The set \( \{ x \in X | \alpha(x) \geq t \} \) is called the support of \( \alpha \) and is generally denoted by \( \text{Supp}(\alpha) \). For any \( t \in [0, 1] \), the set \( \{ x \in X | \alpha(x) \geq t \} \) is called the \( t \)-cut (level set) of \( \alpha \) and is denoted by \( \alpha_t \). Further, \( L_\alpha \) will represent the family \( \{ \alpha_t | t \in \text{Im}(\alpha) \} \) and \( L^*_\alpha \) by \( \{ \alpha_t | t \in \text{Im}(\alpha) - \{0\} \} \).

Definition 1.2.3. [64] Let \( \alpha, \beta \in \mathcal{FP}(X) \). If \( \alpha(x) \leq \beta(x) \), for all \( x \in X \), then \( \alpha \) is said to be contained in \( \beta \) and we write it as \( \alpha \subseteq \beta \). If \( \alpha \subseteq \beta \) and \( \alpha \neq \beta \), then \( \alpha \) is said to be properly contained in \( \beta \) and we write it as \( \alpha \subset \beta \).

Definition 1.2.4. [64] Let \( \alpha, \beta \in \mathcal{FP}(X) \). The union and the intersection \( \alpha \cup \beta \) and \( \alpha \cap \beta \in \mathcal{FP}(X) \) are respectively defined as \( (\alpha \cup \beta)(x) = \alpha(x) \lor \beta(x) \) and \( (\alpha \cap \beta)(x) = \alpha(x) \land \beta(x) \), for all \( x \in X \), where \( \land \) denotes the minimum and \( \lor \) denotes the maximum.

Theorem 1.2.5. [49] Let \( \alpha, \beta \in \mathcal{FP}(X) \). Then the following properties hold:

(i) \( t_1 \geq t_2 \) implies that \( \alpha_{t_1} \subseteq \alpha_{t_2} \), for \( t_1, t_2 \in [0, 1] \).

(ii) \( (\alpha \cap \beta)_t = \alpha_t \cap \beta_t \) and \( (\alpha \cup \beta)_t = \alpha_t \cup \beta_t \), for all \( t \in [0, 1] \).

(iii) \( \alpha \subseteq \beta \) iff \( \alpha_t \subseteq \beta_t \), for all \( t \in [0, 1] \).

(iv) \( \alpha_t = \cap_{s \leq t} \alpha_s \), for all \( t \in [0, 1] \).
We have the following decomposition theorems \[49\] for a given fuzzy set in terms of its level sets.

**Theorem 1.2.6.** \[49\] Let \( \alpha \in \mathcal{FP}(X) \). Then
\[
(i) \quad \alpha = \bigcup_{t \in [0,1]} \alpha^t, \text{ where } \alpha^t = t \cdot \chi_{\alpha, t}.
(ii) \quad \alpha = \bigcup_{t \in \text{Im}(\alpha)} \alpha^t.
\]

**Definition 1.2.7. (Extension principle)\[49\]** Let \( f : X \to Y \) be a function, \( \alpha \in \mathcal{FP}(X) \) and \( \beta \in \mathcal{FP}(Y) \). Define the fuzzy sets \( f(\alpha) \in \mathcal{FP}(Y) \) and \( f^{-1}(\beta) \in \mathcal{FP}(X) \) by
\[
f(\alpha)(y) = \begin{cases} 
\{ \alpha(x) \mid x \in X, f(x) = y \}, & \text{if } f^{-1}(y) \neq \phi, \\
0, & \text{otherwise.}
\end{cases}
\]
and \( f^{-1}(\beta)(x) = \beta(f(x)) \), for all \( x \in X \).

Generally \( f(\alpha) \) is called the image of \( \alpha \) under \( f \) and \( f^{-1}(\beta) \) is the inverse image of \( \beta \) under \( f \).

**Definition 1.2.8.** \[81\] Let \( S \) be a groupoid. A fuzzy set \( \alpha \) of \( S \) is called as **fuzzy groupoid** (or fuzzy semigroup) of \( S \), if for all \( x, y \in S \), \( \alpha(xy) \geq \alpha(x) \land \alpha(y) \). A fuzzy groupoid \( \alpha \) of a group \( G \) is called a **fuzzy group** of \( G \), if for all \( x \in G \), \( \alpha(x^{-1}) \geq \alpha(x) \). The set of all fuzzy groups of \( G \) will be denoted by \( \mathcal{F}(G) \).

**Definition 1.2.9.** A fuzzy group \( \alpha \) of \( G \) is said to be of an **isolated tip** if \( \alpha(x) < \alpha(e), \forall x \in G, x \neq e \).

**Theorem 1.2.10.** \[22\] If \( \alpha \in \mathcal{F}(G) \), then for any \( t \in [0,1] \) with \( \alpha(e) \geq t \), the \( t \)-cut \( \alpha_t \) is a subgroup of \( G \).

**Lemma 1.2.11.** \[71\] If \( \alpha \) is a fuzzy groupoid of a finite group \( G \), then \( \alpha \) is a fuzzy group.

**Lemma 1.2.12.** \[71\] Let \( \alpha \in \mathcal{F}(G) \) and \( x \in G \). Then
\[
\alpha(xy) = \alpha(y), \forall y \in G \iff \alpha(x) = \alpha(e).
\]
Lemma 1.2.13. [22] Let $G$ be a group and $\alpha$ be a fuzzy group of $G$. Two level subgroups $\alpha_{t_1}$ and $\alpha_{t_2}$ (with $t_1 < t_2$) of $\alpha$ are equal if and only if there is no $x \in G$ such that $t_1 \leq \alpha(x) < t_2$.

If $\alpha$ is a fuzzy group of $G$ and $Im(\alpha) = \{t_0, t_1, \ldots, t_n\}$ with $t_0 > t_1 > \ldots > t_n$, then the family of level subgroups of $G$ forms a chain $\alpha_{t_0} \subseteq \alpha_{t_1} \subseteq \ldots \subseteq \alpha_{t_n} = G$, where $\alpha(e) = t_0$.

Definition 1.2.14. [48] Let $\alpha$ be a fuzzy group of a group $G$. The smallest positive integer $n$ (if it exists) such that $\alpha(x^n) = \alpha(e)$ is called the fuzzy order of $x$ with respect to $\alpha$ and is denoted by $FO_\alpha(x)$. If for all $x \in G$, $FO_\alpha(x)$ is a power of a fixed prime number $p$, then we say that $\alpha$ is a fuzzy $p$-group of $G$.

Definition 1.2.15. [64] Let $\alpha$ and $\beta$ be fuzzy sets of $G$. (i) The product $\alpha \ast \beta$ of $\alpha$ and $\beta$ is defined by

$$(\alpha \ast \beta)(x) = \lor \{\alpha(y) \land \beta(z) | y, z \in G \text{ such that } x = y \ast z\}, \text{ for all } x \in G.$$  

(ii) The inverse $\alpha^{-1}$ of $\alpha$ is defined by $\alpha^{-1}(x) = \alpha(x^{-1})$, for all $x \in G$.

Definition 1.2.16. [70] Let $\alpha, \beta$ be two fuzzy groups of $G$. $\alpha$ is said conjugate to $\beta$, if for some $x \in G$, we have $\alpha(g) = \beta(x^{-1}gx)$, $\forall g \in G$.

Definition 1.2.17. [64] Let $\alpha \in \mathcal{F}(G)$. Then $\alpha$ is called a normal fuzzy group of $G$, if $\alpha(xy x^{-1}) \geq \alpha(y)$, for all $x, y \in G$. The set of all normal fuzzy groups of $G$ will be denoted by $\mathcal{FN}(G)$.

Lemma 1.2.18. [64] Let $\alpha \in \mathcal{FP}(G)$. Then following assertions are equivalent

(i) $\alpha \in \mathcal{FN}(G)$.

(ii) $\alpha(xy) = \alpha(yx)$, for all $x, y \in G$. i.e. $\alpha$ is abelian fuzzy group of $G$.

(iii) $\alpha \ast \beta = \beta \ast \alpha$, for all $\beta \in \mathcal{FP}(G)$.

Theorem 1.2.19. [71] Let $G$ be a finite group. If $\alpha \in \mathcal{F}(G)$, such that $\alpha_t$ is normal subgroup of $G$, for all $t \in [0,1]$, then $\alpha \in \mathcal{FN}(G)$.
Theorem 1.2.20. Let $\alpha \in \mathcal{FN}(G)$ and $t \in [0, 1]$ be such that $t \leq \alpha(e)$, where $e$ denotes the identity of $G$. Then $\alpha_t$ is normal subgroup of $G$.

Theorem 1.2.21. Let $f$ be a homomorphism of the group $G$ into the group $K$. If $\alpha$ is a normal fuzzy group of $G$, then $f(\alpha)$ is also a normal fuzzy group of $K$. Also if $\beta$ is a normal fuzzy group of $K$ and $f$ is onto, then $f^{-1}(\beta)$ is normal fuzzy group of $G$.

Let $x, y \in G$, then we denote $[x, y] = x^{-1}y^{-1}xy$. Thus

Lemma 1.2.22. Let $\alpha \in \mathcal{F}(G)$. Then $\alpha \in \mathcal{FN}(G)$ if and only if $\alpha([x, y]) = \alpha(e)$, for all $x, y \in G$.

Lemma 1.2.23. Let $\alpha \in \mathcal{FN}(G)$. Let $H = \{x \in G \mid \alpha(x) = \alpha(e)\}$. Then $H$ is a normal subgroup of $G$ and $G' \subseteq H$, where $G'$ is the commutator subgroup of $G$.

Lemma 1.2.24. Let $G$ be a finite group and $\alpha \in \mathcal{F}(G)$. If $H = \{x \in G \mid \alpha(x) = \alpha(e)\}$ is an abelian subgroup of $G$, then $\alpha \in \mathcal{FN}(G)$.

Definition 1.2.25. Let $\alpha$ be a fuzzy normal group of a group $G$. For any $x, y \in G$, define a binary relation $\sim$ on $G$ by $x \sim y \iff \alpha(xy^{-1}) = \alpha(e)$, where $e$ is an identity element of $G$.

Then $\sim$ is a congruence relation of $G$. The equivalence class containing $x$ is denoted by $\alpha_x$. We shall denote $G/\alpha$ by the set $\{\alpha_x \mid x \in G\}$.

Lemma 1.2.26. If $\alpha$ is a fuzzy normal group of $G$, then $G/\alpha$ is a group with the binary operation $\alpha_x \alpha_y = \alpha_{xy}$.

The group $G/\alpha$ is generally called the quotient group or factor group of $G$ relative to $\alpha$.

Theorem 1.2.27. Let $\alpha \in \mathcal{F}(G)$ and $N$ be a normal subgroup of $G$. Define $\xi \in \mathcal{FP}(G/N)$ as follows:

$\xi(xN) = \vee \{\alpha(z) \mid z \in xN\}$, for all $x \in G$. Then $\xi \in \mathcal{F}(G/N)$. 

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Definition 1.2.28. Let $\alpha, \beta \in \mathcal{F}(G)$ and $\alpha \subseteq \beta$. Then $\alpha$ is called a normal fuzzy group of the fuzzy group $\beta$, if $\alpha(xy^{-1}) \geq \alpha(y) \land \beta(x)$, for all $x, y \in G$.

Lemma 1.2.29. Let $\alpha, \beta \in \mathcal{F}(G)$ and $\alpha$ be a normal fuzzy group of $\beta$. Then $\text{Supp}(\alpha)$ is a normal subgroup $\text{Supp}(\beta)$.

Definition 1.2.30. Let $\alpha$ and $\beta$ be fuzzy groups of the groups $G$ and $H$ respectively. A homomorphism (isomorphism) $f$ of $G$ onto $H$ is called a homomorphism (isomorphism) from $\alpha$ to $\beta$, if $f(\alpha) = \beta$.

Definition 1.2.31. For a group $G$, a nonempty collection $\mathcal{D}$ of endomorphisms of $G$ is called an operator domain on $G$. A subset $S$ of $G$ is admissible (or invariant) under $\mathcal{D}$, if $f(S) \subseteq S$, for all $f \in \mathcal{D}$. A subgroup $H$ of $G$ is said to be a characteristic subgroup of $G$, if it is $\text{Aut}(G)$-invariant, where $\text{Aut}(G)$ is a group of automorphisms on $G$. A fuzzy set $\lambda$ of $G$ is $\mathcal{D}$-admissible, if $f(\lambda) \subseteq \lambda$, for all $f \in \mathcal{D}$.

Lemma 1.2.32. A subset $S$ of $G$ with the operator domain $\mathcal{D}$ is $\mathcal{D}$-admissible if and only if $\chi_S$ is $\mathcal{D}$-admissible fuzzy set of $G$, where $\chi_S$ is the characteristic function of $S$.

1.3 Automata theory

In this section we discuss preliminary concepts of automata theory, which we are combining with (fuzzy) group theory throughout the thesis.

Definition 1.3.1. An automaton is a triple $A = (S, X, M)$, where $S$ is a nonempty finite set called the set of states, $X$ is a nonempty finite set called the set of inputs and $M$ is a mapping of $S \times X$ into $S$, called the state transition function. $M$ can be extended to a mapping $M^*$ of $S \times X^*$ into $S$ by $M^*(s, \epsilon) = s$ and $M^*(s, xa) = M(M^*(s, x), a)$, for all $s \in S$ and $x \in X^*, a \in X$. Here, $X^*$ is the free monoid generated by $X$ and $\epsilon$ is its identity. In literature, generally, $M$ is used for both $M$ and $M^*$ without any confusion.
Definition 1.3.2. Let $A_1 = (S_1, X_1, M_1)$ and $A_2 = (S_2, X_2, M_2)$ be automata. An automata homomorphism (or generalised $X_1X_2$-homomorphism) of $A_1$ into $A_2$ is a pair $(f, g)$ of mappings $f : S_1 \to S_2$ and $g : X_1 \to X_2$ such that $f(M_1(s, a)) = M_2(f(s), g(a))$, for all $s \in S$ and $a \in X_1$. If both $f$ and $g$ are bijective mappings, then $(f, g)$ is called as automata isomorphism.

An automata homomorphism (isomorphism) of $A$ into $A$ is said to be automata endomorphism (automorphism) on $A$. The set of all automaton endomorphisms (automorphisms) will be denoted by $END(A)$ ($AUT(A)$). Further, we denote $End(A) = \{(f, g) \in END(A) \mid g(x) = x, x \in X\}$ and $Aut(A) = \{(f, g) \in AUT(A) \mid g(x) = x, x \in X\}$.

Lemma 1.3.3. Let $A = (S, X, M)$ be an automaton. For any $(f_1, g_1), (f_2, g_2) \in END(A)$, define $(f_1, g_1) \circ (f_2, g_2) = (f_1 \circ f_2, g_1 \circ g_2)$.

Then following statements hold:
(1) $END(A)$ is monoid and $End(A)$ is a submonoid of $END(A)$.
(2) $Aut(A)$ and $AUT(A)$ are groups.

Definition 1.3.4. An automaton $A = (S, X, M)$ is said to be a permutation automaton, if $M(s, a) = M(t, a)$, where $s, t \in S$ and $a \in X$ implies that $s = t$.

Lemma 1.3.5. Let $A = (S, X, M)$ be an automaton. Then the following three conditions are equivalent:
(i) $A$ is a permutation automaton.
(ii) $M(s, x) = M(t, x)$, where $x \in X^*$, implies that $s = t$.
(iii) For every $x \in X^*$, we have $M(S, x) = S$, where $M(S, x) = \{M(s, x) \mid s \in S\}$.

In the following definitions, the set of inputs is considered as a semigroup.

Definition 1.3.6. An automaton, $A$, is an ordered triple $A = (S, I, M)$, where $S$ is a nonempty set of states, $I$ is a nonempty semigroup of inputs and $M$ is a function of $S \times I$ into $S$, satisfying $M(s, xy) = M[M(s, x), y]$, for all $s \in S$ and $x, y \in I$. 

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Definition 1.3.7. [6] Let $A = (S, I, M)$ and $s, s' \in S$. Then $s'$ is said to be reachable from $s$, if $s' \in r(s) = \{M(s, x) | x \in I\} \cup \{s\}$.

Define relation $\sim$ on $S$ by $s \sim s'$ if each is reachable from the other. The relation $\sim$ is an equivalence relation on $S$. Arbib [6] called $s$ and $s'$ communicate, if $s \sim s'$.

$A$ is said to be strongly connected, if $S$ has only one equivalence class with respect to $\sim$.

Lemma 1.3.8. $A = (S, I, M)$ is strongly connected if and only if given any $s_1, s_2 \in S$, there exists $x \in I$ such that $M(s_1, x) = s_2$.

We say $s \in S$ is initial, if $s \in r(s') \Rightarrow s' \in r(s)$ [6].

Lemma 1.3.9. [6] At least one element of $S$ is initial. If $S_i$, an equivalence class under $\sim$, contains an initial element, then all its elements are initial.

Choose one element from each equivalence class of initial elements and form a set. We call any such set, a set of generators for $A$. A component of $A$ is any set $r(s)$ for an initial $s$. Any $s$ such that $C = r(s)$ is a generator of $C$.

Lemma 1.3.10. [6] Let $C_1$ and $C_2$ be components of $A$. Then $C_1 \cap C_2$ is either empty or is a union of $\sim$ equivalence classes all elements of which are reachable from any generator of either $C_1$ or $C_2$.

[6] Let $C_1, C_2, \ldots, C_n$ be distinct components of $A$. Each component $C_i$ defines an automaton $A_i$ as $A_i = (C_i, I, M_i)$, where $M_i = M|_{C_i \times I}$, for all $i = 1, 2, \ldots, n$. Bavel, in [11], call these automata as primaries of $A$.

Definition 1.3.11. [27, 107] Let $A = (S, I, M)$ and $B = (T, I, N)$ be two automata. A function $h : S \rightarrow T$ is called operation preserving or homomorphism from $A$ to $B$, if $h(M(s, x)) = N(h(s), x)$, for all $x \in I$. In addition to this property, if $h$ is one-to-one and onto, then $h$ is called an isomorphism from $A$ to $B$. 

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Theorem 1.3.12. Let \( A = (S, I, M) \) be an automaton. Then the set of all isomorphisms from \( A \) to \( B \) forms a group. We shall denote it by \( G(A) \).

**Definition 1.3.13.** An automaton \( A = (S, I, M) \), is called an **abelian**, if \( M(s, xy) = M(s, yx) \), for all \( s \in S \) and \( x, y \in I \). If in addition \( A \) is strongly connected, then \( A \) is called **perfect automaton**.

**Definition 1.3.14.** Let \( A = (S, I, M) \) be an automaton and \( x, y \in I \). Then \( x \) is equivalent to \( y \), \( x \sim y \), if \( M(s, x) = M(s, y) \), for all \( s \in S \). We denote by \( \pi_A \) the set of all \( y \in I \) such that \( x \sim y \) and by \( T_A \) the set of all such classes. Under the natural operation \( T_A \) forms a semigroup called the **characteristic semigroup** of \( A \).

**Definition 1.3.15.** \( A \) is said to be a **state independent automaton**, if for all \( s, t \in S \) and \( x, y \in I \) one has \( M(s, x) = M(s, y) \) if and only if \( M(t, x) = M(t, y) \). A state independent automaton is said to be a **group-type automaton**, if \( T_A \) is a group. A strongly connected, group type automaton is called a **quasiperfect automaton**.

**Definition 1.3.16.** Let \( A = (S, I, M) \) be an automaton. An automaton \( B = (T, I, M') \) is called a **subautomaton** of \( A \) (symbolically \( B \ll A \)), if \( T \subseteq S \) and \( M' \) is a restriction of \( M \) to \( T \times I \).

The set of successors of \( s \in S \) is the set \( M(s) = \{ M(s, x) | x \in I \} \). The automaton generated by \( s \in S \) is, denoted by \( A(s), A(s) = (M(s), I, M') \), where \( M' \) is restriction of \( M \) to \( M(s) \). An automaton \( A \) is singly generated, if there exists \( s \in S \) such that \( A = A(s) \). A maximal singly generated subautomaton of \( A \) is called a **primary** of \( A \).

### 1.4 Fuzzy automata theory

Since the introduction of fuzzy automaton as a generalization of classical automaton by Wee [105], many researchers have developed fuzzy automata theory in
various directions. Here in this section we recall few basic definitions and results, that are needed for us throughout the thesis. Most of the concepts of fuzzy automata are covered in the book by Mordeson and Malik [65], hence we use [65] as a basis for our work.

**Definition 1.4.1.** [65] A *fuzzy finite state machine* (or *fuzzy automaton*) is a triple \( A = (Q, \Sigma, \mu) \), where \( Q \) and \( \Sigma \) are finite non-empty sets and \( \mu \) is a fuzzy set of \( Q \times \Sigma \times Q \), i.e. \( \mu : Q \times \Sigma \times Q \rightarrow [0, 1] \). An extension \( \mu^* \) of \( \mu \) to \( \Sigma^* \) is defined by

\[
\mu^*(p, \epsilon, q) = \begin{cases} 
1, & \text{if } p = q \\
0, & \text{if } p \neq q.
\end{cases}
\]

and \( \mu^*(p, xa, q) = \vee\{\mu^*(p, x, r) \land \mu(r, a, q) \mid r \in Q\} \), for all \( x \in \Sigma^* \) and \( a \in \Sigma \).

Let \( A = (Q, \Sigma, \mu) \) be a fuzzy automaton and \( M \subseteq Q \). Then the *successor* of \( M \) is the set \( S(M) = \{p \in Q \mid \mu(q, x, p) > 0, \text{ for some } (q, x) \in M \times \Sigma^*\} \). We denote the *x-successor* of \( M \) for fixed \( x \in \Sigma^* \), by the set \( S_x(M) = \{p \in Q \mid \mu(q, x^k, p) > 0, \text{ for some } q \in M \text{ and } k \in \mathbb{N} \cup \{0\}\} \), where \( x^0 = \epsilon \). The successor of \( \{q\} \) is generally denoted by \( S(q) \) and the *x-successor* of \( \{q\} \) by \( S_x(q) \). Similarly, the *predecessor* of \( M \) is the set \( D(M) = \{p \in Q \mid \mu(p, x, q) > 0, \text{ for some } (q, x) \in M \times \Sigma^*\} \) and for fixed \( x \in \Sigma^* \) the *x-predecessor* of \( M \) is the set \( D_x(M) = \{p \in Q \mid \mu(p, x^k, q) > 0, \text{ for some } q \in M \text{ and } k \in \mathbb{N} \cup \{0\}\} \).

**Definition 1.4.2.** [57] A fuzzy automaton \( B = (R, \Sigma, \lambda) \) is called a *subautomaton* of an automaton \( A = (Q, \Sigma, \mu) \), if \( R \subseteq Q \), \( S(R) = R \) and \( \mu|_{R \times \Sigma \times R} = \lambda \). This subautomaton is called *separated*, if \( S(Q - R) \cap R = \emptyset \).

**Definition 1.4.3.** [57] A fuzzy automaton \( A = (Q, \Sigma, \mu) \) is called

(i) *connected*, if \( A \) has no proper separated subautomaton.

(ii) *strongly connected*, if \( \forall p, q \in Q \), we have \( q \in S(p) \).

(iii) *abelian*, if \( \mu(p, xy, q) = \mu(p, yx, q) \), \( \forall x, y \in \Sigma^* \) and \( p, q \in Q \).
Definition 1.4.4. A connected fuzzy automaton $A = (Q, \Sigma, \mu)$ is said to be an inverse fuzzy automaton if for all $x \in \Sigma^*$, there exists unique $y \in \Sigma^*$ such that $\mu(p, x y x, q) = \mu(p, x, q)$ and $\mu(p, y x y, q) = \mu(p, y, q)$, for all $p, q \in Q$.

Definition 1.4.5. An inverse semigroup is a semigroup $S$ with the property that, for each $a \in S$ there is a unique element $a^{-1} \in S$ such that $a = a a^{-1} a$ and $a^{-1} = a^{-1} a a^{-1}$. If $S$ has an identity $1$, we refer to it as an inverse monoid.

Define a congruence $\theta_A$ on $\Sigma^*$ as $x \theta_A y$ iff $\mu(p, x, q) = \mu(p, y, q)$, for all $p, q \in Q$.

Thus

Lemma 1.4.6. A fuzzy automaton $A = (Q, \Sigma, \mu)$ is an inverse fuzzy automaton if and only if $\Sigma^*/\theta_A$ is an inverse monoid.

Definition 1.4.7. Let $A_1 = (Q_1, \Sigma_1, \mu_1)$ and $A_2 = (Q_2, \Sigma_2, \mu_2)$ be fuzzy automata. A pair $(f, g)$ of mappings $f : Q_1 \to Q_2$ and $g : \Sigma_1 \to \Sigma_2$ is called a homomorphism, written $(f, g) : A_1 \to A_2$, if $\mu_1(p, x, q) \leq \mu_2(f(p), g(x), f(q))$, for all $p, q \in Q, x \in \Sigma_1$. The pair $(f, g)$ is called a strong homomorphism, if $\mu_2(f(p), g(x), f(q)) = \vee \{\mu_1(p, x, r) \mid r \in Q_1, f(r) = f(q)\}$, for all $p, q \in Q, x \in \Sigma_1$. A homomorphism (strong homomorphism) $(f, g) : A_1 \to A_2$ is called an isomorphism (strong isomorphism), if $f$ and $g$ are both one-one and onto mappings.

Theorem 1.4.8. Let $A_1 = (Q_1, \Sigma_1, \mu_1)$ and $A_2 = (Q_2, \Sigma_2, \mu_2)$ be fuzzy automata. Then following statements hold. i) If $(f, g) : A_1 \to A_2$ is homomorphism, then $\mu_1^*(p, x, q) \leq \mu_2^*(f(p), g(x), f(q))$, for all $p, q \in Q, x \in \Sigma_1^*$.

ii) If $(f, g) : A_1 \to A_2$ is strong homomorphism, then $f$ is one-one $\iff \mu_1^*(p, x, q) = \mu_2^*(f(p), g(x), f(q))$, for all $p, q \in Q, x \in \Sigma_1^*$.

Das [21] was the first who discuss fuzzy automaton over a group. We recite few concepts from [21] for self contentedness.

Definition 1.4.9. A fuzzy set $\alpha$ of a group $Q$ is called a fuzzy subsemiautomaton of a fuzzy automaton $A = (Q, \Sigma, \mu)$, if
(i) $\alpha$ is a fuzzy group of $Q$.

(ii) $\alpha(p) \geq \mu(q, x, p) \land \alpha(q)$, for all $p, q \in Q$ and $x \in \Sigma$.

We shall denote $\text{sub}^F(A)$ for the set of all fuzzy subsemiautomaton of $A$.

**Definition 1.4.10.** [21] Let $A = (Q, \Sigma, \mu)$ be a fuzzy automaton. A fuzzy set $\alpha$ of a group $Q$ is called a **fuzzy kernel** of $A$, if

(i) $\alpha$ is a fuzzy normal group of $Q$.

(ii) $\alpha(p \ast r^{-1}) \geq \mu(q \ast k, x, p) \land \mu(q, x, r) \land \alpha(k)$ for all $p, q, k, r \in Q$, $x \in \Sigma$.

**Lemma 1.4.11.** [21] Let $A = (Q, \Sigma, \mu)$ be a fuzzy automaton. If $\alpha$ is a fuzzy kernel and $\beta$ is a fuzzy subsemiautomaton of $A$, then $\alpha \ast \beta$ is a fuzzy subsemiautomaton of $A$.

**Lemma 1.4.12.** [21, 65] If $\alpha$ and $\beta$ are a fuzzy kernels of $A$, then $\alpha \ast \beta$ is a fuzzy kernel of $A$.

### 1.5 New concepts for fuzzy automata theory

In this section, beginning with concepts in [51], we introduce new definitions and notations and use them throughout the thesis.

**Definition 1.5.1.** [51] Let $A$ and $B$ be sets. A **fuzzy relation** from $A$ to $B$ is a fuzzy set $R$ of $A \times B$. The number $R(a, b)$ denote the degree to which $a$ is related to $b$.

**Definition 1.5.2.** [51] A fuzzy relation $R$ from $A$ to $B$ is said to be **complete**, if for each $a \in A$, there exists $b \in B$ such that $R(a, b) > 0$. A fuzzy relation $R$ is said to be **fuzzy function**, if for each $a \in A$, there is unique $b \in B$ such that $R(a, b) > 0$.

Since $\text{Supp}(R) = \{(a, b) \mid R(a, b) > 0\}$ is a (crisp) function, the above definition resembles to that of the definition of the (crisp) function, in the sense of unique image for each element of the domain.
Definition 1.5.3. A fuzzy automaton is a triplet \( A = (Q, \Sigma, \mu) \), where \( Q \) is a non-empty finite set called the set of states, \( \Sigma \) is a non-empty finite set called the set of inputs and \( \mu \) is a fuzzy function from \( Q \times \Sigma \) to \( Q \) called fuzzy transition function.

If \( A = (Q, \Sigma, \mu) \) is a fuzzy automaton, then \( \Sigma^* \) denotes the set of all strings of symbols in \( \Sigma \) including the empty string \( \epsilon \) and the fuzzy function \( \mu \) is extended to a fuzzy function \( \mu^* \) from \( Q \times \Sigma^* \) to \( Q \) as follows: for all \( p, q \in Q \), \( a \in \Sigma \) and \( x \in \Sigma^* \) we have \( \mu^*(p, ax, q) = \mu(p, a, r) \land \mu^*(r, x, q) \), where \( r \in Q \) is such that \( \mu(p, a, r) > 0 \) and

\[
\mu^*(p, \epsilon, q) = \begin{cases} 
1, & \text{if } p = q; \\
0, & \text{otherwise}. 
\end{cases}
\]

Here, onward in this thesis we write \( \mu \) for both \( \mu \) and \( \mu^* \) without any ambiguity.

Definition 1.5.4. Let \( A_1 = (Q_1, \Sigma_1, \mu_1) \) and \( A_2 = (Q_2, \Sigma_2, \mu_2) \) be two fuzzy automata. A pair \( (h, k) \) of maps, where \( h : Q_1 \to Q_2 \), \( k : \Sigma_1 \to \Sigma_2 \), is called a fuzzy automaton homomorphism from \( A_1 \) to \( A_2 \), symbolically \( (h, k) : A_1 \to A_2 \), if for \( p, q \in Q_1 \) and \( x \in \Sigma_1^* \), \( \mu_2(h(p), k(x), h(q)) = \mu_1(p, x, q) \).

Definition 1.5.5. A pair of maps \( (h, k) : A_1 \to A_2 \), where \( h : Q_1 \to Q_2 \), \( k : \Sigma_1 \to \Sigma_2 \), is said to be weak fuzzy automaton homomorphism, if for \( p, q \in Q_1 \) and \( x \in \Sigma_1^* \), \( \mu_1(p, x, q) > 0 \Rightarrow \mu_2(h(p), k(x), h(q)) > 0 \)

Remark 1.5.6. Every fuzzy automaton homomorphism is a weak fuzzy automaton homomorphism, but not conversely.

A (weak) fuzzy automaton homomorphism \( (h, k) \) from \( A_1 \) to \( A_2 \) is said to be (weak) fuzzy automaton isomorphism, if both \( h \) and \( k \) are bijective functions.

We shall adopt the following notations for simplicity throughout the thesis.

N1 : \( \mathcal{H}^f (A \to B) \) : The set of all fuzzy automaton homomorphisms from \( A \) to \( B \).
N2 : $\mathcal{WF}(A \to B)$ : The set of all weak fuzzy automaton homomorphisms from $A$ to $B$.

N3 : $\mathcal{IF}(A \to B)$ : The set of all fuzzy automaton isomorphisms from $A$ to $B$.

N4 : $\mathcal{WI}(A \to B)$ : The set of all weak fuzzy automaton isomorphisms from $A$ to $B$.

N5 : $\mathcal{EF}(A)$ : The set of all fuzzy automaton endomorphisms on $A$.

N6 : $\mathcal{WE}(A)$ : The set of all weak fuzzy automaton endomorphisms on $A$.

N7 : $\mathcal{GF}(A)$ : The set of all fuzzy automaton automorphisms on $A$.

N8 : $\mathcal{WG}(A)$ : The set of all weak fuzzy automaton automorphisms on $A$.

In case, if $\Sigma_1 = \Sigma_2 = \Sigma$ and $k$ is the identity function on $\Sigma$, then we shall denote the homomorphism $(h, k)$ simply by the function $h : A_1 \to A_2$. Then all the above notations will be denoted respectively by $HF(A \to B), WHF(A \to B)$ and so on.

**Definition 1.5.7.** A fuzzy automaton $A = (Q, \Sigma, \mu)$ is said to be weakly abelian, if $\mu(p, xy, q) > 0 \Leftrightarrow \mu(p, yx, q) > 0$, for $x, y \in \Sigma^*$ and $p, q \in Q$.

**Definition 1.5.8.** For a fuzzy automaton $A = (Q, \Sigma, \mu)$, the center of $A$, denoted by $Z(A)$, is the set $\{x \in \Sigma^* |$ for $q, s \in Q$, we have $\mu(q, xy, s) > 0 \Leftrightarrow \mu(q, yx, s) > 0, \forall y \in \Sigma^*\}$. Clearly

**Theorem 1.5.9.** $Z(A) = \Sigma^*$ if and only if $A$ is weakly abelian.

**Definition 1.5.10.** Let $A = (Q, \Sigma, \mu)$ be a fuzzy automaton and $q \in Q$. Let $S(q) = \{t \in Q| \mu(q, x, t) > 0, x \in \Sigma^*\}$. Then the fuzzy automaton generated by $q$ is, denoted by $A(q)$, $A(q) = (S(q), \Sigma, \mu')$, where $\mu'$ is a restriction of $\mu$ to $S(q) \times \Sigma \times S(q)$. A fuzzy automaton $A$ is said to be singly generated, if there exists $q \in Q$ such that $A = A(q)$. Moreover, $A$ is said to be strict
singly generated, if there exists $q \in Q$ such that $A = A^+(q)$, where $A^+(q) = (S^+(q), \Sigma, \mu')$ and $S^+(q) = \{ t \in Q | \mu(q, x, t) > 0, x \in \Sigma^+ \}$.

The set of generators of $A(q)$ is the set $\text{gen}A(q) = \{ r \in S^A(q) | A(r) = A(q) \}$.

Recall that $B = (Q', \Sigma, \mu')$ is a subautomaton of $A = (Q, \Sigma, \mu)$, if $Q' \subseteq Q$, $S(Q') = Q'$ and $\mu' = \mu|_{Q' \times \Sigma \times Q'}$. We shall denote it by $B \ll A$.

**Definition 1.5.11.** A subautomaton $B$ of $A$ is called a **primary** of $A$, if
(i) there exists $q \in Q$ such that $B = A(q)$ and
(ii) for any $t \in Q$, if $B \ll A(t)$, then $B = A(t)$

**Definition 1.5.12.** Let $A = (Q, \Sigma, \mu)$ be a fuzzy automaton. Then $A$ is called **quasi-strongly connected**, if $A(q)$ is strongly connected, for each $q \in Q$.

**Definition 1.5.13.** A maximal connected fuzzy subautomaton $A' = (Q', \Sigma', \mu')$ of a fuzzy automaton $A = (Q, \Sigma, \mu)$ is called a **block** of $A$. We shall denote $Q'$ as a **component** of $Q$ (or $A$).

**Definition 1.5.14.** Let $A = (Q, \Sigma, \mu)$ be a fuzzy automaton and let $q \in Q$, $y, z \in \Sigma^*$. Then $y$ is **$q$-fuzzy equivalent** to $z$, if $\mu(q, y, p) > 0$ and $\mu(q, z, p) > 0$, for some $p \in Q$. We shall denote it by $y \equiv^F_q z$.

**Remark 1.5.15.** The relation $\equiv^F_q$ is an equivalence relation of finite index on $Q$ (i.e. on $A$). We shall denote the equivalence class of $x \in \Sigma^*$ with respect to this relation by $[x]_q$.

**Definition 1.5.16.** A fuzzy automaton $A = (Q, \Sigma, \mu)$ is said to be **discrete fuzzy automaton**, if $\mu(p, a, q) = 0$, for all $p, q \in Q$ and $a \in \Sigma$.

**Definition 1.5.17.** A fuzzy automaton $A = (Q, \Sigma, \mu)$ is said to be **permutation fuzzy automaton**, if the following condition holds: $\mu(p_1, a, q) \land \mu(p_2, a, q) > 0$, for some $p_1, p_2, q \in Q$ and $a \in \Sigma$ implies that $p_1 = p_2$.

**Definition 1.5.18.** A fuzzy automaton $A = (Q, \Sigma, \mu)$ is said to be an **inverse fuzzy automaton**, if for any $x \in \Sigma^*$, there exists unique $y \in \Sigma^*$ such that $xyx \equiv^F x$ and $yxy \equiv^F y$. In this case, we say $y = x^{-1}$. 

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Definition 1.5.19. A fuzzy automaton $A = (Q, \Sigma, \mu)$ is said to be faithful, if the following condition holds: for any $a, b \in \Sigma$, $\mu(p, a, q) \land \mu(p, b, q) > 0$ for all $p, q \in Q \Rightarrow a = b$.

Definition 1.5.20. Let $A = (Q, \Sigma, \mu)$ be any fuzzy automaton and $q \in Q, x \in \Sigma^*$. Then the $x$-path of $q$ is the subautomaton $O_x(q) = (S_x(q), \{x\}, \mu')$, where $\mu'$ is the restriction of $\mu$ to $S_x(q) \times \{x\} \times S_x(q)$.

Definition 1.5.21. The $x$ circle of $q$ is the subautomaton $C_x(q) = (S^c_x(q), \{x\}, \mu')$, where $S^c_x(q) = \{ t \in S_x(q) : \mu(q, x^k, t) > 0 \text{ and } \mu(q, x^m, t) > 0, \text{ for some integer } m > k \}$ and $\mu'$ is the restriction of $\mu$ to $S^c_x(q) \times \{x\} \times S^c_x(q)$.

The $x$-path of $q$, $O_x(q)$ is said to be circular, if $O_x(q) = C_x(q)$.