CHAPTER 5

INVENTORY SYSTEM WITH SERVICE FACILITY AND
FINITE SOURCE

5.1 Introduction

In the previous two chapters, we assumed the customers are generated from an infinite source. In many practical situations, it is important to take into account the fact that the rate of generation of new customers decreases as the number of customers in the system increases. This can also be executed with the help of finite source queueing systems. Usually, queues with finite sources are modeled by assuming quasi-random inputs. Retrial queueing systems with finite source of customers can be observed in telephone registration systems at several universities, at call centers, ethernet systems, mobile communications, magnetic disk memory systems, local-area networks, etc., For example, in the ethernet system, the number of terminals are finite and usually not large enough to approximate the system with infinite source models. Kornyshev [1969], was the first paper devoted to finite source retrial queues, there has been a rapid growth in the literature of this topic. Alfa and Isotupa [2004] analyzed the $M/PH/k$ retrial queue with a finite number of sources.

An inventory modeling with the continuous review finite-source inventory system, the first paper was by Sivakumar [2009], which assumed the exponential life time for the items, exponential lead time for the supply of the ordered items and exponential retrial rate for the customers in the orbit. Yadavalli et al. [2012] considered multi-server inventory system with service facility in finite source. Shophia Lawrence et al. [2013] discussed perishable inventory system with service facility and finite source, they have
assumed the service time and lead time have $PH$ distribution. In this chapter, we assume finite source inventory system with service facility and feedback customers.

We organize this chapter as follows: In section 5.2, the mathematical model is described. The analysis of the model made in section 5.3. The steady state analysis of the model is presented in section 5.4. In section 5.5, we derive various system performance measures under the steady state and we calculate the total expected cost rate in the steady state case. The convexity of the total expected cost rate is shown in a numerical example in section 5.6.

### 5.2 Mathematical Modelling

Consider a finite source inventory system with single server in which the demanded items are delivered after some random time due to the service. The maximum capacity of the inventory is $S$. The arrival of customers are generated by the finite number $(N)$ of homogeneous sources. The arrival process follows quasi-random distribution with parameter $\alpha$. That is, the probability that any particular source generates a demand in any interval $(t, t+dt)$ is $\alpha dt + o(dt)$ as $dt \to 0$. We assume the capacity of waiting hall is $N$, which includes the one who is receiving the service. Customer are served under the first come first served (FCFS) discipline. Service time for primary customer has exponential distribution with parameter $\mu_1$. After the service completion, the customer either joins the orbit for additional service with probability $p_1$ ($0 \leq p_1 \leq 1$) or leaves the system with probability $\bar{p}_1$. An $(s,S)$ ordering policy is adopted with positive lead time. The ordering quantity $Q = S - s > s + 1$ is placed whenever the inventory level drops to $s$ and the items are received only after a random time which is distributed as exponential with parameter $\beta(> 0)$.

We assume that the customers waiting in the orbit send out a signal to the server according to independent exponential distribution with parameter $\theta(> 0)$. The retrial policy is classical policy. If the server is idle and there is no customer in the waiting hall or the inventory level is zero or both, the retrial customer enters into the service point. The retrial customer requires only the service not an item. The service time of the retrial customer is assumed to be exponential with parameter $\mu_2(> 0)$. We assume non-preemptive priority rule, (ie) the primary customer arrive during the feedback customer service, he/she will wait upto the service completion of feedback customer. After the service completion to the retrial customer, he/she either joins the orbit for additional
service or leaves the system. We assume that the orbiting customer enters into the orbit after the service completion according to Bernoulli trials with probability \( p_2 (0 \leq p_2 \leq 1) \) and with probability \( \bar{p}_2 \), the customer leave the system. In figure 5.1 represent the typical picture of the model.

![Graphical representation of the model](image)

**Figure 5.1:** Graphical representation of the model

### 5.3 Analysis

Let \( L(t), X_1(t) \) and \( X_2(t) \), respectively, denote the inventory level, the number of customers in the waiting hall and the number of customers in the orbit at time \( t \). Denote the status of the server by \( Y(t) \) and it is defined as follows:

\[
Y(t) = \begin{cases} 
0, & \text{if the server is idle} \\
1, & \text{if the server is busy serving feedback customer} \\
2, & \text{if the server is busy serving primary customer.}
\end{cases}
\]

From the assumption made on the input and output processes, it may be verified that the stochastic process \( \{(L(t), X_1(t), Y(t), X_2(t)), t \geq 0\} \) is a continuous time Markov chain.
with state space $E$, which is given by

$$
E = \{(0, i_2, 0, i_4) : i_2 \in E^0_N, i_4 \in E^0_{N-i_2}\} \cup \{(i_1, 0, 0, i_4) : i_1 \in E^1_S, i_4 \in E^0_N\}
\cup \{(i_1, i_2, 1, i_4) : i_1 \in E^0_S, i_2 \in E^0_{N-1}, i_4 \in E^0_{N-1-i_2}\} \cup \{(i_1, i_2, 2, i_4) : i_1 \in E^1_S, i_2 \in E^1_N, i_4 \in E^0_{N-i_2}\}.
$$

The infinitesimal generator $A$ of this process may be expressed conveniently as a block partitioned matrix with entries

$$
[A]_{i_1j_1} = \begin{cases}
B_1, & j_1 = i_1 - 1, \quad i_1 = 1, \\
B_2, & j_1 = i_1 - 1, \quad i_1 \in E^2_S, \\
C_0, & j_1 = i_1 + Q, \quad i_1 = 0, \\
C_1, & j_1 = i_1 + Q, \quad i_1 \in E^1_S, \\
A_0, & j_1 = i_1, \quad i_1 = 0, \\
A_1, & j_1 = i_1, \quad i_1 \in E^1_S, \\
A_2, & j_1 = i_1, \quad i_1 \in E^1_{n+1}, \\
0, & \text{otherwise}
\end{cases}
$$

$$
[C_0]_{i_2j_2} = \begin{cases}
D_{i_2}, & j_2 = i_2, \quad i_2 \in E^0_N, \\
0, & \text{otherwise}
\end{cases}
$$

$$
D_0 = \beta I_{(2N+1)}.
$$

For $i_2 \in E_N$,

$$
D_{i_2} = \begin{pmatrix} 0 & D^{(i_2)}_1 \\ D^{(i_2)}_2 & 0 \end{pmatrix} D^{(i_2)}_1 = \beta I_{(N+1-i_2)}, \quad D^{(i_2)}_2 = \beta I_{(N-i_2)};
$$

$$
C_1 = \beta I_{(N+1)^2}.
$$

For $i_2 \in E^1_N$,

$$
G_{i_2} = \begin{pmatrix} 0 & 0 \\ G^{(i_2)}_1 & 0 \end{pmatrix}, \quad \begin{bmatrix} G^{(i_2)}_1 \end{bmatrix}_{i_4j_4} = \begin{cases}
\bar{p}_1 \mu_1, & j_4 = i_4, \quad i_4 \in E^0_{N-i_2}, \\
\bar{p}_1 \mu_1, & j_4 = i_4 + 1, \quad i_4 \in E^0_{N-i_2}, \\
0, & \text{otherwise}
\end{cases}
$$

$$
[B_2]_{i_2j_2} = \begin{cases}
G_1, & j_2 = i_2 - 1, \quad i_2 = 1, \\
H_{i_2}, & j_2 = i_2 - 1, \quad i_2 \in E^2_N, \\
0, & \text{otherwise}
\end{cases}
$$

For $i_2 \in E^2_N$,

$$
H_{i_2} = \begin{pmatrix} 0 & 0 \\ 0 & H^{(i_2)}_1 \end{pmatrix}, \quad \begin{bmatrix} H^{(i_2)}_1 \end{bmatrix}_{i_4j_4} = \begin{cases}
\bar{p}_1 \mu_1, & j_4 = i_4, \quad i_4 \in E^0_{N-i_2}, \\
\bar{p}_1 \mu_1, & j_4 = i_4 + 1, \quad i_4 \in E^0_{N-i_2}, \\
0, & \text{otherwise}
\end{cases}
$$

$$
[A_0]_{i_2j_2} = \begin{cases}
J_{i_2}, & j_2 = i_2 + 1, \quad i_2 \in E^0_{N-1}, \\
K_{i_2}, & j_2 = i_2, \quad i_2 \in E^0_N, \\
0, & \text{otherwise}
\end{cases}
$$

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For \( i_2 \in E_{N-1}^0 \),

\[
J_{i_2} = \begin{pmatrix} J_{1i_2}^{(i_2)} & 0 \\ 0 & J_{2i_2}^{(i_2)} \end{pmatrix}, \quad [J_{1i_2}^{(i_2)}]_{i_4j_4} = \begin{cases} (N - i_2 - i_4)\alpha, & j_4 = i_4, \quad i_4 \in E_{N-1-i_2}^0, \\ 0, & \text{otherwise}. \end{cases}
\]

\[
[J_{2i_2}^{(i_2)}]_{i_4j_4} = \begin{cases} (N - 1 - i_2 - i_4)\alpha, & j_4 = i_4, \quad i_4 \in E_{N-2-i_2}^0, \\ 0, & \text{otherwise}. \end{cases}
\]

For \( i_2 \in E_{N-1}^0 \),

\[
K_{i_2} = \begin{pmatrix} K_{1i_2}^{(i_2)} & K_{2i_2}^{(i_2)} \\ K_{3i_2}^{(i_2)} & K_{4i_2}^{(i_2)} \end{pmatrix}, \quad [K_{1i_2}^{(i_2)}]_{i_4j_4} = \begin{cases} i_4\theta, & j_4 = i_4 - 1, \quad i_4 \in E_{N-i_2}^1, \\ 0, & \text{otherwise}. \end{cases}
\]

\[
[K_{2i_2}^{(i_2)}]_{i_4j_4} = \begin{cases} -((N - i_2)\alpha + \beta), & j_4 = i_4, \quad i_4 = 0, \\ -((N - i_2 - i_4)\alpha + i_4\theta + \beta), & j_4 = i_4, \quad i_4 \in E_{N-i_2}^1, \\ 0, & \text{otherwise}. \end{cases}
\]

\[
[K_{3i_2}^{(i_2)}]_{i_4j_4} = \begin{cases} \tilde{p}_2\mu_2, & j_4 = i_4, \quad i_4 \in E_{N-i_2}^0, \\ \tilde{p}_2\mu_2, & j_4 = i_4 + 1, \quad i_4 \in E_{N-i_2}^0, \\ 0, & \text{otherwise}. \end{cases}
\]

\[
[K_{4i_2}^{(i_2)}]_{i_4j_4} = \begin{cases} -((N - 1 - i_2 - i_4)\alpha + \mu_2 + \beta), & j_4 = i_4, \quad i_4 \in E_{N-i_2}^0, \\ 0, & \text{otherwise}. \end{cases}
\]

\[
K_N = -\beta.
\]

\[
[A_1]_{i_2j_2} = \begin{cases} L_{i_2}, & j_2 = i_2 + 1, \quad i_2 \in E_{N-1}^0, \\ K_{i_2}, & j_2 = i_2, \quad i_2 = 0, \\ M_{i_2}, & j_2 = i_2, \quad i_2 \in E_{N-1}^0, \\ 0, & \text{otherwise}. \end{cases}
\]

\[
L_0 = \begin{pmatrix} 0 & L_{10}^0 \\ L_{20}^0 & 0 \end{pmatrix}, \quad [L_{10}^0]_{i_4j_4} = \begin{cases} (N - i_4)\alpha, & j_4 = i_4, \quad i_4 \in E_{N-1}^0, \\ 0, & \text{otherwise}. \end{cases}
\]

\[
[L_{20}^0]_{i_4j_4} = \begin{cases} (N - 1 - i_4)\alpha, & j_4 = i_4, \quad i_4 \in E_{N-2}^0, \\ 0, & \text{otherwise}. \end{cases}
\]
For \( i_2 \in E_{N-1}^1 \),

\[
L_{i_2} = \begin{pmatrix}
L_1^{(i_2)} & 0 \\
0 & L_2^{(i_2)}
\end{pmatrix}, \quad L_1^{(i_2)} = J_2^{(i_2)}, \quad L_2^{(i_2)} = J_1^{(i_2)}.
\]

For \( i_2 \in E_{N-1}^1 \),

\[
M_{i_2} = \begin{pmatrix}
M_1^{(i_2)} & M_2^{(i_2)} \\
0 & M_3^{(i_2)}
\end{pmatrix}, \quad M_2^{(i_2)} = K_3^{(i_2)}, \quad M_1^{(i_2)} = K_4^{(i_2)}.
\]

\[
[M_3^{(i_2)}]_{i_4 j_4} = \begin{cases}
-((N - i_2 - i_4) \alpha + \mu_1 + \beta), & \text{if } j_4 = i_4, \quad i_4 \in E_{N-1}^0 \\
0, & \text{otherwise}
\end{cases}
\]

\[
M_N = -(\mu_1 + \beta)
\]

\[
[A_2]_{i_2 j_2} = \begin{cases}
L_{i_2}, & j_2 = i_2 + 1, \quad i_2 \in E_{N-1}^0 \\
R_{i_2}, & j_2 = i_2, \quad i_2 = 0, \\
R_{i_2}, & j_2 = i_2, \quad i_2 \in E_N^1, \\
0, & \text{otherwise}
\end{cases}
\]

\[
R_{(0)} = \begin{pmatrix}
R_1^{(0)} & R_2^{(0)} \\
R_3^{(0)} & R_4^{(0)}
\end{pmatrix}, \quad R_2^{(0)} = K_2^{(0)}, \quad R_3^{(0)} = K_3^{(0)}
\]

\[
[R_1^{(0)}]_{i_4 j_4} = \begin{cases}
-N \alpha, & j_4 = i_4, \quad i_4 = 0, \\
-((N - i_4) \alpha + i_4 \theta), & j_4 = i_4, \quad i_4 \in E_N^1, \\
0, & \text{otherwise}
\end{cases}
\]

\[
[R_4^{(0)}]_{i_4 j_4} = \begin{cases}
-((N - 1 - i_4) \alpha + \mu_2), & j_4 = i_4, \quad i_4 \in E_{N-1}^0, \\
0, & \text{otherwise}
\end{cases}
\]

For \( i_2 \in E_{N-1}^1 \),

\[
R_{i_2} = \begin{pmatrix}
R_1^{(i_2)} & M_2^{(i_2)} \\
0 & R_2^{(i_2)}
\end{pmatrix}
\]

\[
[R_1^{(i_2)}]_{i_4 j_4} = \begin{cases}
-((N - 1 - i_2 - i_4) \alpha + \mu_2), & j_4 = i_4, \quad i_4 \in E_{N-1-i_2}^0, \\
0, & \text{otherwise}
\end{cases}
\]

\[
[R_2^{(i_2)}]_{i_4 j_4} = \begin{cases}
-((N - i_2 - i_4) \alpha + \mu_1), & j_4 = i_4, \quad i_4 \in E_{N-i_2}^1, \\
0, & \text{otherwise}
\end{cases}
\]

\[
R_N = -\mu_1.
\]

The Table 5.1, we give the dimensions of the sub matrices.
5.4 Steady State Analysis

It can be seen from the structure of $A$ that the homogeneous continuous time Markov chain

$$\{(L(t), X_1(t), Y(t), X_2(t)), \ t \geq 0\}$$

defined on the finite state space $E$ is irreducible. Hence the limiting distribution

$$\pi^{(i_1,i_2,i_3,i_4)} = \lim_{t \to \infty} Pr\{L(t) = i_1, X_1(t) = i_2, Y(t) = i_3, X_2(t) = i_4 \mid L(0), X_1(0), Y(0), X_2(0)\}$$

exists.

Let $\Pi = (\pi^{(0)}, \pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(S)})$

where

$$\pi^{(i_1)} = (\pi^{(i_1,0)}, \pi^{(i_1,1)}, \pi^{(i_1,2)}, \ldots, \pi^{(i_1,N)}); \ i_1 \in E^0_S;$$

$$\pi^{(i_1,i_2)} = \begin{cases} 
\pi^{(i_1,i_2,0)}, & i_1 = 0; \ i_2 \in E^0_N; \\
\pi^{(i_1,i_2,0)}, & i_1 \in E^1_S; \ i_2 = 0; \\
\pi^{(i_1,i_2,1)}, & i_1 \in E^0_S; \ i_2 \in E^0_N; \\
\pi^{(i_1,i_2,2)}, & i_1 \in E^1_S; \ i_2 \in E^1_S.
\end{cases}$$

$$\pi^{(i_1,i_2,i_3)} = \begin{cases} 
(\pi^{(i_1,i_2,i_3,0)}, \ldots, \pi^{(i_1,i_2,i_3,N-i_2)}), & i_1 = 0; \ i_2 \in E^0_N; \ i_3 = 0; \\
(\pi^{(i_1,i_2,i_3,0)}, \ldots, \pi^{(i_1,i_2,i_3,N)}), & i_1 \in E^0_S; \ i_2 = 0; \ i_3 = 0; \\
(\pi^{(i_1,i_2,i_3,0)}, \ldots, \pi^{(i_1,i_2,i_3,N-1-i_2)}), & i_1 \in E^0_S; \ i_2 \in E^0_{N-1}; \ i_3 = 1; \\
(\pi^{(i_1,i_2,i_3,0)}, \ldots, \pi^{(i_1,i_2,i_3,N-i_2)}), & i_1 \in E^1_S; \ i_2 \in E^1_N; \ i_3 = 2.
\end{cases}$$

Then the vector of limiting probability $\Pi$ satisfies

$$\Pi A = 0 \quad \text{and} \quad (5.1)$$

$$\Pi e = 1 \quad (5.2)$$

The equation (5.1) of the above yields the following set of equations

$$\pi^{(i)} A_0 + \pi^{(i+1)} B_1 = 0 \quad i = 0,$$

$$\pi^{(i)} A_1 + \pi^{(i+1)} B_2 = 0 \quad i \in E_1,$$

$$\pi^{(i)} A_2 + \pi^{(i+1)} B_2 = 0 \quad i \in E_{Q-1}^1,$$

$$\pi^{(0)} C_0 + \pi^{(i)} A_2 + \pi^{(i+1)} B_2 = 0 \quad i = Q,$$

$$\pi^{(i-Q)} C_1 + \pi^{(i)} A_2 + \pi^{(i+1)} B_2 = 0 \quad i \in E_{Q-1}^1,$$

$$\pi^{(i-Q)} C_1 + \pi^{(i)} A_2 = 0 \quad i = S.$$

The limiting probability distribution $\pi^{(i)}, i \in E^0_S$, can be obtained using the following algorithm.
Algorithm:
1. Solve the following system of equations to find the value of $\pi^{(Q)}$

$$
\pi^{(Q)} \left[ (-1)^m (B_2 A_2^{-1})^{(Q-(s+1))} (B_2 A_1^{-1})^s (B_1 A_0^{-1}) (C_0) + A_2 
+ \left( (-1)^m \sum_{l=0}^{s-1} (B_2 A_2^{-1})^{(S+(s-(Q+2)-l))} (B_2 A_1^{-1})^{(l+1)} (C_1 A_2^{-1}) (B_2) \right) \right] = 0
$$

and

$$
\pi^{(Q)} \left[ (-1)^m (B_2 A_2^{-1})^{(Q-(s+1))} (B_2 A_1^{-1})^s (B_1 A_0^{-1}) + \sum_{i=1}^{s} (-1)^m (B_2 A_2^{-1})^{(Q-(s+1))} (B_2 A_1^{-1})^{(s+1-i)} 
+ \sum_{i=s+1}^{Q-1} (-1)^m (B_2 A_2^{-1})^{(Q-i)} + I 
+ \left( \sum_{i=Q+1}^{S} (-1)^{2Q+1-i} \sum_{l=0}^{s-1} (B_2 A_2^{-1})^{(S+(s-(i+1)-l))} (B_2 A_1^{-1})^{(l+1)} (C_1 A_2^{-1}) \right) \right] e = 1
$$

2. Compute the values of

$$
\Omega_i = \begin{cases} 
(-1)^m (B_2 A_2^{-1})^{(Q-(s+1))} (B_2 A_1^{-1})^s (B_1 A_0^{-1}), & i = 0, \\
(-1)^m (B_2 A_2^{-1})^{(Q-(s+1))} (B_2 A_1^{-1})^{((s+1)-i)}, & i \in E_s, \\
(-1)^m (B_2 A_2^{-1})^{(Q-i)}, & i \in E^{s+1}_{Q-1}, \\
I, & i = Q, \\
\left( (-1)^{2Q+1-i} \sum_{l=0}^{s-1} (B_2 A_2^{-1})^{(S+(s-(i+1)-l))} (B_2 A_1^{-1})^{(l+1)} (C_1 A_2^{-1}) \right), & i \in E^{Q+1}_S. 
\end{cases}
$$

3. Using $\pi^{(Q)}$ and $\Omega_i, i \in E^0_S$ calculate the value of $\pi^{(i)}, i \in E^0_S$. That is,

$$
\pi^{(i)} = \pi^{(Q)} \Omega_i, \quad i \in E^0_S.
$$

5.5 System Performance Measures

In this section, we derive some performance measures of the system. Using these measures, we can construct the total expected cost per unit time.

5.5.1 Expected Inventory Level

Let $\eta_i$ denote the expected inventory level in the steady state. Since $\pi^{(i)}$ is the steady state probability vector that there are $i$ items in the inventory with each component
represents a particular combination of the number of customers in the system, the status of the server and the number of customers in the orbit, \( \pi^{(i_1)} e \) gives the probability of \( i_1 \) item in the inventory in the steady state. Hence \( \eta_I \) is given by

\[
\eta_I = \sum_{i_1=1}^{S} i_1 \pi^{(i_1)} e.
\]

5.5.2 Expected Reorder Rate

Let \( \eta_R \) denote the expected reorder rate in the steady state. A reorder is placed when the inventory level drops from \( s + 1 \) to \( s \). It may occur when the inventory level is \( s + 1 \) and the server completes a service for a primary customer. Hence we get

\[
\eta_R = \mu_1 \sum_{i_2=1}^{N} \pi^{(s+1,i_2,2)} e.
\]

5.5.3 Expected Queue Length

Let \( \eta_{EP} \) be the expected queue length for primary customers in the waiting hall. It is given by

\[
\eta_{EP} = \sum_{i_2=1}^{N} i_2 \pi^{(0,i_2,0)} e + \sum_{i_1=0}^{S} \sum_{i_2=1}^{N-1} i_2 \pi^{(i_1,i_2,1)} e + \sum_{i_1=1}^{S} \sum_{i_2=1}^{N} (i_2 - 1) \pi^{(i_1,i_2,2)} e.
\]

Let \( \eta_{EF} \) be the expected queue length for feedback customers in the orbit. We have

\[
\eta_{EF} = \sum_{i_2=0}^{N-1} \sum_{i_4=1}^{N-i_2} i_4 \pi^{(0,i_2,0,i_4)} + \sum_{i_1=1}^{S} \sum_{i_4=1}^{N} i_4 \pi^{(i_1,0,0,i_4)} + \sum_{i_1=0}^{S} \sum_{i_2=0}^{N-2} \sum_{i_4=1}^{N-1-i_2} i_4 \pi^{(i_1,i_2,1,i_4)} + \sum_{i_1=1}^{S} \sum_{i_2=1}^{N-1} \sum_{i_4=1}^{N-i_2} i_4 \pi^{(i_1,i_2,2,i_4)}.
\]

5.5.4 The Overall Rate of Retrials

Let \( \eta_{OR} \) denote the over all retrial rate for the feedback customers in the orbit in the steady state. We get

\[
\eta_{OR} = \sum_{i_2=0}^{N-1} \sum_{i_4=1}^{N-i_2} i_4 \theta \pi^{(0,i_2,0,i_4)} + \sum_{i_1=1}^{S} \sum_{i_4=1}^{N} i_4 \theta \pi^{(i_1,0,0,i_4)} + \sum_{i_1=0}^{S} \sum_{i_2=0}^{N-2} \sum_{i_4=1}^{N-1-i_2} i_4 \theta \pi^{(i_1,i_2,1,i_4)} + \sum_{i_1=1}^{S} \sum_{i_2=1}^{N-1} \sum_{i_4=1}^{N-i_2} i_4 \theta \pi^{(i_1,i_2,2,i_4)}.
\]
5.5.5 The Successful Rate of Retrials

Let $\eta_{SR}$ denote the successful rate of retrial in the steady state. Hence we get

$$\eta_{SR} = \sum_{i_2=0}^{N-1} \sum_{i_4=1}^{N-i_2} i_4 \theta \pi^{(i_2, 0, i_4)} + \sum_{i_1=1}^{S} \sum_{i_4=1}^{N} i_4 \theta \pi^{(i_1, 0, i_4)}.$$ 

5.5.6 Fraction of Successful Rate of Retrials

Let $\eta_{FSR}$ denote the fraction of successful rate of retrial in the steady state. We have

$$\eta_{FSR} = \frac{\eta_{SR}}{\eta_{OR}}.$$ 

5.5.7 Expected Total Cost Rate

The long-run expected cost rate for this model is defined to be

$$TC(s, S) = c_h \eta_{I} + c_s \eta_{R} + c_{wp} \eta_{EP} + c_{wf} \eta_{EF}$$

where

- $c_h$ — Inventory carrying cost per unit per unit time
- $c_s$ — Setup cost per order
- $c_{wp}$ — Waiting cost of a primary customer per unit time
- $c_{wf}$ — Waiting cost of a feedback customer per unit time

Substituting the values of $\eta$’s, we get

$$TC(s, S) = c_h \sum_{i_1=1}^{S} i_1 \pi^{(i_1)} e + c_s \mu_1 \sum_{i_2=1}^{N} \pi^{(s+1, i_2, 2)} e$$

$$+ c_{wp} \sum_{i_2=1}^{N-1} N-i_2 \sum_{i_4=1}^{i_2} i_4 \pi^{(0, i_2, i_4)} e + \sum_{i_1=1}^{S} \sum_{i_2=1}^{N-1} i_2 \pi^{(i_1, i_2, 1)} e$$

$$+ c_{wp} \sum_{i_1=1}^{S} \sum_{i_2=0}^{N-2} N-i_2 \sum_{i_4=1}^{N-i_2} i_4 \pi^{(0, i_2, i_4)} e + \sum_{i_1=1}^{S} \sum_{i_2=1}^{N-1} \sum_{i_4=1}^{(i_2-1)} i_4 \pi^{(i_1, i_2, 1, i_4)} e$$

$$+ \sum_{i_1=1}^{S} \sum_{i_2=1}^{N-1} \sum_{i_4=1}^{(i_2, 2, i_4)} i_4 \pi^{(i_1, i_2, 2, i_4)} e$$

$$+ \sum_{i_1=1}^{S} \sum_{i_2=1}^{N-1} \sum_{i_4=1}^{N-i_2} i_4 \pi^{(i_1, i_2, 2, i_4)} e e$$

$$+ \sum_{i_1=1}^{S} \sum_{i_2=1}^{N-1} \sum_{i_4=1}^{N-i_2} i_4 \pi^{(i_1, i_2, 2, i_4)} e e$$

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5.6 Numerical Illustrations

A three dimensional plot of $TC(s, S)$ is presented in Figure 5.2. We use simple numerical search procedure to get the optimal values of $TC, S$ and $s$ (say $TC^*, S^*$ and $s^*$ respectively). The minimum value of $TC = 2.715910$ is obtained at $(S^*, s^*) = (34, 5)$ for fixed $N = 10; \alpha = 2.5; \theta = 3; \mu_1 = 5; \beta = 1; \mu_2 = 4; p_1 = 0.55; p_2 = 0.7; c_h = 0.02; c_s = 5; c_{wp} = 4; c_{wf} = 0.1$.

Figure 5.2: A three dimensional plot of total cost rate per unit time
Table 5.1: The submatrices and their dimensions

<table>
<thead>
<tr>
<th>Matrices</th>
<th>Dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_0, C_1, B_1, B_2, A_0, A_1, A_2$</td>
<td>$(N + 1)^2 \times (N + 1)^2$</td>
</tr>
<tr>
<td>$D_{i_2}$, $i_2 \in E_N^0$, $K_{i_2}$, $i_2 \in E_{N-1}^0$, $M_{i_2}$, $i_2 \in E_{N-1}^0$, $R_{i_2}$, $i_2 \in E_{N-1}^0$</td>
<td>$2(N + 1 - i_2) - 1 \times 2(N + 1 - i_2) - 1$</td>
</tr>
<tr>
<td>$G_{i_2}$, $i_2 \in E_N^0$, $H_{i_2}$, $i_2 \in E_N^0$</td>
<td>$2(N + 1 - i_2) - 1 \times 2(N + 1 - (i_2 - 1)) - 1$</td>
</tr>
<tr>
<td>$J_{i_2}$, $i_2 \in E_{N-1}^0$, $L_{i_2}$, $i_2 \in E_{N-1}^0$</td>
<td>$2(N + 1 - i_2) - 1 \times 2(N + 1 - (i_2 + 1)) - 1$</td>
</tr>
<tr>
<td>$D_{i_2}^{(1)}, i_2 \in E_{N-1}^0$, $K_{i_2}^{(1)}, i_2 \in E_{N-1}^0$, $M_{i_2}^{(1)}$, $i_2 \in E_{N-1}^0$, $R_{i_2}^{(1)}$, $i_2 \in E_{N-1}^0$</td>
<td>$(N + 1 - i_2) \times (N + 1 - i_2)$</td>
</tr>
<tr>
<td>$D_{i_2}^{(2)}, i_2 \in E_{N-1}^0$, $K_{i_2}^{(2)}, i_2 \in E_{N-1}^0$, $M_{i_2}^{(2)}$, $i_2 \in E_{N-1}^0$, $R_{i_2}^{(2)}$, $i_2 \in E_{N-1}^0$</td>
<td>$(N - i_2) \times (N - i_2)$</td>
</tr>
<tr>
<td>$G_{i_2}^{(1)}, i_2 \in E_N^1$, $H_{i_2}^{(1)}, i_2 \in E_N^2$</td>
<td>$(N + 1 - i_2) \times (N + 1 - (i_2 - 1))$</td>
</tr>
<tr>
<td>$J_{i_2}^{(1)}, i_2 \in E_{N-1}^0$, $L_{i_2}^{(1)}$, $i_2 = 0$, $L_{i_2}^{(2)}, i_2 \in E_{N-1}^1$, $K_{i_2}^{(2)}, i_2 \in E_{N-1}^0$</td>
<td>$(N + 1 - i_2) \times (N - i_2)$</td>
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<tr>
<td>$J_{i_2}^{(2)}, i_2 \in E_{N-1}^0$, $L_{i_2}^{(2)}$, $i_2 = 0$, $L_{i_2}^{(1)}$, $i_2 \in E_{N-1}^1$, $K_{i_2}^{(1)}$, $i_2 \in E_{N-1}^0$</td>
<td>$(N - i_2) \times (N - 1 - i_2)$</td>
</tr>
<tr>
<td>$K_{i_2}^{(1)}, i_2 \in E_{N-1}^0$, $M_{i_2}^{(1)}, i_2 \in E_{N-1}^1$</td>
<td>$(N - i_2) \times (N + 1 - i_2)$</td>
</tr>
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</table>