Chapter 5

Regular Class Domination Partition in Graphs

Color Class domination has been introduced by Prof. E. Sampathkumar. It is a proper color partition in which every color class is dominated by a vertex with the assumption that a color class which is a singleton is dominated by that vertex. This concept is extended to regular partition and is studied in this chapter.

5.1 Introduction

In dominator color partition studied in ([13] [14]), the requirement is that each vertex of \( V(G) \) must dominate a color class. That is, each vertex either appears as a singleton in the partition (or) it is adjacent to every vertex of some color class. Prof. E. Sampathkumar suggested that instead
of vertex domination of color classes, we may stipulate that each color class
is dominated by some vertex. This concept has been named as color class
domination and a study of this has been made in [23]. Though there is a
strong suspicion that dominator coloring is a concept stronger than color class
domination, it has been found that there is no relationship between the two
parameters namely dominator coloring number and color class domination
number.

Extending this idea to regular partition, we have a new parameter called
regular class domination number. The subject matter of study of this chapter
is regular class domination in graphs.

5.2 Color Class Domination

E.Sampathkumar defined color class domination in graphs as follows:

Let \( \pi \) be a proper color partition of \( V(G) \). This partition is said to be color
class domination partition if every member of \( \pi \) is dominated by a vertex of
\( G \). Let \( V(G) = \{v_1, v_2, \cdots, v_n\} \). Then \( \pi = \{\{v_1\}, \{v_2\}, \cdots, \{v_n\}\} \) is a proper
color class domination partition of \( G \), since each element of the partition is
a singleton which is dominated by itself.
The minimum cardinality of a color class domination partition of $G$ is denoted by $\chi_{cd}(G)$.

Generalizing this concept, we get the following:

**Definition 5.2.1** Let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a partition of $V(G)$. $\pi$ is called class domination partition if each $V_i$ is dominated by some vertex in $V$. For any graph $G$ on $n$ vertices with $V(G) = \{v_1, v_2, \ldots, v_n\}$, the partition $\{\{v_1\}, \{v_2\}, \ldots, \{v_n\}\}$ is a class domination partition of $G$. The class domination partition number of $G$ denoted by $\pi_{cd}(G)$ is the minimum value of $k$ for which the graph $G$ has a class domination partition of size $k$.

**Definition 5.2.2** A class domination partition is minimal if the partition obtained by forming the union of any two classes $\pi$ is no longer a class domination partition. The upper class domination partition number can be defined as the maximum value of $k$ for which the graph has a minimal class domination partition of size $k$.

**Definition 5.2.3** Let $G = (V, E)$ be a graph. Let $\pi = \{V_1, V_2, \ldots, V_n\}$ be a proper color partition of $G$. $\pi$ is called a color class domination partition of $G$ if $\forall i, 1 \leq i \leq k$, there exists a vertex $u \in V(G)$ such that $V_i$ is dominated
by $u$ or $V_i = \{u\}$.

**Remark 5.2.4** Let $V(G) = \{\{v_1\}, \{v_2\}, \ldots, \{v_n\}\}$. Then each color class $\{v_i\}$ is dominated by $\{v_i\}$.

**Definition 5.2.5** The minimum cardinality of a color class domination partition is called proper color class domination partition number of $G$ and is denoted by $\chi_{cd}(G)$.

**Remark 5.2.6** If $u$ is an isolate of $G$, then $\{u\}$ is a color class in any color class domination partition. Hence, $\chi_{cd}(G) = \chi_{cd}(H) + k$, where $k$ is the number of isolates of $G$ and $H = G - \{\text{isolates of } G\}$. So, it is enough if we consider graphs without isolates.

**Remark 5.2.7** Let $G$ be a graph with components $G_1, G_2, \ldots, G_k$. Then $\chi_{cd}(G) = \sum_{i=1}^{k} \chi_{cd}(G_i)$. Hence, it is enough to consider connected graphs for the study of color class domination.

**Remark 5.2.8** Let $G_1, G_2$ be two simple graphs. Let $G = G_1 + G_2$. Then $\chi_{cd}(G) = \chi(G_1) + \chi(G_2)$.
Remark 5.2.9 A dominator color partition of $G$ (i.e. a proper color partition of $G$ in which every vertex dominates a color class) need not be a proper color class domination partition of $G$.

Example 5.2.10

Then $\pi = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5, 6\}\}$ is a dominator color partition of $G$, but not a proper color class domination partition of $G$.

Consider $\pi' = \{\{1, 3, 5\}, \{2, 4, 6\}\}$. $\pi'$ is not a dominator coloring, but $\pi$ is a proper color class domination partition of $G$.

Here $\chi_{cd}(G) = 2; \chi_d(G) = 4. \pi' = \{\{1, 5\}, \{2\}, \{3\}, \{4, 6\}\}$ is a minimum dominator color partition of $G$.

Remark 5.2.11 $\max\{\chi(G), \gamma(G)\} \leq \chi_{cd}(G)$.
Remark 5.2.12 There is no relation between $\chi_d(G)$ and $\chi_{cd}(G)$.

Example 5.2.13 In $D_{r,s}$, $\chi_{cd}(G) = 2 < \chi_d(G) = 3$.

Example 5.2.14 $\chi_{cd}(C_{20}) = 10 > \chi_d(C_{20}) = 8$, since $\chi_d(C_n) \equiv \left\lceil \frac{n}{3} \right\rceil + 2$.

Remark 5.2.15 If a graph $G$ has a full degree vertex, then

$\chi(G) = \chi_{cd}(G) = \chi_d(G)$.

5.3 Regular Class Domination Partition

Definition 5.3.1 Let $G = (V, E)$ be a simple graph. Let $\pi = \{V_1, \cdots, V_k\}$ be a partition of $V(G)$ where each $V_i$ is regular and for every $V_i$, there exists a vertex $u_i \in V(G)$ such that $u_i$ is adjacent to every vertex of $V_i, 1 \leq i \leq k$. $\pi$ is called a regular class domination partition of $G$. $\pi = \{\{u_1\}, \{u_2\}, \cdots, \{u_n\}\}$ where $V(G) = \{u_1, u_2, \cdots, u_n\}$ is a regular partition in which each class is dominates by a vertex. The minimum cardinality of a regular class domination partition of $G$ is called the regular class domination number of $G$ and is denoted by $\chi_{cd^r}(G)$. Since any color class domination partition is a regular class dominates partition, we get that $\chi_{cd^r}(G) \leq \chi_{cd}(G)$. 

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\( \chi_{cd}^r(G) \) for standard graphs

1. \( \chi_{cd}^r(K_n) = 2 \)

2. \( \chi_{cd}^r(K_{1,n}) = 2 \)

3. \( \chi_{cd}^r(K_{m,n}) = 2 \)

4. \( \chi_{cd}^r(P_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil, & \text{if } n \equiv 0, 3(\text{mod}4) \\ \left\lfloor \frac{n}{2} \right\rfloor + 1, & \text{if } n \equiv 1, 2(\text{mod}4) \end{cases} \)

5. \( \chi_{cd}^r(C_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil, & \text{if } n \equiv 0, 3(\text{mod}4) \\ \left\lfloor \frac{n}{2} \right\rfloor + 1, & \text{if } n \equiv 1, 2(\text{mod}4) \end{cases} \)

**Definition 5.3.2** If \( u \) is an isolate of \( G \), then \( \{u\} \) is a class in any regular class dominating partition. Therefore, \( \chi_{cd}^r(G) = \chi_{cd}^r(H) + k \), where \( k \) is the number of isolates of \( G \) and \( H \) is the subgraph of \( G \) obtained by removing all the isolates of \( G \). Hence it is enough if we consider graphs without isolates.

**Remark 5.3.3** \( \chi_{cd}^r(K_n) = 2 < \chi_{cd}(K_n) = n \). Given any positive integer \( k \), there exists a connected graph \( G \) such that \( \chi_{cd}(G) - \chi_{cd}^r(G) = k \) (Take \( G = K_{k+2} \)).

**Observation 5.3.4** There is no relationship between \( \chi_{cd}^r(G) \) and \( \chi_{dr}^r(G) \).
Example 5.3.5

\[ \chi_d^r(G) = 4; \chi_{cd}^r(G) = 2. \{\{1, 5\}, \{4, 6\}, \{2\}, \{3\}\} \text{ is a minimum regular dominator partition of } G. \{\{1, 3, 5\}, \{2, 4, 6\}\} \text{ is a minimum regular class domination partition of } G. \chi_{cd}^r(G) < \chi_d^r(G). \]

Example 5.3.6 \( \chi_d^r(K_n) = \chi_{cd}^r(K_n) = 2. \)

Example 5.3.7 \( \chi_{cd}^r(C_{20}) = 10 < \chi_d^r(C_{20}) (\text{since } \chi_d^r(C_{20}) \leq 8). \)

Observation 5.3.8 If \( G \) has a full degree vertex, then \( \chi(G) = \chi_{cd}(G) = \chi_d(G). \) But \( \chi_r(G) \) need not be equal to \( \chi_{cd}^r(G). \)

Example 5.3.9

\[ \chi_d^r(G) = 4; \chi_{cd}^r(G) = 2. \{\{1, 5\}, \{4, 6\}, \{2\}, \{3\}\} \text{ is a minimum regular dominator partition of } G. \{\{1, 3, 5\}, \{2, 4, 6\}\} \text{ is a minimum regular class domination partition of } G. \chi_{cd}^r(G) < \chi_d^r(G). \]

\[ \chi_d^r(K_n) = \chi_{cd}^r(K_n) = 2. \]

\[ \chi_{cd}^r(C_{20}) = 10 < \chi_d^r(C_{20}) (\text{since } \chi_d^r(C_{20}) \leq 8). \]

\[ \text{Observation 5.3.8 If } G \text{ has a full degree vertex, then } \chi(G) = \chi_{cd}(G) = \chi_d(G). \text{ But } \chi_r(G) \text{ need not be equal to } \chi_{cd}^r(G). \]

\[ \chi_d^r(G) = 4; \chi_{cd}^r(G) = 2. \{\{1, 5\}, \{4, 6\}, \{2\}, \{3\}\} \text{ is a minimum regular dominator partition of } G. \{\{1, 3, 5\}, \{2, 4, 6\}\} \text{ is a minimum regular class domination partition of } G. \chi_{cd}^r(G) < \chi_d^r(G). \]
Remark 5.3.10 If $G$ has a full degree vertex, then $\chi_r(G) \leq \chi_{cd}(G) = \chi_{cd}^r(G) \leq \chi_r(G) + 1$.

Proof: Let $u$ be a full degree vertex of $G$.

Case(i) There exists a $\chi_r$-partition in which the regular class containing $u$ is dominated by a vertex not in the class containing $u$. Let $\pi = \{V_1, V_2, \cdots, V_{\chi_r(G)}\}$ be such a partition. Let without loss of generality $u \in V_1$. Then by hypothesis, $V_1$ is dominated by some vertex $v \notin V_1$ and $V_2, V_3, \cdots, V_{\chi_r(G)}$ are dominated by $u$. Therefore, $\pi$ is a regular class domination partition of $G$. That is $\chi_{cd}(G) \leq |\pi| = \chi_r(G)$. But $\chi_r(G) \leq \chi_{cd}(G)$. Therefore, $\chi_{cd}(G) = \chi_r(G)$.

Case(ii) There exists a $\chi_r(G)$-partition in which $\{u\}$ is a class of the partition. Then it is clear that $\chi_r(G) = \chi_{cd}(G) = \chi_d^r(G)$.

Case(iii) There exists no $\chi_r$-partition such that the class containing $u$ is dominated by a vertex not belonging to the class. Let $\pi = \{V_1, \cdots, V_{\chi_r(G)}\}$ be a $\chi_r$-partition of $G$ and let $u \in V_1$. Then $\pi = \{V_1 - \{u\}, \{u\}, V_2, \cdots, V_{\chi_r(G)}\}$ is a regular class domination partition partition as well as regular dominator partition. Therefore, $\chi_{cd}(G) = \chi_d^r(G) = \chi_r(G) + 1$.  

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Remark 5.3.11  It may happen that $\chi_r(G) = \chi_d^r(G) = \chi_{cd^r}(G)$ even if $G$
has no full degree vertex.

Example 5.3.12  In $G = C_4$, $\chi_d(G) = \chi_{cd^r}(G) = \chi_d^r(G) = 2$.

Result 5.3.13  $\left\lceil \frac{n}{\Delta(G)} \right\rceil \leq \left\lceil \frac{n}{\Delta_r(G)} \right\rceil \leq \chi_{cd^r}(G)$.

Proof: Let $\pi = \{V_1, V_2, \cdots, V_{\chi_{cd^r}(G)}\}$ be a minimum regular class domination partition of $G$. Since each element of $\pi$ is dominated by a vertex, $|V_i| \leq \Delta_r(G)$, for all $i$. Therefore, $n = \sum_{i=1}^{\chi_{cd^r}(G)} |V_i| \leq \chi_{cd^r}(G)\Delta_r(G)$. Hence the result.

Remark 5.3.14  The bound is sharp as seen in $K_n$, $(\Delta_r(K_n) = n - 1$ and $\chi_{cd^r}(K_n) = 2$).

Result 5.3.15  $\max\{\gamma(G), \chi_r(G)\} \leq \chi_{cd^r}(G)$.

Proof: Let $\pi = \{V_1, V_2, \cdots, V_{\chi_{cd^r}(G)}\}$ be a minimum regular class domination partition of $G$. Choose $x_i \in V$, such that $V_i$ is dominated by $x_i$, $1 \leq i \leq \chi_{cd^r}(G)$. Let $S = \{x_1, x_2, \cdots, \chi_{cd^r}(G)\}$. Let $v \in (V - S)$. Then $v \in V_j$ for some $j$, $1 \leq j \leq \chi_{cd^r}(G)$. $V_j$ is dominated by $x_j$. Therefore, $v$ is adjacent to $x_j$. 

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Therefore, $S$ is a dominating set of $G$. Therefore, $\gamma(G) \leq |S| = \chi_{cdr}(G)$.

Also, $\chi_r(G) \leq \chi_{cdr}(G)$. Hence $\max\{\gamma(G), \chi_r(G)\} \leq \chi_{cdr}(G)$. ■

Remark 5.3.16 $\chi_{cdr}(G)$ need not be less than or equal to $\gamma_r(G) + \chi_r(G)$.

Example 5.3.17 $\chi_{cdr}(C_{20}) = 10$, $\gamma_r(C_{20}) = 7$, $\chi_r(C_{20}) = 2$. Therefore, $\chi_{cdr}(C_{20}) > \gamma_r(C_{20}) + \chi_r(C_{20})$.

Result 5.3.18 Let $G$ be a bi-partite graph without isolates. Then $\gamma(G) = \chi_{cdr}(G)$.

Proof: Let $\{u_1, u_2, \ldots, u_r\}$ be a $\gamma$-set of $G$. Let $(X, Y)$ be the bi-partition of $G$. Without loss of generality, let $u_1, u_2, \ldots, u_k \in X$ and $u_{k+1}, \ldots, u_r \in Y$.

Let $V_i = N(u_i) - \bigcup_{j=1}^{i-1} N(u_j)$, $1 \leq i \leq r$. Clearly, $V_i$'s are independent and disjoint and $V = V_1 \cup V_2 \cup \cdots \cup V_r$. Each $V_i$ is dominated by $u_i$. Therefore, $\pi = \{V_1, \cdots, V_r\}$ is a regular class domination partition of $G$. Therefore, $\chi_{cdr}(G) \leq |\pi| = r = \gamma(G)$. But $\gamma(G) \leq \chi_{cdr}(G)$. Hence the result. ■

Corollary 5.3.19 For any tree $T$, $\chi_{cdr}(T) = \gamma(T)$.

Theorem 5.3.20 Let $G$ be a graph without isolates. $\chi_{cdr}(G) \leq n - \omega(G) + 2$. 139
**Proof:** Let $S$ be a maximum clique of $G$. Let $u \in S$. Let $\pi = \{V_1, V_2, \ldots, V_k\}$ where $V_1 = S - \{u\}$, $V_2 = \{u\}$, $V_3, \ldots, V_t$ are singletons from $(V - S)$ such that $t - 2 = n - \omega(G)$. Clearly $\pi$ is a regular class domination partition of $G$. Therefore, $\chi_{cd^r}(G) \leq n - \omega(G) + 2$. 

**Remark 5.3.21** Given a positive integer $k \geq 2$, there exists a connected graph $G$ with $\chi_{cd^r}(G) = k$.

**Proof:** Let $H = K_2 \cup K_3 \cup \cdots \cup K_{k-1}$. Let $G$ be the graph obtained from $H$ by adding a new vertex $u$ and joining $u$ with exactly one vertex of each of the component of $H$. Then $\chi_{cd^r}(G) = k$. The upper bound is reached in $K_n$. 

**Illustration 5.3.22**

![Illustration of a graph](image)

$k = 5$. $\pi = \{\{u\}, \{1, 4, 8\}, \{2, 3\}, \{5, 6, 7\}, \{9, 10, 11, 12\}\}$ is a minimum regular class dominator partition of $G$. 

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