Chapter 2

Regular Domination in graphs

The Chapter aims at the study of a new concept called regular domination in graphs introduced by Prof. E.Sampathkumar. Regular domatic and antidomatic partitions are defined and studied. Regular domination saturation number is defined and determined.

2.1 Introduction

Let $G = (V, E)$ be a simple graph. There are certain properties of subsets of $V(G)$ which are also true in $\overline{G}$. For example, a subset of $V(G)$ in which no two vertices have the same degree in the graph is called a highly irregular set. A subset which is highly irregular in $G$ is also highly irregular in $\overline{G}$. If a subset of $V(G)$ induces a regular subgraph in $G$, then it also induces a regular subgraph in $\overline{G}$. Thus, a dominating set whose induced subgraph is
regular is also regular in $\overline{G}$.

In practical sense, a dominating set whose induced subgraph is regular may be used to represent a council whose members are homogeneous in the sense that they are acquainted with the same number of people in the council. In an organization, it is desirable if the executive council is homogeneous. Even an independent set of members constitutes a homogeneous set. A study of regular dominating sets and the corresponding parameter, regular domination number, is made in this chapter. This will help one to determine the minimum number of members required in a homogeneous council.

### 2.2 Regular Domination

**Definition 2.2.1** A dominating set $D$ of $V(G)$ is called a regular dominating set if $< D >$ is regular. The minimum cardinality of a regular dominating set is called regular domination number of $G$ and is denoted by $\gamma_r(G)$.

**Remark 2.2.2** Since any maximal independent set is a regular dominating set, the existence of a regular dominating set is always guaranteed in any graph.

**Remark 2.2.3** $\gamma(G) \leq \gamma_r(G) \leq i(G)$. 

24
Example 2.2.4

Here $\gamma(G) = 2$, $i(G) = 3$ and $\gamma_r(G) = 2$. Thus, $\gamma(G) = \gamma_r(G) < i(G)$.

Example 2.2.5

$\gamma(G) = 2$, $i(G) = k + 1$ and $\gamma_r(G) = 2$. Here also, $\gamma(G) = \gamma_r(G) < i(G)$.

Example 2.2.6

$\gamma(G) = 3$, $i(G) = k + 2$ and $\gamma_r(G) = 4$. Here, $\gamma(G) < \gamma_r(G) < i(G)$. 

25
Example 2.2.7

\[\gamma(G) = 4; i(G) = k + 3 \text{ and } \gamma_r(G) = 6. \] Here, \(\gamma(G) < \gamma_r(G) < i(G).\)

Example 2.2.8

\[\gamma_r(G) = 3, \gamma_r(G) = 4 = i(G). \] That is \(\gamma(G) < \gamma_r(G) \leq i(G).\)

\(\gamma_r\) for standard graphs

1. \(\gamma_r(K_n) = 1\)

2. \(\gamma_r(K_{1,n}) = 1\)

3. \(\gamma_r(K_{m,n}) = 2\)
4. $\gamma_r(W_n) = 1$

5. $\gamma_r(P_n) = \lceil \frac{n}{3} \rceil = \gamma(P_n)$

6. $\gamma_r(C_n) = \lceil \frac{n}{3} \rceil = \gamma(C_n)$

7. $\gamma_r(P) = 3$, where $P$ is the Petersen graph.

**Observation 2.2.9** $\gamma_r(G) = 1$ if and only if $G$ has a full-degree vertex.

**Theorem 2.2.10** $\gamma_r(G) = n$ if and only if $G = K_n$.

**Proof:**

Suppose $\gamma_r(G) = n$. Suppose there exists $u, v \in V(G)$ such that $u$ and $v$ are adjacent. Then $\beta_0(G) \leq (n - 1)$. Therefore, $\gamma_r(G) \leq \beta_0(G) \leq (n - 1)$, a contradiction. Therefore, $G = K_n$. 

**Theorem 2.2.11** Let $a, b$ and $c$ be three positive integers such that $a < b < c$ and $b < 3a$. Then there exists a connected graph $G$ such that $\gamma(G) = a; \gamma_r(G) = b; i(G) = c$.

**Proof:**

**Case (i)** $3a - b$ is odd. Consider the cycle $C_b$. Attach $k_1, k_2, \cdots, k_t$ pendant vertices at each of $\frac{3a-b+1}{2}(= t)$ consecutive vertices of $C_b$. Without loss of
generality, let the pendent vertices be attached at the consecutive vertices $u_1, u_2, \cdots, u_t$, where $V(C_b) = \{u_1, \cdots, u_b\}$. Since $a < b$, $\frac{3a-b+1}{2} \leq b$.

$$\gamma(G) = t + \left\lceil \frac{b-t-2}{3} \right\rceil$$

$$= \frac{3a-b+1}{2} + \left\lceil \frac{2b-(3a-b+1)-4}{6} \right\rceil$$

$$= \frac{3a-b+1}{2} + \left\lceil \frac{3b-3a-5}{6} \right\rceil$$

$$= \frac{3a-b+1}{2} + \left\lceil \frac{b-a-1}{2} - \frac{1}{3} \right\rceil$$

(since $3a - b$ is odd, we get that $b - a$ is odd. Therefore, $b - a - 1$ is even)

$$= \frac{3a-b+1}{2} + \frac{b-a-1}{2}$$

Therefore, $\gamma(G) = a$.

To find $i(G)$.

If $t$ is odd, then $u_1$ and $u_t$ are necessarily to be chosen and the pendent vertices at $u_2, u_4, \cdots, u_{t-1}$ are chosen. For the vertices $u_{t+2}, \cdots, u_{b-1}(u_{t+1}$ is dominated by $u_t$ and $u_b$ is dominated by $u_1$), We select $\left\lceil \frac{b-t-2}{3} \right\rceil$ vertices.

$$\left\lceil \frac{b-t-2}{3} \right\rceil = \left\lceil \frac{b-(3a-b+1)-2}{3} \right\rceil = \left\lceil \frac{3b-3a-5}{6} \right\rceil = \left\lceil \frac{b-a-1-5}{6} \right\rceil = \frac{b-a-1}{2} = l \text{(say)}.$$

If $t$ is even, then only one of $u_1, u_t$ can be chosen. Assume $k_1 > k_t$. Choose $u_1$ and pendent vertices at $u_2, u_4, \cdots, u_t$. For the vertices $u_{t+1}, \cdots, u_{b-1} (u_b$ is dominated by $u_1$ and $u_{t+1}$ is not dominated since $u_t$ is not chosen), we select $\left\lceil \frac{b-t-1}{3} \right\rceil$ vertices. $\left\lceil \frac{b-t-1}{3} \right\rceil = \left\lceil \frac{3b-3a-3}{6} \right\rceil = \frac{b-a-1}{2} = l$. Choose $k_1, k_2, \cdots, k_t$
such that the independent domination number of the graph formed by the $t$ consecutive vertices along with their pendants is $(c - l)$. Therefore, $i(G) = (c - l) + l = c$. Clearly, the regular domination number of $G$ is $b$.

Case (ii) $(3a - b)$ is even. Consider the cycle $C_b$. Attach $k_1, k_2, \ldots, k_t$ pendent vertices at each of $\frac{3a - b}{2}(=t)$ consecutive vertices of $C_b$. Without loss of generality, let the pendent vertices be attached at $u_1, u_2, \ldots, u_t$ where $V(C_b) = \{u_1, u_2, \ldots, u_t\}$. It can be shown as in case (i) that $i(G) = c$ and $\gamma_r(G) = b$. ■

Remark 2.2.12 The complement of a $\gamma_r$ -set need not contain a regular dominating set.

Example 2.2.13

\begin{center}
\begin{tikzpicture}
    \node at (1,0) (1) {$1$};
    \node at (2,0) (2) {$2$};
    \node at (3,0) (3) {$3$};
    \node at (1,-1) (4) {$4$};
    \node at (2,-1) (5) {$5$};
    \node at (3,-1) (6) {$6$};
    \draw (1) -- (2);
    \draw (2) -- (3);
    \draw (3) -- (4);
    \draw (4) -- (5);
    \draw (5) -- (6);
\end{tikzpicture}
\end{center}

$G$:

$D = \{4, 5, 6\}$ is a regular dominating set and $(V - D)$ does not contain a regular dominating set.
Remark 2.2.14 Let $G$ be a graph with a full degree vertex. Then $\gamma(G) = \gamma_r(G)$ and every $\gamma$-set is a $\gamma_r$-set. The converse is not true since in $C_4$, every $\gamma$-set is a $\gamma_r$-set but $C_4$ has no full degree vertex.

Observation 2.2.15 Let $D$ be a regular dominating set. Let $u \in D$.

(i) If $\deg_{<D>}(u) = 1$, then $D = tK_2$.

(ii) If $\deg_{<D>}(u) \geq 2$, then $D$ contains a cycle.

(iii) If $u, v, w \in D$ such that $v$ is adjacent to $u$ and $w$, then $D$ contains a cycle passing through $u, v, w$.

(iv) If $G$ contains an isolate, then $D$ is an independent dominating set of $G$.

Therefore, $\gamma_r(G) = i(G)$. The converse is not true. For example,

Example 2.2.16

\[
G: \quad \begin{array}{c}
1 \\
2 \\
3 \\
4
\end{array} \begin{array}{c}
2 \\
\end{array} \begin{array}{c}
3 \\
\end{array} \begin{array}{c}
4 \\
5
\end{array}
\]

$\gamma_r(G) = i(G)$.

Observation 2.2.17 If $G$ is a claw-free graph, then $\gamma(G) = i(G)$ and hence $\gamma(G) = \gamma_r(G) = i(G)$. Hence, in any line graph $G$, $\gamma(G) = \gamma_r(G) = i(G)$. 
Observation 2.2.18 If $G$ is a graph not containing $K_{1,3}$ or the A-L graph, then $\gamma(G) = \gamma_r(G) = i(G)$.

Observation 2.2.19 Regular domination is not a super hereditary property.

Observation 2.2.20 A regular dominating set $D$ is 1-minimal if and only if for any $u \in D$ one of the following holds.

(i) $u$ is an isolate of $D$.

(ii) $u$ has a private neighbour in $V - D$.

(iii) $u$ is not a full degree vertex in any of the components of $<D>$.

Remark 2.2.21 Let $D$ be a $\gamma_r$-set of $G$. Then there may exists a vertex $u \in D$ such that neither $u$ is an isolate of $D$ nor $u$ has a private neighbour.

Example 2.2.22

\[ G : \]

\{1, 2, 3\} is a minimum dominating set but it is not regular. $D = \{1, 2, 3, 4\}$
is a minimum regular dominating set and 4 is neither an isolate of $D$ nor a private neighbour.

**Observation 2.2.23** If $D$ is a regular dominating set such that either $D$ is independent or $< D >$ is non-complete then $D$ is 1-minimal.

**Observation 2.2.24** Let $u \in D$. Then $D - \{u\}$ is regular if and only if $\deg_{< D >}(u) = 0$ or $|D| - 1$.

**Proof:** If $\deg_{< D >}(u) = 0$ or $|D| - 1$, then $D - \{u\}$ is regular. Conversely, suppose $D - \{u\}$ is regular and $1 \leq \deg_{< D >}(u) < |D| - 1$. Then there exists vertices $v_1, v_2 \in D$ such that $v_1$ is adjacent to $u$ and $v_2$ is not adjacent to $u$. Hence $\deg_{D-\{u\}}(v_1) < \deg_{D-\{u\}}(v_2)$, a contradiction.

Thus the result holds. ■

**Theorem 2.2.25** Let $G$ be a simple $(n, m)$ graph. $\gamma_r(G) \leq \begin{cases} \frac{2m}{k} & \text{for all } k \\ \frac{2(m-n)}{(k-2)} & \text{for all } k \geq 3 \end{cases}$ where $k$ is the degree of any vertex of $D$ in $< D >$.

**Proof:**

The number of edges in $< D > = \frac{k\gamma_r(G)}{2} \leq m$. Therefore, $\gamma_r(G) \leq \frac{2m}{k}$.

Let $k \geq 3$. Then, $\frac{k\gamma_r}{2} + (n - \gamma_r(G)) \leq m$. Hence $\gamma_r(G)(k - 2) + 2n \leq 2m$.

That is $\gamma_r(G) \leq \frac{2(m-n)}{(k-2)}$. ■

32
Remark 2.2.26 When \( k = 1 \), \( \frac{\gamma_r(G)}{2} + (n - \gamma_r(G)) \leq m \). Therefore, \( \gamma_r(G) \geq 2(n - m) \).

Observation 2.2.27 \( k \leq (\gamma_r(G) - 1) \).

Observation 2.2.28 If \( k = 0 \), then \( \gamma_r(G) = i(G) \).

Result 2.2.29 For any graph \( G \), \( i(G) \leq n - \Delta(G) \).

Proof:

Let \( u \) be a vertex of maximum degree \( \Delta \). Let \( D = V - N(u) \). Then \( |D| = n - \Delta(G) \) and \( v \in D \). Consider a maximal independent set \( S \) in \( < D > \) containing \( v \). Then \( S \) is a dominating set of \( < D > \). Since \( v \) dominates \( N(v) \), \( S \) dominate \( N(v) \). Therefore, \( S \) is a dominating set of \( G \).

Therefore, \( |S| \leq |D| = n - \Delta(G) \). \( i(G) \leq |S| \leq n - \Delta(G) \).

Remark 2.2.30 \( \frac{n}{1 + \Delta(G)} \leq \gamma(G) \leq \gamma_r(G) \leq i(G) \leq n - \Delta(G) \).

Theorem 2.2.31 Let \( G \) be a simple graph. Then \( \frac{n}{\Delta - k + 2} \leq \gamma_r(G) \), where \( k \) is the maximum regularity of a minimum regular dominating set of \( G \).

Proof:
Let \( D \) be a minimum regular dominating set of regularity \( 't' \). There are at least \( (n - \gamma_r(G)) \) edges from \( (V - D) \) to \( D \). Therefore, \( (n - \gamma_r(G)) \leq \sum_{v \in D} (\text{deg}(v) - (t - 1)) \leq \sum_{v \in D} (\Delta(G) - t + 1) = \gamma_r(G)(\Delta(G) - t + 1) \) which gives \( n \leq \gamma_r(G)(\Delta(G) - t + 2) \). Therefore, \( \frac{n}{\Delta(G) - t + 2} \leq \gamma_r(G) \). As \( k \) is the maximum of \( t \) over all minimum regular dominating sets. We get \( \frac{n}{\Delta(G) - k + 2} \leq \gamma_r(G) \). The bound is sharp as seen in \( K_n \). □

**Observation 2.2.32** Let \( D \) be a maximal independent set of \( G \) such that no vertex in \( (V - D) \) is adjacent to every vertex of \( D \). Then \( D \) is a dominating set of \( \overline{G} \).

**Observation 2.2.33** Let \( D \) be a maximal independent set of \( G \) such that there exists at most two adjacent vertices \( u,v \) in \( (V-D) \) which are adjacent to every vertex of \( D \) and for any vertex \( w \in (V - D), w \) is adjacent to either \( u \) or \( v \) and not both. Then \( D \) is a dominating set of \( \overline{G} \).

**Observation 2.2.34** Any induced paired dominating set is a regular dominating set. Therefore, \( \gamma_r(G) \leq \gamma_{ip}(G) \).

**Remark 2.2.35** \( \gamma(\mu(G)) \leq \gamma_r(\mu(G)) \leq i(\mu(G)) = i(G) + 1 \). But \( \gamma(\mu(G)) = \gamma(G) + 1 \) and so \( \gamma(G) + 1 \leq \gamma_r(\mu(G)) \leq i(G) + 1 \).
Observation 2.2.36

(i) Let $G$ have a full degree vertex.
Then $\gamma_r(\mu(G)) = 2$. Therefore, the upper bound $i(G)+1$ is sharp.

(ii) Let $G = K_2$. Then $\mu(G) = C_5$. $\gamma_r(\mu(G)) = 2 = \gamma(G) + 1$. Therefore, lower bound is sharp.

Observation 2.2.37 Let $H$ be the shadow graph of $G$. Let any regular dominating set of $G$ of minimum cardinality is independent. Then $\gamma_r(H) = 2\gamma_r(G)$ and $\gamma_r(\mu(G)) = \gamma_r(G) + 1$.

Proposition 2.2.38 Let $D$ be a $\gamma_r$-set of $G$ and $D$ be not independent. Then $\gamma_r(H) \leq \gamma_r(G)$, where $H$ is the shadow graph $G$.

Proof:

Let $D = \{u_1, u_2, \ldots, u_{\gamma_r}\}$. Since $\deg_{<D>}(u_i) \geq 1$, there exists $u_j \in D$ adjacent to $u_i$. Therefore, $u'_i$ is dominated by $u_j$. Therefore, $u'_1, u'_2, \ldots, u'_{\gamma_r}$ are all dominated by $D$. Let $x' \in (V' - \{u'_1, u'_2, \ldots, u'_{\gamma_r}\})$. Then $x \in (V - D)$ and their exists $u_t \in D$ adjacent to $x$. Therefore, $x'$ is adjacent to $u_t$. Therefore, $x'$ is dominated by $D$. Therefore, $D$ is a regular dominating set for $H$. Therefore, $\gamma_r(H) \leq \gamma_r(G)$. ■
Theorem 2.2.39 Let \( G \) be a graph in which every \( \gamma_r \)-set is independent. Then \( \gamma_r(S(G)) > \gamma_r(G) \).

Proof:

Let \( D \) be an independent \( \gamma_r \)-set of \( S(G) \). Suppose \( \{u_1, u_2, \ldots, u_t\} = D \cap V(G) \). Then \( u'_1, u'_2, \ldots, u'_t \) also belong to \( D \). For: suppose \( u'_i \notin D \). Then \( u'_i \) is dominated by \( D \) and since \( u'_i \) is not adjacent to any \( u'_r, u'_t \) is adjacent to some \( u_j \). Therefore, \( u_i \) is adjacent to \( u_j \), a contradiction to the fact that \( u_1, u_2, \ldots, u_t \) are independent. Let \( D = \{u_1, u_2, \ldots, u_t, u'_1, u'_2, \ldots, u'_t, u'_{t+1}, u'_{t+2}, \ldots, u'_{t+k}\} \).

Let \( D_1 = \{u_1, u_2, \ldots, u_t, u_{t+1}, \ldots, u_{t+k}\} \). Let \( x \in (V - D_1) \). If \( x \) is adjacent to \( u_i \) or \( u'_{t+i} \), then \( x \) is adjacent to \( u_i \) or \( u_{t+i} \). Therefore, \( x \) is dominated by \( D_1 \). If \( x \) is adjacent to \( u'_j \) for some \( j, 1 \leq j \leq t \), then \( x \) is adjacent to \( u_j \). Therefore, \( x \) is dominated by \( D_1 \). Therefore, \( D_1 \) is an independent dominating set of \( G \). Therefore, \( \gamma_r(G) \leq |D_1| < |D| = \gamma_r(S(G)) \). \( \blacksquare \)

Corollary 2.2.40 Under the hypothesis of the theorem,

\[ \gamma_r(G) < \gamma_r(S(G)) \leq \gamma_r(\mu(G)). \] That is \( \gamma_r(G) + 1 \leq \gamma_r(\mu(G)) \).

Theorem 2.2.41 \( \gamma_r(S(G)) = \gamma_r(G) \) if \( G \) has a \( \gamma_r \)-set without isolates.
Proof:

Let $G$ have a $\gamma_r$-set say $D$ such that $<D>$ has no isolates. Then $D$ is a $\gamma_r$-set of $S(G)$. For, let $D = \{u_1, u_2, \ldots, u_t\}$. Consider $u'_i \in V'(G), 1 \leq i \leq t$. As $u_i$ is adjacent to some $u_j, 1 \leq j \leq t$, $u'_i$ is adjacent to $u_j$. Therefore, $\{u'_1, u'_2, \ldots, u'_t\}$ are dominated by $D$. Let $x' \in (V - \{u'_1, u'_2, \ldots, u'_t\})$. $x \in (V - D)$ and $x$ is dominated by say $u_i \in D$. Therefore, $x'$ is adjacent to $u_i$. Therefore, $x'$ is dominated by $D$. Therefore, $D$ is a regular dominating set of $S(G)$. Therefore, $\gamma_r(S(G)) \leq \gamma_r(G)$. Suppose $S = \{u_1, u_2, \ldots, u_r, u'_{r+1}, \ldots, u'_{r+k}\}$ be a $\gamma_r$-set of $S(G)$.

Case (i) $S$ is independent. Then $S_1 = \{u_1, u_2, \ldots, u_r, u'_{r+1}, \ldots, u'_{r+k}\}$ is an independent dominating set of $G$.

Therefore, $\gamma_r(G) \leq |S_1| = |S| = \gamma_r(S(G))$.

Case (ii) $S$ is not independent. Here for it can be proved that $S_1$ is a regular dominating set of $G$. Therefore, $\gamma_r(G) \leq \gamma_r(S(G))$. Therefore, $\gamma_r(G) = \gamma_r(S(G))$. ■

Theorem 2.2.42 If every $\gamma_r$-set of $G$ independent, then

$\gamma_r(\mu(G)) = \gamma_r(G) + 1$. 

37
Proof:

Let \( \{u_1, u_2, \cdots, u_t\} \) be an independent \( \gamma_r \)-set of \( G \). Then \( \{u_1, u_2, \cdots, u_t, w\} \) is a regular dominating set of \( \mu(G) \). Therefore, \( \gamma_r(\mu(G)) \leq (t + 1) \). Suppose \( \gamma_r(\mu(G)) = t \). Suppose \( D_1 = \{v_1, v_2, \cdots, v_k, v'_{k+1}, \cdots, v'_t\} \) is a \( \gamma_r \)-set of \( \mu(G) \). Then \( D_2 = \{v_1, v_2, \cdots, v_k, v_{k+1}, \cdots, v_t\} \) is a regular dominating set of \( G \) and of cardinality \( t = \gamma_r(G) \). Since, every \( \gamma_r \)-set of \( G \) is independent, \( D_2 \) is independent. Therefore, \( \{v_1, v_2, \cdots, v_k\} \) are independent. Then \( v'_1 \) is not dominated by \( D_1 \) in \( \mu(G) \); a contradiction. Suppose, \( \gamma_r(\mu(G)) < t \). Suppose \( D_3 = \{v_1, v_2, \cdots, v_k, v'_{k+1}, \cdots, v'_s\} \) be a \( \gamma_r \)-set of \( \mu(G) \) where \( s \leq \gamma_r(\mu(G)) \). Then \( \{v_1, v_2, \cdots, v_k, v_{k+1}, \cdots, v_s\} \) is a regular dominating set of \( G \) of cardinality \( s < \gamma_r(\mu(G)) < t = \gamma_r(G) \), a contradiction. Therefore, \( \gamma_r(\mu(G)) = \gamma_r(G) + 1 \).  

**Observation 2.2.43** There exist connected graph \( G \) for which

\[ \gamma_r(\mu(G)) > \gamma_r(G) + 1. \]
Example 2.2.44

\[ G : \]

\[ \gamma_r(G) = 2, \gamma_r(\mu(G)) = 4. \]

Observation 2.2.45 Let \( G \) be a graph having a \( \gamma_r \)-set without isolates. Then

\[ \gamma_r(G) = \gamma_r(S(G)) \leq \gamma_r(\mu(G)). \]

Theorem 2.2.46 Let \( G \) be a graph having a \( \gamma_r \)-set \( D \) of regularity \( \geq 2 \) (That is \( d_{<D>}(u) \geq 2 \), for all \( u \in D \)). Suppose there exists at least one vertex in \( D \) which has no private neighbours. Then \( \gamma_r(G) = \gamma_r(\mu(G)) \).

Proof: Let \( V(\mu(G)) = V(G) \cup \{u'_1, u'_2, \cdots, u'_n, v\} \), where \( V(G) = \{u_1, u_2, \cdots, u_n\} \). Let \( D = \{u_1, u_2, \cdots, u_t\} \) be a \( \gamma_r \)-set of \( G \) and \( d_{<D>}(u) \geq 2 \), for all \( u \in D \). Let \( u_1 \in D \) have no private neighbourhood. Consider \( D_1 = \{u'_1, u'_2, \cdots, u'_t\}. v \in V(\mu(G)) \) dominated by \( u'_1 \). \( u_1 \) is dominated by some \( u_i, 2 \leq i \leq t \) where \( u_i \) is adjacent to \( u_1 \in D \). Let \( x \in (V - D) \). If \( x \) is
dominated by \( u_i \in D \), then \( x \) is dominated by \( u_i' \in D_1 \). If \( x \) is dominated by \( u_i, \ 2 \leq i \leq t \), then \( x \) is dominated by \( D_1 \) (as \( u_i \in D_1 \)). Consider \( u_j', \ 2 \leq j \leq t \).

Since \( D \) is of regularity \( \geq 2 \), there exists \( u_{i_1}, u_{i_2} \in D \) which are adjacent to \( u_j \). Therefore, there exists at least one vertex \( u_i \in D, i \neq 1 \) such that \( u_i \) is adjacent to \( u_j \). Moreover \( u_i \in D_1 \) and \( u_j' \) is adjacent to \( u_i \). Therefore, \( u_j' \) is dominated by \( D_1 \). Let \( x \in \{ u_{t+1}', \ldots, u_n' \} \). Let \( x = u_{t+i}' \), \( u_{t+i} \) is dominated by \( D \). Suppose \( u_{t+i} \) is dominated by \( u_1 \). Since \( u_1 \) has no private neighbour, \( u_{t+i} \) is dominated by some \( u_k, 2 \leq k \leq t \). Therefore, \( u_k' \) is adjacent to \( u_k \). Hence \( x \) is dominated by \( D_1 \) and \( D_1 \) is a dominating set of \( \mu(G) \). Clearly, \( \langle D_1 \rangle \) is regular. Therefore, \( \gamma_r(\mu(G)) \leq \gamma_r(G) \). Let \( S = \{ v_1, v_2, \ldots, v_k, v_{k+1}', \ldots, v'_s \} \) be a \( \gamma_r \)-set \( \mu(G) \). Then \( \{ v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_s \} \) is a regular dominating set of \( G \). Therefore, \( \gamma_r(G) \leq \gamma_r(\mu(G)) \). Suppose \( S = \{ v_1, v_2, \ldots, v_k, v \} \) is a \( \gamma_r \)-set of \( \mu(G) \). Then \( \{ v_1, v_2, \ldots, v_k \} \) is a regular dominating set of \( G \).

Therefore, \( \gamma_r(G) \leq \gamma_r(\mu(G)) \). Therefore, \( \gamma_r(\mu(G)) = \gamma_r(G) \).  

Illustration 2.2.47

![Diagram of a graph](image.png)

\( G : \)

40
\[ D = \{1, 2, 3, 4\} \] is a \( \gamma_r \)-set of \( G \).

\[ < D_1 >: \]

\[ D_1 = \{1, 2, 3, 4'\} \] is a \( \gamma_r \)-set of \( \mu(G) \).

**Observation 2.2.48** The regular domination number of the Kneser graph \( K(n, 2) \), for all \( n \geq 3 \) is 3.

**Proof:** Let \( u_1 = \{1, 2\} \), \( u_2 = \{2, 3\} \), \( u_3 = \{1, 3\} \). Then \( \{u_1, u_2, u_3\} \) is a dominating set of \( K(n, 2) \). Therefore, \( \gamma_r(K(n, 2)) \leq 3 \).

But \( 3 = \gamma(K(n, 2)) \leq \gamma_r(K(n, 2)) \). Hence \( \gamma_r(K(n, 2)) = 3 \). \( \square \)

### 2.3 Regular domination saturation number

**Theorem 2.3.1** Given a graph \( G \) of order \( n \) and regular domination number \( t \), every \( t \) subset of \( V(G) \) is a regular dominating set if and only if \( G = K_n \).
(or) $K_n$ (or) $(\frac{n}{2})K_2$.

Proof:

Let $G$ be a $(n, m)$ graph with $\gamma_r(G) = t$. Suppose every subset of $V(G)$ of cardinality $t$ is a regular dominating set.

Case (i) $t = 1$. Then $G = K_n$.

Case (ii) $t = n$. Then $G = \overline{K}_n$.

Case (iii) $1 < t < n$. Therefore, $t \geq 2$. Let $u \in V(G)$. Suppose $d_{\overline{G}}(u) \geq t$. Then any set $D$ of $t$ vertices from $N_{\overline{G}}(u)$ will not dominate $u$ in $G$, a contradiction. Therefore, $d_{\overline{G}}(u) < t$ for all $u \in V(G)$. Therefore, $d_G(u) \geq n - t$ for all $u \in V(G)$. Therefore, $2m = \sum_{u \in V(G)} (d_G(u)) \geq n(n - t)$. Therefore, $m \geq \left\lceil \frac{n(n - t)}{2} \right\rceil \rightarrow (1)$. Dankelman et al [10] proved the following: If $i(G)$ is the independent domination number for $G$, then $m \leq \frac{n(n - i(G))}{2} - \frac{r(i(G) - r)}{2}$ where $r$ is given by $n = qi(G) + r$, $0 \leq r \leq i(G)$. Since $\gamma_r(G) \leq i(G)$, $m \leq \frac{n(n - \gamma_r(G))}{2} - \frac{r(\gamma_r(G) - r)}{2}$.

Subcase (i) $r > 0$. Then $m < \frac{n(n - \gamma_r(G))}{2}$, a contradiction to (1).

Subcase (ii) $r = 0$. $m \leq \frac{n(n - i(G))}{2}$. In this case, the equality is reached if and only if $G$ is complete multipartite with cardinality of each partite
set being $i(G)$. For such a graph $\gamma_r(G) = 2$. Suppose $i(G) \geq 3$. Then $m \leq \frac{n(n-3)}{2} < \frac{n(n-2)}{2}$, a contradiction to (1). Therefore, $i(G) \leq 2$. If $i(G) = 1$, then $\gamma_r(G) = 1$, a contradiction. Therefore, $i(G) = 2 = \gamma_r(G)$. Therefore, $\Delta(G) \leq n - i(G) = n - 2$. But $d_G(u) \geq n - t = n - 2$, for all $u \in V(G)$. Therefore, $\Delta(G) \geq n - 2$. Therefore, $\Delta(G) = n - 2$ and $d_G(u) = n - 2$, for all $u \in V(G)$. Therefore, $G$ is $(n - 2)$-regular. Therefore, $G = (\frac{n}{2})K_2$. ■

**Theorem 2.3.2** Given a graph $G$ of order $n$ and independent domination number $t$, every $t$ subset of $V(G)$ is an independent dominating set if and only if $G = K_n$ (or) $\overline{K_n}$ (or) $(\frac{n}{2})\overline{K_2}$.

**Proof:**

Proceeding as in the proof of the previous theorem, we get that $m \geq \left\lceil \frac{n(n-t)}{2} \right\rceil$. But $m \leq \frac{n(n-i(G))}{2} - \frac{r(i(G)-r)}{2}$, $0 \leq r < i(G)$.

**Case(i)** $r > 0$. Then $m < \frac{n(n-i(G))}{2}$, a contradiction.

**Case(ii)** $r = 0$. $G$ is a multi partite graph with $\frac{n}{i(G)}$ partite sets each containing $i(G)$-vertices. If $i(G) \geq 4$, then $m \geq \left\lceil \frac{n(n-3)}{2} \right\rceil$ and $m \leq \frac{n(n-4)}{2}$, a contradiction. Therefore, $i(G) \leq 3$.

If $i(G) = 3$, then as $G$ is multipartite with each partite set containing
\( \frac{n}{i(G)} \) elements, any 3-element set need not be an independent dominating set as a set containing one element from one partite set and two elements from another partite set dominates \( G \) but it is not independent. Therefore, \( i \leq 2 \).

Proceedings as in the previous theorem, we get that \( G = K_n \) (or) \( \overline{K}_n \) (or) \((\frac{n}{2})K_2\).

\[ \textbf{Theorem 2.3.3} \] For any graph \( G \), \( \gamma_r(G) \leq n - \kappa(G) \).

\[ \textbf{Proof:} \]

\[ \gamma_r(G) \leq i(G) \leq n - \Delta(G) \leq n - \delta(G) \leq n - \kappa(G) \] (since \( \delta(G) \leq \Delta(G) \))

and \( \kappa(G) \leq \delta(G) \).

\[ \textbf{Result 2.3.4} \] \( \gamma_r(G) = n - \kappa(G) \) if and only if \( G = K_n \) or \( \overline{K}_n \) or \( (\frac{n}{2})K_2 \).

\[ \textbf{Proof:} \] If \( \gamma_r(G) = 1 \) or \( n \) , then \( G = K_n \) or \( \overline{K}_n \). Let \( 2 \leq \gamma_r(G) \leq n - 1 \).

Clearly, \( \gamma_r(G) \leq n - \Delta(G) \to (1) \). Therefore, \( n - \kappa(G) \leq n - \Delta(G) \).

Therefore, \( \Delta(G) \leq \kappa(G) \leq \delta(G) \). Therefore, \( \Delta(G) = \delta(G) \). Therefore, \( G \) is regular. Let \( t \) be the regularity of \( G \), then \( n - \gamma_r(G) = \kappa(G) \leq t \). Therefore, \( \gamma_r(G) \geq h - t \). \( \gamma_r(G) \leq n - t \) (from (1)). Therefore, \( \gamma_r(G) = n - t \).

Therefore, \( \kappa(G) = n - \gamma_r(G) = t \). Since \( G \) is regular, \( m = \frac{nt}{2} = \frac{n(n - \gamma_r(G))}{2} \).

But \( q \leq \frac{n(n - \gamma_r(G))}{2} - \frac{r(\gamma_r(G) - r)}{2} \).
Case (i) $r > 0$. Therefore, $q < \frac{n(n-\gamma_r(G))}{2}$, a contradiction.

Case (ii) $r = 0$. Then $\gamma_r(G) = 2$. Proceeding as in the above theorem, we get that $G = \left(\frac{n}{2}\right)K_2$. The converse is obvious.

\begin{corollary}
For any graph $G$ of order $n$ and $\gamma_r(G) = t$, the following are equivalent:

(i) every $t$-subset of $V(G)$ is regular dominating.

(ii) $\gamma_r(G) = n - \kappa(G)$.

(iii) $G = K_n$ or $\overline{K_n}$ or $\left(\frac{n}{2}\right)K_2$.
\end{corollary}

\begin{proposition}
Suppose a new parameter called the strong regular domination number of a graph $G$ and denoted by $s_r(G)$ is defined as the least positive integer $s$ such that every $s$-subset of $V(G)$ is a regular dominating set of $G$. Then the existence of $s_r(G)$ is not guaranteed. For example:

There is no positive integer $s$ with $1 \leq s \leq 7$ such that every $s$-subset of $G$ is a regular dominating set.
\end{proposition}
\( V(G) \) is a regular dominating set of \( G \). This prompts us to define the strong regular domination number in the case of regular graphs. In a regular graph, the strong regular domination number \( s_r(G) \) is the least positive integer \( s \) such that every \( s \)-subset of \( V(G) \) is a regular dominating set of \( G \).

**Observation 2.3.7** For any regular graph \( G \) of order \( n \), \( \gamma_r(G) \leq s_r(G) \leq n \) and the bounds are sharp. The lower bound is attained in \( K_n \) of the upper bound is attained in \( K_{m,m} \), \( m \geq 3 \).

**Proposition 2.3.8** Let \( G \) be a \((m, n)\)-graph with \( s_r(G) = s \). Then \( m \geq \left\lceil \frac{1}{2}n(n - s) \right\rceil \) and the bound is sharp.

**Proof:**

Proceeding as in the above theorem, we get that \( m \geq \left\lceil \frac{1}{2}n(n - s) \right\rceil \). The bounds are attained in \( K_n \) and \( \overline{K_n} \).

**Result 2.3.9** There are connected graphs which are not complete and \( m > \left\lceil \frac{1}{2}n(n - s) \right\rceil \).
Example 2.3.10

\[ G : \]

1  2

1  2

\[ m = 5, n = 4, s = 2. \quad \left\lceil \frac{1}{2}n(n - s) \right\rceil = \left\lceil \frac{1}{2} \times 4 \times 2 \right\rceil = 4. \]

Theorem 2.3.11 Suppose \( G \) is a connected non-regular graph in which \( s_r(G) \) exists. Then \( s_r(G) = 2. \)

Proof:

Suppose \( s_r(G) \geq 3. \) Let \( s_r(G) = k \geq 3. \) Since \( G \) is non-regular, \( s_r(G) \leq n - 1. \) Let \( D \) be a subset of \( V(G) \) of cardinality \( k. \)

Case (i) \( D \) is not independent. Then \( < D > \) is regular of regularity say \( t \) and \( D \) is dominating. Let \( D = \{ u_1, u_2, \ldots, u_k \} \). Let \( v \in (V - D) \). Since \( k \leq n - 1, V - D \neq \phi. \) Then \( v \) is adjacent to some vertex of \( D \) say \( u_1. \)

Subcase (i) \( < D > \) is not complete. Let \( k \geq 3. \) \( u_1 \) is not adjacent to some vertex \( u_i \in D \) and there exists \( u_j \in D \) such that \( u_i \) and \( u_j \) are adjacent. Consider \( D_1 = \{ u_1, u_2, \ldots, \hat{u_i}, \ldots, u_k, v \} \) Then \( d_{<D_1>}(u_1) = d_{<D>}(u_1) + 1. \) Also,
$d_{<D_1>}(u_j) \leq d_{<D>}(u_j)$. Therefore, $<D_1>$ is not regular, a contradiction.

Since any set of $s_r$ vertices is a regular dominating set.

**Subcase (ii)** $<D>$ is complete. Suppose there exists $v \in (V - D)$ such that $v$ is not adjacent to at least two vertices say $u_{i_1}, u_{i_2} \in D$ ($1 \leq i_1, i_2 \leq k$). Since $D$ is dominating there exists $u_j \in D$ such that $v$ is adjacent to $u_j$. $D_1 = \{u_1, \cdots, \hat{u}_j, \cdots, u_k, v\}$ is a subset of $V(G)$ of cardinality $k$ but not regular (since $d_{<D_1>}(v) \leq k - 3$, $d_{<D_1>}(u_1) \geq k - 2$). Suppose $v$ is not adjacent to exactly one vertex of $D$ say $u_{i_1}, 1 \leq i_1 \leq k$, there exists $u_j \in D$ such that $u_j$ and $v$ are adjacent. Consider $D_1 = \{u_1, u_2, \cdots, \hat{u}_j, \cdots, u_k, v\}$. $d_{<D_1>}(u_{i_1}) = k-2, d_{<D_1>}(u_t) = k-1, t \neq i_1, j$. Therefore, $D_1$ is not regular.

Suppose every vertex of $(V - D)$ is adjacent to every vertex of $D$. Since, $s_r(G) \geq 3$, $G$ is not complete. Therefore, $<(V - D)>$ is not complete.

Therefore, there exists $v_1, v_2 \in (V - D)$ such that $v_1$ and $v_2$ are not adjacent. Let $D_1 = \{u_1, \cdots, u_{k-2}, v_1, v_2\}$. $d_{<D_1>}(u_1) = k-1, d_{<D_1>}(v_1) = k-2$. $D_1$ is not regular.

**Case (ii)** $D$ is independent. Let $D = \{u_1, u_2, \cdots, u_k\}$. Let $v \in (V - D)$. There exists $u_i \in D$ such that $v$ and $u_i (1 \leq i \leq k)$ are adjacent.
Subcase (i) Suppose \( v \) is not adjacent to at least two vertices say \( u_{j_1}, u_{j_2} \in D \). Then \( D_1 = \{u_1, u_2, \cdots, u_{j_1}, \cdots, u_k, v\} \). \( d_{<D_1>}(v) \geq 1; \ d_{<D_1>}(u_{j_2}) = 0 \). Therefore, \( D_1 \) is not regular.

Subcase (ii) Suppose \( v \) is not adjacent to exactly one vertex say \( u_j \in D \). Then \( D_1 = \{u_1, u_2, \cdots, u_{j_1}, \cdots, u_k, v\} \). \( d_{<D_1>}(v) = k - 1 \geq 2 \). \( d_{<D_1>}(u_1) = 1 \). Therefore, \( D_1 \) is not regular.

Subcase (iii) Suppose \( v \) is adjacent to all the vertices of \( D \). Then \( D_1 = \{u_1, \cdots, u_{k-1}, u_k, v\} \). \( d_{<D_1>}(v) = k - 1 \geq 2 \). \( d_{<D_1>}(u_1) = 1 \). Therefore, \( D_1 \) is not regular.

Therefore, \( s_r(G) \leq 2 \).

If \( s_r(G) = 1 \), then \( G = K_n \), a contradiction as \( G \) is non-regular. Therefore, \( s_r(G) = 2 \).

Corollary 2.3.12 Let \( G \) be a connected non-regular graph in which \( s_r(G) \) exist. Then \( s_r(G) = 2 \) if and only if \( G \) is a non-complete graph with a full degree vertex.

Proof:

Let \( G \) be a connected non-regular graph in which \( s_r(G) \) exists.
Then $s_r(G) = 2$.

Case (i) $\gamma(G) = 2, s_r(G) = 2$. Then any two element set is a minimum regular dominating set. Therefore, $G = \frac{n}{2}K_2$, a contradiction, since $G$ is non-regular.

Case (ii) $\gamma(G) = 1, s_r(G) = 2$. Therefore, $G \neq K_n$ and $G$ has a full degree vertex.

Therefore, $s_r(G) = 2$ if and only if $G$ has a full degree vertex and $G \neq K_n$.

\[\blacksquare\]

Adopting the definitions of $r$-level domination and dom saturation number given in [1], we introduce the following:

**Definition 2.3.13** Let $D_t(G) = \{D : D$ is a regular dominating set of $G$ with $|D| = t\}$ and $DC_t(G) = \{u \in V(G) : u \in D$ for some $D \in D_t(G)\}$. $G$ is said to be $t$-level regular dom saturated if $DC_t(G) = V(G)$. The least positive integer $t$ such that $DC_t(G) = V(G)$ is called the regular dom saturation number of $G$ and is denoted by $ds_r(G)$.

**Remark 2.3.14** Given any vertex $u \in V(G)$ a maximal independent set of $G$ containing $u$ is a regular dominating set of $G$ and hence every vertex is
contained in some regular dominating set of $G$. Therefore, $d_{sr}(G)$ exists in any graph.

**Remark 2.3.15** $d_{sr}(G) \leq \beta_0(G)$.

**Proposition 2.3.16** For any regular graph $G$, $\gamma_r(G) \leq d_{sr}(G) \leq s_r(G)$ and the bounds are best possible in $C_4$, $(\gamma_r(C_4) = d_{sr}(C_4) = s_r(C_4) = 2)$.

**$d_{sr}$ for standard graphs**

1. $d_{sr}(K_n) = 1$

2. $d_{sr}(K_{1,n}) = 2$

3. $d_{sr}(K_{m,n}) = 2$

4. $d_{sr}(C_n) = \left\lfloor \frac{n}{3} \right\rfloor$

5. $d_{sr}(P_n) = \left\lceil \frac{n}{3} \right\rceil$

6. $d_{sr}(W_n) = 2$

7. $d_{sr}(P) = 3$
8. $ds_r(\mu(G)) \leq 2\beta_0(G)$. The bound is sharp as seen in $C_4$. (For: consider $u \in V(G)$. Let $D$ be a maximal independent set of $G$ containing $u$. Let $V(\mu(G)) = \{V(G) \cup V(G') \cup \{v\}\}$. Therefore, $D \cup \{v\}$ is a regular dominating set containing $u$ as well as $v$. Let $D' = \{u' : u \in D\}$. $D \cup D'$ is an independent dominating set of $\mu(G)$ containing $u'$.)

### 2.4 Regular domatic partition

**Definition 2.4.1** A partition of $V(G)$ into regular dominating sets is called a **regular domatic partition** of $G$. The maximum cardinality of a regular domatic partition of $G$ is called the **regular domatic number** of $G$ and is denoted by $d_r(G)$.

**Observation 2.4.2** Not all graphs have regular domatic partition. For example $(K_1 \cup K_2)$ has no regular domatic partition. In fact, any graph $G \neq \overline{K_n}$ of order $\geq 3$ with an isolated vertex has no regular domatic partition.

**Observation 2.4.3** A graph $G$ with an isolated vertex has a regular domatic partition if and only if $G = \overline{K_n}$. 

52
Observation 2.4.4

1. \( 1 \leq d_r(G) \leq n \).

2. \( d_r(G) = n \) if and only if \( G = K_n \).

3. If \( G = \overline{K_n} \) or \( C_{2n+1} \) then \( d_r(G) = 1 \).

4. If \( d_r(G) = 1 \) then \( G \) is regular.

5. \( d_r(G) \leq d(G) \) (since any regular domatic partition is a domatic partition of \( G \)).

6. For any graph \( G \), \( d_r(G) \leq \delta(G) + 1 \).

Result 2.4.5 *If* \( G \) *has a pendent vertex, then* \( d_r(G) = d(G) = \delta_r(G) + 1 = \delta(G) + 1 \).

**Proof:** Let \( \pi = \{V_1, V_2, \cdots, V_{d_r(G)}\} \) be a regular domatic partition of \( G \). Let \( u \) be a pendent vertex with support \( v \). Without loss of generality let \( u \in V_1 \). If \( v \in V_1 \) then \( V_2 \) cannot dominate \( u \). If \( v \in V_2 \), then any vertex in \( V_i, i \geq 3 \) cannot dominate \( u \). Therefore, \( |\pi| \leq 2 \). If \( |\pi| = 1 \), then \( G \) is regular. As \( G \) has a pendent vertex, \( G = tK_2 \) (\( t \geq 1 \)). In this case, \( d_r(G) = 2 = \delta_r(G) + 1 = \delta(G) + 1 \). If \( |\pi| > 1 \), then \( |\pi| = 2 \). Hence \( d_r(G) = d(G) = \delta_r(G) + 1 = \delta(G) + 1 \). \[\blacksquare\]
Corollary 2.4.6 For any tree $T$, $d_r(T) = 2$.

Remark 2.4.7 This result may not true if $G$ has no pendent vertex. For example:

Example 2.4.8

For this graph, $\delta_r(G) = 1$ and $d_r(G) = 3$ and hence $d_r(G) > \delta_r(G) + 1$.

$\pi = \{\{1,3\}, \{2,5\}, \{4,6\}\}$ is a minimum regular domatic partition of $G$.

Also, $\delta(G) > \delta_r(G)$.

$d_r(G)$ for standard graphs

1. $d_r(C_n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}$
2. \( d_r(P_n) = 2 \)

3. \( d_r(K_{1,n}) = d_r(D_{r,s}) = 2 \)

4. \( d(W_n) = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 3 & \text{if } n \text{ is even} \end{cases} \)

5. \( d_r(K_{m,n}) = \begin{cases} n & \text{if } m = n \\ 2 & \text{if } m \neq n \end{cases} \)

6. \( d_r(P) = 3 \)

\[
\{\{1, 3, 7\}, \{2, 4, 8\}, \{5, 6, 9, 10\}\} \text{ is regular domatic partition of } G.
\]

**Definition 2.4.9** \( d_i(G) \) is the maximum number of disjoint independent dominating sets in \( G \).

**Remark 2.4.10**

(1) \( d_i(G) \leq d_r(G) \leq d(G) \).

(2) \( d_r(G) \leq \frac{n}{\gamma_r(G)} \).
Example 2.4.11

\[ G : \quad K_j \xrightarrow{K_1} K_1 \xrightarrow{K_1} K_j \]

In this graph, \( d_r(G) = d_i(G) = d(G) = 2j + 1 \).

Example 2.4.12

\[ G : \quad \overline{K_j} \xrightarrow{K_j} K_j \xrightarrow{K_j} K_j \xrightarrow{\overline{K_j}} \]

In this graph, \( d_r(G) = d(G) = j + 1 \).

Result 2.4.13 Let \( G_1 \) and \( G_2 \) be two graphs for which \( d_r(G_1) \) and \( d_r(G_2) \) exist. Then \( d_r(G_1 \cup G_2) \) exists if and only if there exist partitions \( \pi_1 = \{V_1, \ldots, V_{d_r(G_1)}\} \) and \( \pi_2 = \{U_1, \ldots, U_{d_r(G_2)}\} \) such that (i) if \( d_r(G_1) = d_r(G_2) \) then, for each \( V_i \) there exists \( U_i \) such that \( V_i \) and \( U_i \) are having the same regularity for each \( i \), \( 1 \leq i \leq d_r(G_1) = d_r(G_2) \) (ii) if \( d_r(G_1) < d_r(G_2) \) then \( V_i \)
and $U_i$ are having the same regularity for each $i$, $1 \leq i \leq d_r(G_1)$ and $V_{d_r(G_1)} + 1, \ldots V_{d_r(G_2)}$ must have the same regularity with any one $V_1, V_2, \ldots, V_{d_r(G_1)}$.

Observation 2.4.14 $d_r(P_2 \times P_n) = 2$.

2.5 Regular anti-domatic partition

Definition 2.5.1 Let $G$ be a graph. A partition $\pi = \{V_1, V_2, \ldots, V_k\}$ of $V(G)$ is called a regular anti-domatic partition of $G$ if for each $V_i$, $< V_i >$ is regular and $V_i$ is a non dominating set of $G$. The minimum cardinality of such a partition is called the regular anti-domatic number of $G$ and is denoted by $d_r(G)$.

Example 2.5.2

In this graph, $\{\{1, 2, 3\}, \{4, 6, 7\}, \{5, 8\}\}$ is a minimum regular anti-domatic partition of $G$. Thus $d_r(G) = 3$. 

57
Remark 2.5.3  (i) If $G$ has a full degree vertex, then $G$ has no regular anti-domatic partition. Therefore, we assume that $G$ has no full degree vertex.

(ii) Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then $\pi = \{\{v_1\}, \{v_2\}, \ldots, \{v_n\}\}$ is a regular anti-domatic partition of $G$.

(iii) $\overline{d}(G) \leq \overline{d}_r(G)$, where $\overline{d}(G)$ is the minimum cardinality of an anti-domatic partition of $G$.

$\overline{d}_r(G)$ for standard graphs

1. $\overline{d}_r(K_{m,n}) = 4$

2. $\overline{d}_r(D_{r,s}) = 3$

3. $\overline{d}_r(P_n) = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even and } n \geq 4 \\ \lceil \frac{n}{2} \rceil & \text{if } n \text{ is odd and } n \geq 4 \end{cases}$

4. $\overline{d}_r(C_n) = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even and } n \geq 6 \\ \lceil \frac{n}{2} \rceil & \text{if } n \text{ is odd} \\ 4 & \text{if } n = 4 \end{cases}$

5. $\overline{d}_r(P) = 4$. Where $P$ is the Petersen graph (since $\{\{1, 3\}, \{2, 4, 6\}, \{5, 8, 9\}, \{7, 10\}\}$ is a regular anti-domatic partition of $P$).

Proposition 2.5.4  For every graph $G$ without full degree vertex, $\overline{d}_r(G) \geq 2$.

Proof: If $\overline{d}_r(G) = 1$, then $V(G)$ is regular and not dominating, a contradiction.■

58
Proposition 2.5.5 If \( d_r(G) = 2 \), then \( \text{diam}(G) \geq 3 \). The converse is not true.

Proof: Let \( d_r(G) = 2 \). Let \( \pi = \{V_1, V_2\} \) be a regular anti-domatic partition of \( G \). There exists \( v_1 \in V_1 \) and \( v_2 \in V_2 \) such that \( v_1 \) is not adjacent to any vertex of \( V_2 \) and \( v_2 \) is not adjacent to any vertex of \( V_1 \). Suppose \( N(v_1) \cup N(v_2) \neq \phi \). Let \( x \in N(v_1) \cap N(v_2) \). If \( x \in V_1 \), then \( x \) dominates \( v_2 \) and if \( x \in V_2 \) then \( x \) dominates \( v_1 \), a contradiction. Therefore, \( d(v_1, v_2) \geq 3 \).

Therefore, \( \text{diam}(G) \geq 3 \). \( \blacksquare \)

Example 2.5.6 Let \( G = C_6 \). \( \text{diam}(G) = 3 \) and \( d_r(G) = 3 \).

Proposition 2.5.7 Let \( \text{diam}(G) \geq 3 \). Let \( x, y \in V(G) \) be such that \( d(x, y) = \text{diam}(G) \) and \( N[x] \) and \( (V - N[x]) \) are regular. Then \( d_r(G) = 2 \).

Proof: Clearly \( N[x] \) and \( (V - N[x]) \) are not dominating. \( \blacksquare \)

Proposition 2.5.8 Let \( G \) be a disconnected graph with components \( G_1, G_2, \ldots, G_k \).

Then \( d_r(G) = 2 \) if and only if for some \( i, 1 \leq i \leq k - 1, G_1, G_2, \ldots, G_i \) are \( r_1 \)-regular and \( G_{i+1}, \ldots, G_k \) are \( r_2 \)-regular.
Proof: Let $G$ be a disconnected with components $G_1, G_2, \cdots, G_k$ graphs such that for some $i$, $1 \leq i \leq k - 1$, $G_1, G_2, \cdots, G_i$ are $r_1$-regular and $G_{i+1}, \cdots, G_k$ are $r_2$-regular. Let $\pi = \{V(G_1) \cup V(G_2) \cup \cdots \cup V(G_i), V(G_{i+1}) \cup \cdots \cup V(G_k)\}$. Then $\pi$ is a regular anti-domatic partition of $G$. The converse is obvious. ■

Proposition 2.5.9 Let $G$ be a graph without full-degree vertices. Then
\[
\left\lceil \frac{n}{n - 1 - \delta(G)} \right\rceil \leq \overline{d}_r(G).
\]

Proof:

Let $\pi = \{v_1, v_2, \cdots, v_k\}$ be a regular anti-domatic partition of $G$. $V_i$ is a non-dominating set of $G$. Therefore, there exists $x \in V(G) - V_i$ such that $x$ is not adjacent to any vertex of $V_i$. Therefore, $\text{deg}(x) \leq n - 1 - |V_i|$. Therefore, $|V_i| \leq n - 1 - \text{deg}(x)$. That is $|V_i| \leq n - 1 - \delta(G)$. That is $n = \sum_{i=1}^{k} |V_i| \leq k(n - 1 - \delta(G))$. Therefore, \[
\left\lceil \frac{n}{n - 1 - \delta(G)} \right\rceil \leq k = \overline{d}_r(G). \]

Proposition 2.5.10 Let $G$ be a graph without full-degree vertices. Let $u$ be a vertex of degree $\delta(G)$ such that $V - N[u]$ is regular. Then $\overline{d}_r(G) \leq \delta(G) + 2$. 

60
Proof:

Let \( N[u] = \{u, v_1, v_2, \ldots, v_\delta\} \). Let \( \pi = \{V - N[u], \{u\}, \{v_1\}, \ldots, \{v_\delta\}\} \).

Then \( \pi \) is regular anti domatic partition of \( G \). Therefore, \( \overline{d_r}(G) \leq |\pi| = \delta(G) + 2 \).

The converse is not true for consider the following examples. In \( C_6 \),
\[
\overline{d_r}(C_6) = 3; \delta(C_6) = 2; \overline{d_r}(C_6) < \delta(C_6) + 2.
\]

Example 2.5.11

\[
\pi = \{\{u, x\}, \{y, u_1\}, \{u_2, u_3\}\} \text{ is a regular anti-domatic partition of } G.
\]
\[
\overline{d_r}(G) = 3. \overline{d_r}(G) = \delta(G) + 2
\]

Illustration 2.5.12 In \( C_{10} \), \( \overline{d_r}(C_{10}) = 5. \delta(C_{10}) + 2 = 4 \). Therefore,
\[
\overline{d_r}(C_{10}) > \delta(C_{10}) + 2.
\]
Note that for any vertex $u \in C_{10}$, $V - N[u]$ is not regular.

**Theorem 2.5.13** Let $G$ be a graph without full-degree vertex. $\overline{d_r}(G) = n$ if and only if $n$ is even and $G = \overline{K}_2 \oplus \overline{K}_2 \cdots (\frac{n}{2})$ times.

**Proof:**

If $n$ is even and $G = \overline{K}_2 \oplus \overline{K}_2 \cdots (\frac{n}{2})$ times, then $\overline{d_r}(G) = n$. Suppose $\overline{d_r}(G) = n$. Then any single vertex is a regular non-dominating set. If there exists a 2-set which is not dominating, then $\overline{d_r}(G) \leq n - 1$. Hence every 2-element set of $G$ is a dominating set and it is obviously regular. Therefore, $\gamma_r(G) = 2$ and any 2-set of $G$ is dominating. Therefore, $G = \frac{n}{2} \overline{K}_2$, by the theorem [2.3.1].

**Proposition 2.5.14** $\overline{d_r}(G_1 \oplus G_2) = \overline{d_r}(G_1) + \overline{d_r}(G_2)$.

**Proof:**

Let $\pi_1 = \{V_1, V_2, \cdots, V_r\}$ be a $\overline{d_r}$-partition of $G_1$ and $\pi_2 = \{W_1, W_2, \cdots, W_s\}$ be a $\overline{d_r}$-partition of $G_2$. Any $W_i$ dominates $V(G_1)$ and any $V_j$ dominates $V(G_2)$. Therefore, $(\pi_1 \cup \pi_2)$ is a regular anti-domatic partition of $(G_1 \oplus G_2)$. Therefore, $\overline{d_r}(G_1 \oplus G_2) \leq (r + s)$. Let $\pi = \{X_1, X_2, \cdots, X_t\}$ be a minimum
regular anti domatic partition of \((G_1 \oplus G_2)\). (That is, \(t = \overline{d}_r(G_1 \oplus G_2)\).

For any \(i, 1 \leq i \leq t\), if \((X_i \cap V(G_1)) \neq \phi\) and \((X_i \cap V(G_2)) \neq \phi\) then \(X_i\) dominates \((G_1 \oplus G_2)\), a contradiction. Therefore, for any \(i, 1 \leq i \leq t\), either \((X_i \cap V(G_1)) = \phi\) or \((X_i \cap V(G_2)) = \phi\). Without loss of generality, let 
\[(X_i \cap V(G_1)) = \phi\] for \(1 \leq i \leq t_1\) and \((X_i \cap V(G_2)) = \phi\) for \(t_1 + 1 \leq i \leq t_2\).

Then \(s \leq t_1\) and \(r \leq t_2\) (since \(\{X_1, X_2, \ldots, X_t\}\) is a regular dominating partition of \(G_2\) and \(\{X_{t_1+1}, \ldots, X_t\}\) is a regular anti dominating partition of \(G_1\)). Therefore, \((r + s) \leq (t_1 + t_2) = t\). That is, \((r + s) \leq \overline{d}_r(G_1 \oplus G_r)\).

Hence the proposition. 

\[\square\]

**Proposition 2.5.15** Let \(k, n\) be integers such that \(2 \leq k \leq (u - 2)\). Then there exists a connected graph \(G\) with \(n\) vertices such that \(\overline{d}_r(G) = k\).

**Proof:**

**Case(i)** \(k\) is even say \(k = 2t\). Let \(G = K_2 \oplus K_2 \cdots (t-1)\) copies \(\oplus K_{n-2t+2}\).

Since \(\overline{d}_r(K_m) = 2\), \(\overline{d}_r(G) = 2(t-1) + 2 = 2t = k\).

**Case(ii)** \(k\) is odd say \(k = 2t + 1 (t \geq 1)\). Let \(G = K_2 \oplus K_2 \cdots (t-2)\) copies \(\oplus C_5 \oplus K_{n-k}\). Then \(\overline{d}_r(G) = 2t - 4 + 3 + 2 = 2t + 1 = k\). 

\[\square\]
Remark 2.5.16 For any graph $G$, $\gamma(G) + \overline{\gamma}(G) \leq \gamma_r(G) + \overline{\gamma_r}(G) \leq \beta_0(G) + \omega(G) \leq (n + 1)$. Therefore, $\gamma_r(G) + \overline{\gamma_r}(G) \leq (n + 1)$.

Proof:

Let $I$ be a maximum independent set of $G$ and $C$ be a maximum clique. Then $|(I \cap C)| \leq 1$. Therefore, $n \geq |(I \cup C)| = |I| + |C| - |(I \cap C)| \geq \beta_0 + \omega - 1$. Therefore, $n + 1 \geq \beta_0(G) + \omega(G)$. That is, $\beta_0(G) + \omega(G) \leq (n + 1)$. $lacksquare$

Remark 2.5.17 $\gamma_r(K_n) + \overline{\gamma_r}(K_n) = n + 1 = \gamma(K_n) + \overline{\gamma}(K_n)$.

Remark 2.5.18 Given any positive integer $k$, there exists a connected graph $G$ with $\omega(G) + \beta_0(G) + k = n$.

Proof:

Take $r = k + 2$. Consider $C^+_r$, $\beta_0(G) = r, w = 2, n = 2r$. Now $\beta_0(G) + \omega(G) + k = r + 2 + r - 2 = 2r = n$. $lacksquare$

Remark 2.5.19 $\gamma(G) + \overline{\gamma}(G) \leq \gamma_r(G) + \overline{\gamma_r}(G) \leq \beta_0(G) + \omega(G)$ and $\beta_0(G)$ and $\omega(G)$ is a better upper bound than $n + 1$.

Remark 2.5.20 It is a known result that if $G$ and $\overline{G}$ have no isolates then $\gamma(G) + \gamma(\overline{G}) \leq \left\lceil \frac{n}{2} \right\rceil + 2$. But $\beta_0(G) + \omega(G)$ may be a better upper bound in
some classes of graphs. For consider, the graph $G = \mu(C_n)$. Add $(k-1)$ vertices ($k \leq n - 2$) and make each of them adjacent to the copy of $C_5$ in $\mu(C_n)$. Let $H$ be the resulting graph $\beta_0(H) = n$. $\omega(H) = 2$. Therefore, $\beta_0(H) + \omega(H) = n + 2$. $|V(H)| = 2n + k$. $\left\lfloor \frac{|V(H)|}{2} \right\rfloor + 2 = n + \left\lfloor \frac{k}{2} \right\rfloor + 2$. Thus, $\beta_0(H) + \omega(H) = \left\lfloor \frac{|V(H)|}{2} \right\rfloor + 2 - \left\lfloor \frac{k}{2} \right\rfloor$. Since $\gamma(H) + \gamma(\overline{H}) \leq \beta_0(H) + \omega(H) < \left\lfloor \frac{|V(H)|}{2} \right\rfloor + 2$, $\beta_0(G) + \omega(G)$ may be a better upper bound in certain classes of graphs.

**Remark 2.5.21** There are graphs for which $\beta_0(G) + \omega(G) > \left\lfloor \frac{n}{2} \right\rfloor + 2$.

**Example 2.5.22**

\begin{align*}
\beta_0(G) = 3, \omega(G) = 2 \text{ and } \left\lfloor \frac{n}{2} \right\rfloor + 2 = \left\lfloor \frac{5}{2} \right\rfloor + 2 = 4. \gamma(G) = 2, \gamma(\overline{G}) = 2.
\end{align*}
Example 2.5.23

\[ \gamma(G) + \overline{\gamma}(G) = 4. \quad \left\lfloor \frac{n}{2} \right\rfloor + 2 = 5. \quad \beta_0(G) = 3, \quad \omega(G) = 3, \quad \gamma(G) = 1, \quad \overline{\gamma}(G) = 3. \]

Remark 2.5.24 Let \( G = K_{2n} \). \( \gamma(G) + \gamma(\overline{G}) = 2n+1. \quad \beta_0(G) + \omega(G) = 2n+1. \)

\[ \left\lfloor \frac{|V(G)|}{2} \right\rfloor + 2 = \left\lfloor \frac{2n}{2} \right\rfloor + 2 = n + 2. \quad \text{Therefore,} \quad \left\lfloor \frac{|V(G)|}{2} \right\rfloor + 2 < \gamma(G) + \gamma(\overline{G}) = \beta_0(G) + \omega(G). \]

Remark 2.5.25 If \( G \) is a triangle free graph with \( \beta_0(G) < \frac{n}{2} \), then \( \beta_0(G) + \omega(G) \leq \left\lfloor \frac{n}{2} \right\rfloor + 2. \)

Remark 2.5.26 Let \( G = C_{4n} \). Draw chords joining diametrically opposite vertices of \( G \). Let \( H \) be the resulting graph. Then \( \beta_0(H) = n, \quad \omega(H) = 2. \)

\[ \left\lfloor \frac{|V(H)|}{2} \right\rfloor + 2 = 2n+2. \quad \left\lfloor \frac{|V(H)|}{2} \right\rfloor + 2 = \beta_0(H) + \omega(H) + n. \quad \text{Note that} \quad H \quad \text{and} \quad \overline{H} \quad \text{have no isolates.} \]

Proposition 2.5.27 Let \( G \) be a graph without full degree vertices. Then

\[ \overline{d}_r(G) \leq 2\chi(G). \]
Proof:

Let \( \pi = \{V_1, V_2, \cdots, V_{\chi(G)}\} \) be a chromatic partition of \( G \). Without loss of generality, let \( V_1, V_2, \cdots, V_i \) be non-dominating and \( V_{i+1}, \cdots, V_{\chi(G)} \) are dominating sets of \( G \). (0 \( \leq i \leq \chi(G) \)). \( |V_j| \geq 2 \), for all \( (i + 1) \leq j \leq \chi(G) \), since \( G \) has no full degree vertex. For every \( V_j \), \( (i + 1) \leq j \leq \chi(G) \), take \( \{V'_j, V''_j\} \) as a partition of \( V_j \). Then \( \pi_1 = \{V_1, \cdots, V_i, V'_i, V''_i, \cdots, V'_{\chi(G)}, V''_{\chi(G)}\} \) is a regular non-dominating partition of \( G \). Hence, \( \overline{d}_r(G) \leq |\pi_1| \leq 2\chi(G) \). \( \blacksquare \)

Remark 2.5.28 If \( G \) is a graph without full degree vertices, then \( \overline{d}(G) \leq \frac{n}{\gamma_r(G) - 1} \). This is not true in the case of regular anti-domatic partition.

Illustration 2.5.29 Consider \( P_{18} \). \( \gamma_r(P_{18}) = 6 \). \( \overline{d}_r(P_{18}) = 9 \).

\[
\frac{n}{\gamma_r(P_{18}) - 1} = \frac{18}{6-1} = \frac{18}{5} = \frac{3}{5}.
\]
Therefore, \( \overline{d}_r(P_{18}) > \frac{n}{\gamma_r(P_{18}) - 1} \)

Remark 2.5.30 Given any positive integer \( k \), there exists a connected graph \( G \) such that \( \overline{d}_r(G) - \left\lceil \frac{n}{\gamma_r(G) - 1} \right\rceil = k \).

Proof:

Consider \( G = P_{2k+8} \), where \( k \geq 1 \). \( \left\lceil \frac{n}{\gamma_r(G) - 1} \right\rceil = 4 \). \( \frac{n}{2} = k + 4 \). Now

\[
\overline{d}_r(G) - \left\lceil \frac{n}{\gamma_r(G) - 1} \right\rceil = k + 4 - 4 = k.
\]

\( \blacksquare \)
Proposition 2.5.31 \( \overline{d_r}(C_n) = \begin{cases} \frac{n}{2} & n \equiv 0 \pmod{4} \\ \frac{n}{2} + 1 & n \equiv 2 \pmod{4} \\ \frac{n+1}{2} & \text{n is odd} \end{cases} \)

Proof:

Since \( \left\lceil \frac{n}{n-1-\delta(G)} \right\rceil \leq \overline{d_r}(G) \) and since \( C_n \) is regular of degree \((n-3)\), we get that \( \left\lceil \frac{n}{2} \right\rceil \leq \overline{d_r}(G) \). Let \( n \equiv 0(\pmod{4}) \). Let \( n = 4k \). Let \( V(C_n) = \{u_1, u_2, \ldots, u_{4k}\} \) Consider \( \pi = \{\{u_1, u_3\}, \{u_2, u_4\}, \{u_5, u_7\}, \ldots, \{u_{4k-2}, u_{4k}\}\} \). \( \pi \) is a regular anti domotic partition. \( \overline{d_r}(C_n) \leq 2k = \frac{n}{2} \). Therefore, \( \overline{d_r}(C_n) = \frac{n}{2} \) if \( n \equiv 0(\pmod{4}) \). Similarly other cases can be proved.

Proposition 2.5.32 Suppose a graph \( H \) exists with \( \overline{d}(H) = n - 1 \). Then \( n \) is odd and \( \overline{d_r}(H) = \overline{d}(H) = n - 1 \).

Proof:

Suppose \( \overline{d}(H) = n - 1 \). Then by Theorem 4 in [5], \( n \) is odd and every non-dominating set has at most two vertices and hecne regular. Therefore, \( \overline{d_r}(H) = n - 1 \).