CHAPTER 3

INVENTORY SYSTEM WITH SERVICE FACILITY AND FINITE ORBIT SIZE

3.1 Introduction

In the literature on inventory system with service facility, all the models studies so far assume the customer leaves the system after completion of service. A detailed survey of the existing literature is provided in Chapter 1. But in many real life problems, the customer requests additional service on their item after a random time. For example, if a customer buy an electronic good such as air conditioner, personal computer etc., he may choose maintenance contracts (MC) or he may not choose MC. For those customer choosing MC may come after a random time for service. In queueing theory these customers are called feedback customers. The concept of feedback customer with queue can be extensively analysed by many researchers in the past and a detailed survey of the existing literature is given in Chapter 1. To the best of our knowledge, we have not found any literature studying an inventory system maintained at service facility with feedback customers.

In this chapter, we introduce the concept of feedback queue in the inventory system with service facility and derive the steady state probability vector. Using the steady state probability vector, we derive some useful system performance measures and calculated the total expected cost rate. We also derive the Laplace-Stieltjes transform for the waiting time distribution of both primary and feedback customers. The rest of the chapter is organized as follows. In the next section, we describe the mathematical model. Analysis of the model is presented in the section 3.3. The section 3.4 presents the steady state distribution of the inventory level and expresses it in a matrix-geometric form. In section
3.5, we have calculated some system performance measures and using a cost structure, we have constructed the total expected cost rate. In section 3.6, we have calculated the Laplace-Stieltjes transform of waiting time distribution of customers in the waiting hall as well as in the orbit. In the final section, we present some numerical examples to illustrate the effect of the system parameters and the costs on the optimal costs and on the expected waiting time of the primary as well as the feedback customer.

3.2 Model Description

We consider an inventory system with a maximum capacity $S$ units at a service facility. The customer arrives at the service facility one at a time, and the arrival time points form a Poisson process with rate $\lambda (> 0)$. We call these customers as primary customers. The primary customer demands single item which is delivered to him after a random time of service which is assumed to be exponential with parameter $\mu_1 (> 0)$. An order for $Q = S - s > s + 1$ items is placed whenever the inventory level drops to $s$ and the items are received only after a random time which has exponential distribution with parameter $\beta (> 0)$. The maximum number of the primary customers allowed to wait in the system is fixed as $N$ (which includes the customer (primary) one who is receiving service). An arriving primary customer who finds the waiting hall is full is considered to be lost. After receiving the demanded item, the primary customer will decide either to join the retrial group (called orbit) for additional service with probability $\bar{p}$ ($0 \leq p \leq 1$) or he leaves the system with probability $p$. The maximum capacity of the orbit is finite and is fixed as $M$. Hence, a customer who wishes to join to orbit finds there are already $M$ customers in the orbit is also considered to be lost. We assume that the customers waiting in the orbit send out a signal to the server, the retrial customer enters into the service point. The retrial rate follows exponential distribution with parameter $\theta (> 0)$. The orbit customer joins the server if there is no primary customers in the system or the inventory level is zero or both. The retrial customer doesn’t demand item. They required only the service. The service time of the retrial customer is assumed to be exponential with parameter $\mu_2 (> 0)$. We assume non - preemptive priority rule. That is the primary customer arrive during the feedback customer service, he/she will wait upto the service completion of feedback customer. After the service completion to the retrial customer, he/she will decide whether to join the orbit for additional service with probability $\bar{q}$ ($0 \leq q \leq 1$) or leave the system with probability $q$. The figure (3.1) shows the typical picture of the model.
Further, we assume that the inter arrival time between two successive primary customers, retrial times, service time for primary and feedback customers, replenishment time of reorders are mutually independent.

Figure 3.1: Graphical representation of the model

### 3.3 Analysis

Let $L(t), X_1(t)$ and $X_2(t)$, respectively, denote the inventory level, the number of customers in the waiting hall and the number of customers in the orbit at time $t$.

Further, let the status of the server $Y(t)$ be defined as follows:

$$ Y(t) = \begin{cases} 
0 & \text{if the server is idle} \\
1 & \text{if the server is busy serving feedback customer} \\
2 & \text{if the server is busy serving primary customer.}
\end{cases} $$

From the assumptions made on the input and output processes, it may be shown that the quadruplet $\{(L(t), X_1(t), X_2(t), Y(t)), t \geq 0\}$ is a continuous time Markov chain with
state space given by

$$\Omega = \{(0,k_2,k_3,0) : k_2 = 0,1,\ldots,N; k_3 = 0,1,\ldots,M\}$$

$$\cup\{(k_1,0,k_3,0) : k_1 = 1,2,\ldots,S; k_3 = 0,1,2,\ldots,M\}$$

$$\cup\{(k_1,k_2,k_3,1) : k_1 = 0,1,\ldots,S; k_2 = 0,1,2,\ldots,N; k_3 = 0,1,\ldots,M\}$$

$$\cup\{(k_1,k_2,k_3,2) : k_1 = 1,2,\ldots,S; k_2 = 1,2,\ldots,N; k_3 = 0,1,\ldots,M\}$$

Let

$$<i,j,k> = \begin{cases} ((i,j,k,0),(i,j,k,1)), & i = 0; j = 0,1,\ldots,N; k = 0,1,\ldots,M; \\ ((i,j,k,0),(i,j,k,1)), & i = 0,1,\ldots; j = 0; k = 0,1,\ldots,M; \\ ((i,j,k),(i,j,k,2)), & i = 0,1,\ldots; j = 1,2,\ldots,N; k = 0,1,\ldots,M; \end{cases}$$

$$\ll i,j\gg = (<i,j,0>,<i,j,1>,\ldots,<i,j,M>), \quad i = 0,1,\ldots,S; \quad j = 0,1,\ldots,N;$$

$$\ll i\gg = (\ll 0\gg,\ll 1\gg,\ldots,\ll N\gg), \quad i = 0,1,\ldots,S.$$

By ordering the state space ($\ll 0\gg,\ll 1\gg,\ldots,\ll S\gg$), the infinitesimal generator $A$ can be conveniently written in a block partitioned matrix with entries

$$A = \begin{pmatrix} A_{0,0} & A_{0,1} & A_{0,2} & \cdots & A_{0,S-1} & A_{0,S} \\ A_{1,0} & A_{1,1} & A_{1,2} & \cdots & A_{1,S-1} & A_{1,S} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{S-1,0} & A_{S-1,1} & A_{S-1,2} & \cdots & A_{S-1,S-1} & A_{S-1,S} \\ A_{S,0} & A_{S,1} & A_{S,2} & \cdots & A_{S,S-1} & A_{S,S} \end{pmatrix}$$

Due to the assumptions made on the demand and replenishment processes, we note that

$$A_{i,j} = 0, \quad \text{for} \quad j \neq i, i-1, i+Q.$$

We first consider the case $A_{i,i+Q}$. This will occur only when the inventory level is replenished. First we consider the inventory level is zero, that is $A_{0,Q}$. For this

**Case (i)** When there is no customer in the waiting hall, at the time of replenishment the state of the system changes from $(0,0,k,l)$ to $(Q,0,k,l)$, $k = 0,1,\ldots,M; l = 0,1,$ with intensity of transition $\beta$. The sub matrix of the transition rates from $\ll 0\gg$ to $\ll Q\gg$, is given by

$$V_0 = \beta I_{(2(M+1))}$$

**Case (ii)** When there is a customer in the waiting hall the replenishment takes the system state from $(0,j,k,0)$ to $(Q,j,k,2)$ or from $(0,j,k,1)$ to $(Q,j,k,1), j =
1, 2, \ldots, N; k = 0, 1, \ldots, M. The sub matrix of this transition rate from \( \ll 0, j \gg \) to \( \ll Q, j \gg , j = 1, 2, \ldots, N \), is given by
\[
V_1 = I_{(M+1)} \otimes J_5,
\]
\[
J_5 = \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix}.
\]

Hence
\[
A_{0,Q} = \begin{cases} 
V_0, & l = k, \quad k = 0, \\
V_1, & l = k, \quad k = 1, 2, \ldots, N, \\
0, & \text{otherwise}.
\end{cases}
\]

We denote \( A_{0,Q} \) as \( C_0 \).

We now consider the case when the inventory level lies between one to \( s \). We note that for this case, only the inventory level changes from \( i \) to \( i + Q \). The other system states do not change. Hence \( A_{i,i+Q} = \beta I_{2(M+1)(N+1)} \) and is denoted by \( C_1 \).

Next, we consider the case \( A_{i,i-1}, i = 1, 2, \ldots, S \). This will occur only when the service completion of the primary customer. For this, we have the following two cases occur:

**Case (i):** If the inventory level is one, at the time of service completion of primary customer both the inventory level and the customer level decreases by one and the server become idle. If the serviced customer wants to require additional service then he joins the orbit and hence the customer level in the orbit increases by one with intensity of transition \( \bar{p}_\mu_1 \). Otherwise there will be no change in the orbit size and hence for the intensity of passage for this transition is given by \( p_\mu_1 \).

Hence \( A_{1,0} \) is given by
\[
A_{1,0} = \begin{cases} 
U_1, & l = k - 1, \quad k = 1, 2, \ldots, N, \\
0, & \text{otherwise}.
\end{cases}
\]

\[
[U_1]_{kl} = \begin{cases} 
J_0, & l = k, \quad k = 0, 1, 2, \ldots, M, \\
J_1, & l = k + 1, \quad k = 0, 1, 2, \ldots, M - 1, \\
0, & \text{otherwise}.
\end{cases}
\]

\[
J_0 = \begin{pmatrix} 0 & 0 \\ 2 \bar{p}_\mu_1 & 0 \end{pmatrix}, \quad J_1 = \begin{pmatrix} 0 & 0 \\ 2 & \bar{p}_\mu_1 \end{pmatrix}
\]

**Case (ii):** If the inventory level is more than one and the number of customer in the buffer is one, then at the time of service completion of primary customer both the
inventory level and the customer level in the waiting hall decreases by one, and the server become idle. If there are more than one customer in the waiting hall then after a service completion server remains busy servicing a primary customer. If the serviced customer wants to require additional service, he joins the orbit and hence the customer level in the orbit increases by one with intensity of transition $\bar{p}\mu_1$. Otherwise, the customer leaves the system and hence there will be no change in the orbit size and the intensity of passage for this transition is given by $p\mu_1$. Hence $A_{i,i-1}$, $i = 2, 3, \ldots, S$, is given by

$$A_{i,i-1} = \begin{cases} U_1, & l = k - 1, \quad k = 1, \\ U_2, & l = k - 1, \quad k = 2, 3, \ldots, N, \\ 0, & \text{otherwise}. \end{cases}$$

$$[U_2]_{kl} = \begin{cases} J_2, & l = k, \quad k = 0, 1, \ldots, M, \\ J_3, & l = k, \quad k = 0, 1, \ldots, M - 1, \\ 0, & \text{otherwise}. \end{cases}$$

$$J_2 = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 0 \\ 1 & \bar{p}\mu_1 \end{pmatrix}$$

We will denote $A_{i,i-1}$, $i = 2, 3, \ldots, S$, as $B_2$.

Finally, we consider the case $A_{i,i}$, $i = 0, 1, \ldots, S$. Here due to each one of the following mutually exclusive cases, a transition results:

- an arrival of a primary customer may occur
- a retrial customer enter into the service may occur
- a service of feedback customer may be completed

When the inventory level is zero we have the following four state changes may arise:

**Case(i):** an arrival of a primary customer, increases the number of customers waiting in the waiting hall by one and the state of the arrival process moves from $(0, j, k, l)$ to $(0, j+1, k, l)$, $j = 0, 1, \ldots, N-1; k = 0, 1, \ldots, M; l = 0, 1$, with intensity of transition $\lambda$. The sub matrix of this transition rate is given by $\ll 0, j \gg$ to $\ll 0, j + 1 \gg$ is $\lambda I_{2(M+1)}$ and is denoted by $F_0$.

**Case(ii):** a feedback customer enters the service which takes the state of the system from $(0, j, k, 0)$ to $(0, j, k - 1, 1)$, $j = 0, 1, \ldots, N; k = 1, 2, \ldots, M$, with the intensity of transition $k\theta$. 

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**Case (iii):** at the time of service completion of a feedback customer, if he decides to leave the system then the state of the system moves from \((0, j, k, 1)\) to \((0, j, k, 0)\), \(j = 0, 1, \ldots, N; k = 0, 1, \ldots, M\), with rate \(q_\mu_2\).

**Case (vi):** at the time of service completion of a feedback customer, if he decides to join the system then the state of the system moves from \((0, j, k, 1)\) to \((0, j, k, 0)\), \(j = 0, 1, \ldots, N; k = 0, 1, \ldots, M - 1\), with rate \(\bar{q}_\mu_2\).

The transition rates for any other transitions not considered above, when the inventory level is zero, are zero. The intensity of passage in the state \((0, j, k, l)\) is given by

\[
- \sum_{(0, j, k, l) \neq (0, j', k', l')} a \left((0, j, k, l); (0, j', k', l')\right).
\]

Using the above arguments, we have constructed the following matrices

\[
R_k = \begin{pmatrix} 0 & 1 \\ 0 & k\theta \\ 1 & 0 \end{pmatrix}, \quad k = 1, 2, \ldots, M, \quad P_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \bar{q}_\mu_2
\]

\[
G^{(0)}_0 = \begin{pmatrix} 0 & -(\lambda + \beta) & 0 \\ 0 & 0 & q\mu_2 \\ 1 & q\mu_2 & -d_1 \end{pmatrix}, \quad d_1 = \mu_2 + \lambda + \beta
\]

\[
G^{(0)}_k = \begin{pmatrix} 0 & -g_{1k} & 0 \\ 0 & 0 & q\mu_2 \\ 1 & q\mu_2 & -d_1 \end{pmatrix}, \quad g_{1k} = k\theta + \lambda + \beta; \quad k = 1, 2, \ldots, M - 1.
\]

\[
G^{(0)}_M = \begin{pmatrix} 0 & -g_{1M} & 0 \\ 0 & 0 & q\mu_2 \\ 1 & q\mu_2 & -d_2 \end{pmatrix}, \quad g_{1M} = M\theta + \lambda + \beta; \quad d_2 = q\mu_2 + \lambda + \beta
\]

\[
G^{(1)}_0 = \begin{pmatrix} 0 & -\beta & 0 \\ 0 & 0 & q\mu_2 \\ 1 & q\mu_2 & -d_3 \end{pmatrix}, \quad d_3 = \mu_2 + \beta
\]

\[
G^{(1)}_k = \begin{pmatrix} 0 & -g_{2k} & 0 \\ 0 & 0 & q\mu_2 \\ 1 & q\mu_2 & -d_3 \end{pmatrix}, \quad g_{2k} = k\theta + \beta; \quad k = 1, 2, \ldots, M - 1.
\]

\[
G^{(1)}_M = \begin{pmatrix} 0 & -g_{2M} & 0 \\ 0 & 0 & q\mu_2 \\ 1 & q\mu_2 & -d_4 \end{pmatrix}, \quad g_{2M} = M\theta + \beta; \quad d_4 = q\mu_2 + \beta.
\]
Combining these matrices in suitable form, we get

\[
[D_0]_{kl} = \begin{cases} 
G_k^{(0)}, & l = k, \quad k = 0, 1, 2, \ldots, M, \\
P_0, & l = k + 1, \quad k = 0, 1, \ldots, M - 1, \\
R_k, & l = k - 1, \quad k = 1, 2, \ldots, M, \\
0, & \text{otherwise.}
\end{cases}
\]

\[
[D_1]_{kl} = \begin{cases} 
P_0, & l = k + 1, \quad k = 0, 1, \ldots, M - 1, \\
R_k, & l = k - 1, \quad k = 1, 2, \ldots, M, \\
G_k^{(1)}, & l = k, \quad k = 0, 1, 2, \ldots, M, \\
0, & \text{otherwise}
\end{cases}
\]

Hence the matrix \(A_0\) is given by

\[
[A_0]_{kl} = \begin{cases} 
F_0, & l = k + 1, \quad k = 0, 1, \ldots, N - 1, \\
D_0, & l = k, \quad k = 0, 1, 2, \ldots, N - 1, \\
D_1, & l = k, \quad k = N, \\
0, & \text{otherwise.}
\end{cases}
\]

and is denoted by \(A_0\). Arguments similar to above yields

For \(i = 1, 2, \ldots, s\),

\[
[A_i]_{kl} = \begin{cases} 
F_1, & l = k + 1, \quad k = 0, \\
F_2, & l = k + 1, \quad k = 1, 2, \ldots, N - 1, \\
D_0, & l = k, \quad k = 0, \\
D_2, & l = k, \quad k = 1, 2, \ldots, N - 1, \\
D_3, & l = k, \quad k = N, \\
0, & \text{otherwise.}
\end{cases}
\]

\[
F_{11} = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}, \quad F_1 = I_{(M+1)} \otimes F_{11}, \quad F_2 = I_{(M+1)} \otimes \lambda I_2.
\]

\[
[D_2]_{kl} = \begin{cases} 
P_1, & l = k + 1, \quad k = 0, 1, 2, \ldots, M - 1, \\
G_0^{(2)}, & l = k, \quad k = 0, 1, 2, \ldots, M - 1, \\
G_1^{(2)}, & l = k, \quad k = M, \\
0, & \text{otherwise.}
\end{cases}
\]

\[
F_1 = \begin{pmatrix} 0 & \bar{q} \mu_2 \\ 2 \bar{q} \mu_2 & 0 \end{pmatrix}, \quad G_0^{(2)} = \frac{1}{2} \begin{pmatrix} -d_1 & q \mu_2 \\ 0 & -f_1 \end{pmatrix}, \quad f_1 = \mu_1 + \lambda + \beta
\]

\[
P_1 = \begin{pmatrix} 1 \\ 2 \bar{q} \mu_2 \end{pmatrix}, \quad G_1^{(2)} = \begin{pmatrix} -d_2 & q \mu_2 \\ 0 & -f_2 \end{pmatrix}, \quad f_2 = (p \mu_1 + \lambda + \beta)
\]

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\[ [D_3]_{kl} = \begin{cases} P_1, & l = k + 1, \ k = 0, 1, 2, \ldots, M - 1, \\ G_0^{(3)}, & l = k, \ k = 0, 1, 2, \ldots, M - 1, \\ G_1^{(3)}, & l = k, \ k = M, \\ 0 & \text{otherwise}. \end{cases} \]

\[ G_0^{(3)}(0) = \begin{pmatrix} \frac{-d_3}{2} & q\mu_2 \\ \frac{1}{2} & \frac{-f_3}{2} \end{pmatrix}, \ f_3 = \mu_1 + \beta; \ d_3 = \mu_2 + \beta \]

\[ G_0^{(3)}(1) = \begin{pmatrix} \frac{-d_3}{2} & q\mu_2 \\ \frac{1}{2} & \frac{-f_3}{2} \end{pmatrix}, \ f_4 = \mu_1 + \beta; \ d_4 = q\mu_2 + \beta. \]

For \( i = s + 1, s + 2, \ldots, S, \)

\[ [A_i]_{kl} = \begin{cases} F_1, & l = k + 1, \ k = 0, \\ F_2, & l = k + 1, \ k = 1, 2, \ldots, N - 1, \\ D_4, & l = k, \ k = 0, \\ D_5, & l = k, \ k = 1, 2, \ldots, N - 1, \\ D_6, & l = k, \ k = N, \\ 0 & \text{otherwise}. \end{cases} \]

\[ [D_4]_{kl} = \begin{cases} P_0, & l = k + 1, \ k = 0, 1, 2, \ldots, M - 1, \\ R_k, & l = k - 1, \ k = 1, 2, \ldots, M, \\ G_0^{(4)}, & l = k, \ k = 0, 1, 2, \ldots, M - 1, M, \\ 0 & \text{otherwise}. \end{cases} \]

\[ G_0^{(4)} = \begin{pmatrix} -\lambda & 0 \\ q\mu_2 & -(\mu_2 + \lambda) \end{pmatrix} \]

\[ G_k^{(4)} = \begin{pmatrix} -(k\theta + \lambda) & 0 \\ q\mu_2 & -(\mu_2 + \lambda) \end{pmatrix}, \ k = 1, 2, \ldots, M - 1, \]

\[ G_M^{(4)} = \begin{pmatrix} -(M\theta + \lambda) & 0 \\ q\mu_2 & -(q\mu_2 + \lambda) \end{pmatrix} \]

\[ [D_5]_{kl} = \begin{cases} P_1, & l = k + 1, \ k = 0, 1, 2, \ldots, M - 1, \\ G_0^{(5)}, & l = k, \ k = 0, 1, 2, \ldots, M - 1, \\ G_1^{(5)}, & l = k, \ k = M, \\ 0 & \text{otherwise}. \end{cases} \]

\[ G_0^{(5)} = \begin{pmatrix} -(\mu_2 + \lambda) & q\mu_2 \\ 2(0 & -(\mu_1 + \lambda)) \end{pmatrix} \]
\( G_1^{(5)} = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \) 
\( [D_6]_{kl} = \begin{cases} P, & l = k + 1, \quad k = 0, 1, 2, \ldots, M - 1, \\ G_0^{(6)}, & l = k, \quad k = 0, 1, 2, \ldots, M - 1, \\ G_1^{(6)}, & l = k, \quad k = M, \\ 0 & \text{otherwise}. \end{cases} \)

We denote \( A_{ii}, \quad i = 1, 2, \ldots, s \) as \( A_1 \) and \( A_{ii}, \quad i = s + 1, s + 2, \ldots, S \) as \( A_2 \). Hence the matrix \( A \) can be written in the following form

\[
[A]_{ij} = \begin{cases} B_1, & j = i - 1, \quad i = 1, \\ B_2, & j = i - 1, \quad i = 2, \ldots, S, \\ C_0, & j = i + Q, \quad i = 0, \\ C_1, & j = i + Q, \quad i = 1, 2, \ldots, s, \\ A_0, & j = i, \quad i = 0, \\ A_1, & j = i, \quad i = 1, 2, \ldots, s, \\ A_2, & j = i, \quad i = s + 1, s + 2, \ldots, S, \\ 0, & \text{otherwise}. \end{cases}
\]

It may be noted that the matrices \( B_1, B_2, C_0, C_1, A_0, A_1 \) and \( A_2 \) are square matrices of order \( 2(N + 1)(M + 1) \) and matrices \( U_1, U_2, D_0, D_1, D_2, D_3, D_4, D_5 \) and \( D_6 \) are square matrices of order \( 2(M + 1) \).

### 3.4 Steady State Analysis

It can be seen from the structure of \( A \) that the homogeneous continuous time Markov chain \( \{(L(t), X_1(t), X_2(t), Y(t)), t \geq 0\} \) defined on the finite state space \( E \) is irreducible. Hence the limiting distribution

\[
\pi^{(k_1, k_2, k_3)} = \lim_{t \to \infty} Pr\{L(t) = k_1, X_1(t) = k_2, X_2(t) = k_3, Y(t) = k_4 \mid L(0), X_1(0), X_2(0), Y(0)\}
\]

exists. Let

\[
\Pi = (\pi^{(0)}, \pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(S)})
\]

where

\[
\pi^{(k_1)} = (\pi^{(k_1,0)}, \pi^{(k_1,1)}, \pi^{(k_1,2)}, \ldots, \pi^{(k_1,N)}); \quad k_1 = 0, 1, \ldots, S;
\]

\[
\pi^{(k_1, k_2)} = (\pi^{(k_1, k_2,0)}, \pi^{(k_1, k_2,1)}, \ldots, \pi^{(k_1, k_2,M)}); \quad k_1 = 0, 1, \ldots, S; \quad k_2 = 0, 1, \ldots, N;
\]
Then the vector of limiting probability \( \Pi \) satisfies

\[
\Pi A = 0 \quad \text{and} \quad \Pi e = 1
\] (3.1) (3.2)

The equation (3.1) of the above yields the following set of equations

\[
\pi^{(k)} A_0 + \pi^{(k+1)} B_1 = 0 \quad k = 0, \quad (3.3)
\]

\[
\pi^{(k)} A_1 + \pi^{(k+1)} B_2 = 0 \quad k = 1, 2, \ldots, s, \quad (3.4)
\]

\[
\pi^{(k)} A_2 + \pi^{(k+1)} B_2 = 0 \quad k = s + 1, s + 2, \ldots, Q - 1, \quad (3.5)
\]

\[
\pi^{(0)} C_0 + \pi^{(k)} A_2 + \pi^{(k+1)} B_2 = 0 \quad k = Q, \quad (3.6)
\]

\[
\pi^{(k-Q)} C_1 + \pi^{(k)} A_2 + \pi^{(k+1)} B_2 = 0 \quad k = Q + 1, Q + 2, \ldots, S - 1, \quad (3.7)
\]

\[
\pi^{(k-Q)} C_1 + \pi^{(k)} A_2 = 0 \quad k = S. \quad (3.8)
\]

The limiting probability distribution \( \pi^{(i)}, i = 0, 1, \ldots, S \), can be obtained using the following algorithm. Algorithm:

1. Solve the following system of equations to find the value of \( \pi^{(Q)} \)

\[
\pi^{(Q)} \left[ (-1)^Q (B_2 A_2^{-1})^{Q-(s+1)} (B_2 A_1^{-1})^{s} (B_1 A_0^{-1}) (C_0) + A_2 
+ (-1)^Q \sum_{l=0}^{s-1} (B_2 A_2^{-1})^{(S+s-(Q+2)-l)} (B_2 A_1^{-1})^{(l+1)} (C_1 A_2^{-1}) (B_2) \right] = 0
\]

and

\[
\pi^{(Q)} \left[ (-1)^Q (B_2 A_2^{-1})^{Q-(s+1)} (B_2 A_1^{-1})^{s} (B_1 A_0^{-1}) + \sum_{k=1}^{s} (-1)^{Q-k} (B_2 A_2^{-1})^{Q-(s+1)} (B_2 A_1^{-1})^{(s+1-k)} + \sum_{k=s+1}^{Q-1} (-1)^{Q-k} (B_2 A_2^{-1})^{Q-k} + I 
+ \sum_{k=Q+1}^{S} (-1)^{Q+1-k} \sum_{l=0}^{S-k} (B_2 A_2^{-1})^{(S+s-(k+1)-l)} (B_2 A_1^{-1})^{(l+1)} (C_1 A_2^{-1}) \right] e = 1
\]

2. Compute the values of

\[
\omega_k = \begin{cases}
(-1)^Q (B_2 A_2^{-1})^{Q-(s+1)} (B_2 A_1^{-1})^{s} (B_1 A_0^{-1}), & k = 0, \\
(-1)^Q (B_2 A_2^{-1})^{Q-(s+1)} (B_2 A_1^{-1})^{(s+1-k)} - k, & k = 1, 2, \ldots, s, \\
(-1)^{Q-k} (B_2 A_2^{-1})^{Q-k}, & k = s + 1, \ldots, Q - 1, \\
I, & k = Q \\
(-1)^{Q+1-k} \sum_{l=0}^{S-k} (B_2 A_2^{-1})^{(S+s-(k+1)-l)} (B_2 A_1^{-1})^{(l+1)} (C_1 A_2^{-1}), & k = Q + 1, Q + 2, \ldots, S.
\end{cases}
\]
3. Using $\pi^{(Q)}$ and $\omega_k, k = 0, 1, \ldots, S$ calculate the value of $\pi^{(i)}$, $i = 0, 1, \ldots, S$. That is,

$$\pi^{(i)} = \pi^{(Q)}\omega_i, \quad i = 0, 1, \ldots, S.$$ 

### 3.5 System Performance Measures

In this section, we derive some measures of system performance in the steady state. Using this, we calculate the total expected cost rate.

#### 3.5.1 Expected Inventory Level

Let $\eta_I$ denote the expected inventory level in the steady state. Since $\pi^{(i_1)}$ is the steady state probability vector that there are $i_1$ items in the inventory with each component represents a particular combination of the number of customers in the system, number of customers in the orbit and the status of the server, $\pi^{(i_1)}e$ gives the probability of $i_1$ item in the inventory in the steady state. Hence $\eta_I$ is given by

$$\eta_I = \sum_{i_1=1}^{S} i_1 \pi^{(i_1)}e$$

#### 3.5.2 Expected Reorder Rate

Let $\eta_R$ denote the expected reorder rate in the steady state. A reorder is placed when the inventory level drops from $s + 1$ to $s$. It may occur when the inventory level is $s + 1$ and the server completes a service for a primary customer. Hence we get

$$\eta_R = \mu_1 \sum_{i_2=1}^{N} \sum_{i_3=0}^{M} \pi^{(s+1,i_2,i_3,2)}.$$ 

#### 3.5.3 Expected Balking Rate

Let $\eta_{BP}$ denote the expected balking rate for the primary customer in the steady state. Any arriving primary customer finds the waiting hall is full and leaves the system without getting service. These customers are considered to be lost. Thus we obtain
\[ \eta_{BP} = \sum_{i_1=0}^{S} \lambda \pi^{(i_1,N)} e \]

Let \( \eta_{BF} \) denote the expected balking rate for the feedback customer in the steady state. This may occur in the following two cases:

- after the service completion of the feedback customer, if he wishes to join the orbit and finds the orbit is full
- after the service completion of the primary customer, if he wishes to join the orbit and finds the orbit is full

These customers are consider to be lost. Hence \( \eta_{BF} \) is given by

\[ \eta_{BF} = \bar{q} \mu_2 \sum_{i_1=0}^{S} \sum_{i_2=0}^{N} \pi^{(i_1,i_2,M,1)} + \bar{p} \mu_1 \sum_{i_1=1}^{S} \sum_{i_2=1}^{N} \pi^{(i_1,i_2,M,2)}. \]

### 3.5.4 Expected Queue Length

Let \( \eta_{EP} \) be the expected queue length for primary customers in the waiting hall. It is given by

\[ \eta_{EP} = \sum_{i_2=1}^{N} \sum_{i_3=0}^{M} i_2 (\pi^{(0,i_2,i_3,0)} + \pi^{(0,i_2,i_3,1)}) + \sum_{i_2=1}^{S} \sum_{i_3=1}^{N} \sum_{i_1=1}^{M} (i_2 \pi^{(i_1,i_2,i_3,1)} + (i_2 - 1) \pi^{(i_1,i_2,i_3,2)}) \]

Let \( \eta_{EF} \) be the expected queue length for feedback customers in the orbit. We have

\[ \eta_{EF} = \sum_{i_1=0}^{S} \sum_{i_2=0}^{N} \sum_{i_3=1}^{M} i_3 \pi^{(i_1,i_2,i_3)} e \]

### 3.5.5 The Overall Rate of Retrials

Let \( \eta_{OR} \) denote the over all retrial rate for the feedback customers in the orbit in the steady state. We get
3.5.6 The Successful Rate of Retrials

Let \( \eta_{SR} \) denote the successful rate of retrial in the steady state. Hence we get

\[
\eta_{SR} = \sum_{i_2=0}^{N} \sum_{i_3=1}^{M} i_3 \theta_{\pi_{(i_1,i_2,i_3)}} + \sum_{i_1=1}^{S} \sum_{i_3=1}^{M} i_3 \theta_{\pi_{(i_1,0,i_3)}} + \sum_{i_1=0}^{S} \sum_{i_2=0}^{N} \sum_{i_3=1}^{M} i_3 \theta_{\pi_{(i_1,i_2,i_3)}}.
\]

3.5.7 Fraction of Successful Rate of Retrials

Let \( \eta_{FSR} \) denote the fraction of successful rate of retrial in the steady state. We have

\[
\eta_{FSR} = \frac{\eta_{SR}}{\eta_{OR}}.
\]

3.5.8 Expected Total Cost Rate

We assume various cost elements associated with different system performance measures are given as follows:

- \( c_h \) – Inventory carrying cost per unit per unit time
- \( c_s \) – Setup cost per order
- \( c_{b_1} \) – Cost per primary customer lost
- \( c_{b_2} \) – Cost per feedback customer lost
- \( c_{qp} \) – Waiting cost of a primary customer per unit time
- \( c_{qf} \) – Waiting cost of a feedback customer per unit time

We construct the function for the expected total cost per unit time as follows:

\[
TC(S,s,M,N) = c_h \eta_{H} + c_s \eta_{R} + c_{b_1} \eta_{BP} + c_{b_2} \eta_{BF} + c_{qp} \eta_{EP} + c_{qf} \eta_{EF}
\]
Using the values of η’s from the above measures of system performance, we obtain

\[
TC(S, s, M, N) = c_h \sum_{i_1=1}^{S} i_1 \pi^{(i_1)} e + c_s \mu_1 \sum_{i_2=1}^{N} \sum_{i_3=0}^{M} \pi^{(s+1, i_2, i_3, 2)} + c_b \sum_{i_1=0}^{S} \lambda \pi^{(i_1, N)} e \\
+ c_{b_2} \left[ \bar{q} \mu_2 \sum_{i_1=0}^{N} \sum_{i_2=0}^{M} \pi^{(i_1, i_2, M, 1)} + \rho \mu_1 \sum_{i_1=1}^{N} \sum_{i_2=1}^{M} \pi^{(i_1, i_2, M, 2)} \right] \\
+ c_{q_p} \left[ \sum_{i_2=1}^{N} \sum_{i_3=0}^{M} i_2 \pi^{(0, i_2, i_3, 0)} + \pi^{(0, i_2, i_3, 1)} \right] \\
+ c_{q_p} \left[ \sum_{i_1=1}^{N} \sum_{i_2=1}^{M} \sum_{i_3=0}^{M} i_2 \pi^{(i_1, i_2, i_3, 1)} + (i_2 - 1) \pi^{(i_1, i_2, i_3, 2)} \right] \\
+ c_{q_f} \sum_{i_1=0}^{S} \sum_{i_2=0}^{M} \sum_{i_3=1}^{M} i_3 \pi^{(i_1, i_2, i_3)} e
\]

### 3.6 Waiting Time Distribution

Generally, system performance measures in inventories are related to the availability of stock but are not customer oriented. However, inventory maintained at service facilities, queues may form and hence the waiting time of the customer cannot be neglected because it gives important information about the system performance from the customers point’s of view. Hence, in this section we derive the Laplace - Stieltjes transform of waiting time distribution for both the primary and feedback customers.

#### 3.6.1 Waiting Time for Primary Customer

In this subsection, our aim is to calculate the waiting time for the primary customer. We deal with the arriving (tagged) customer waiting time, defined as the time between the arrival epoch of a customer till the instant at which the customer request is satisfied. We will represent this continuous random variable as \( W_1 \). The objective is to describe the probability that a customer has to wait, the distribution of the waiting time and \( n^{th} \) order moments. When the buffer size is full, the arriving (tagged) primary customer waiting time is 0, Otherwise the tagged customer waiting time is positive.

In order to get the distribution of \( W_1 \), we will define some auxiliary variables. Let us consider the Markov process at an arbitrary time \( t \) and suppose that it is at state \((k_1, k_2, k_3, k_4)\), \( k_2 > 0 \). We tag any of those waiting primary customer and \((1)W_{(k_1,k_2,k_3,k_4)}\)
denotes the time until the selected primary customer gets the desired item. Let \( W_1^*(y) = E[e^{-yW_1}] \) and \( W_*(k_1, k_2, k_3, k_4) = E[e^{-yW(k_1, k_2, k_3, k_4)}] \) be the corresponding Laplace-Stieltjes transforms for unconditional and conditional waiting time. Obviously we have

\[
W_1^*(y) = \sum_{k_2=0}^{N-1} \sum_{k_3=0}^{M-1} \sum_{k_4=0}^{1} \pi^{(0,k_2,k_3,k_4)}(1) W_{(0,k_2+1,k_3,k_4)}^*(y) + \sum_{k_1=1}^{S} \sum_{k_2=0}^{N-1} \sum_{k_3=0}^{M-1} \pi^{(k_1,k_2,k_3,1)}(1) W_{(k_1,k_2+1,k_3,1)}^*(y)
\]

\[
+ \sum_{k_1=1}^{S} \sum_{k_2=1}^{N-1} \sum_{k_3=0}^{M-1} \pi^{(k_1,k_2,k_3,2)}(1) W_{(k_1,k_2+1,k_3,2)}^*(y)
\]

(3.9)

To study \( W^*_{(k_1,k_2,k_3,k_4)} \), we introduce an auxiliary Markov chain on the state space

\[
E^* = \{*\} \cup \{(0,k_2,k_3,0) : k_2 = 1, \ldots, N; k_3 = 0,1,\ldots,M\}
\]

\[
\cup \{(k_1,k_2,k_3,1) : k_1 = 0,1,\ldots,S; k_2 = 1,2,\ldots,N; k_3 = 0,1,\ldots,M\}
\]

\[
\cup \{(k_1,k_2,k_3,2) : k_1 = 1,2,\ldots,S; k_2 = 1,2,\ldots,N; k_3 = 0,1,\ldots,M\}
\]

where \{*\} represents an absorbing state. The absorption occurs when the primary customer gets his requested item.

Being on the state \((k_1,k_2,k_3,k_4)\), we apply the first step argument in the auxiliary chain (i.e., we condition on the epoch of the next event and next state of this chain) in order to determine the Laplace-Stieltjes transform \( W^*_{(k_1,k_2,k_3,k_4)}(y) \). The functions \( W^*_{(k_1,k_2,k_3,k_4)}(y) \), \((k_1,k_2,k_3,k_4) \in E^* \) are the smallest non-negative solution to the system

For \( k_1 = 0, \quad 1 \leq k_2 \leq N, \quad 0 \leq k_3 \leq M, \)

\[
(y + \lambda \bar{\delta}_{k_2N} + \beta H(s - k_1)\bar{\delta}_{k_20} + \beta H(s - k_1)\delta_{k_20} - k_3\theta \bar{\delta}_{k_30})(1) W_{(0,k_2,k_3,0)}^*(y) - \lambda \bar{\delta}_{k_2N}(1) W_{(0,k_2+1,k_3,0)}^*(y) - \beta H(s - k_1)\delta_{k_20}(1) W_{(Q,k_2,k_3,2)}^*(y) - \beta H(s - k_1)\delta_{k_20}(1) W_{(0,k_2,k_3-1,1)}^*(y) = 0 \quad (3.10)
\]

For \( 0 \leq k_1 \leq S, \quad 1 \leq k_2 \leq N, \quad 0 \leq k_3 \leq M, \)

\[
w_1(1) W_{(k_1,k_2,k_3,1)}^*(y) - \lambda \bar{\delta}_{k_2N}(1) W_{(k_1+1,k_2,k_3,1)}^*(y) - \beta H(s - k_1)(1) W_{(k_1+Q,k_2,k_3,1)}^*(y) - q\mu_2 \delta_{k_10}(1) W_{(k_1,k_2,k_3,0)}^*(y) - \bar{q}\mu_2 \delta_{k_10}(1) W_{(k_1,k_2,k_3+1,0)}^*(y) - q\mu_2 \bar{\delta}_{k_10}(1) W_{(k_1,k_2,k_3,2)}^*(y) - \bar{q}\mu_2 \bar{\delta}_{k_10}(1) W_{(k_1,k_2,k_3,2)}^*(y) = 0 \quad (3.11)
\]

where

\[
w_1 = y + \lambda \bar{\delta}_{k_2N} + \beta H(s - k_1) + q\mu_2 \delta_{k_10} + \bar{q}\mu_2 \delta_{k_10} + q\mu_2 \bar{\delta}_{k_10} + \bar{q}\mu_2 \bar{\delta}_{k_10}
\]
For $1 \leq k_1 \leq S$, $1 \leq k_2 \leq N$, $0 \leq k_3 \leq M$, 
\[
  w_2^{(1)} W_{(k_1,k_2,k_3,2)}^*(y) = \lambda \bar{\delta}_{k_2N} W_{(k_1,k_2+1,k_3,2)}^*(y) - \beta H(s - k_1) W_{(k_1+1,k_2,k_3,2)}^*(y) \\
  - \mu_1 \delta_{k_1} W_{(k_1-1,k_2-1,k_3,0)}^*(y) - \bar{\mu_1} \delta_{k_1} W_{(k_1-1,k_2-1,k_3+1,0)}^*(y) \\
  - \bar{\mu_1} \delta_{k_1} W_{(k_1-1,k_2-1,k_3+1,2)}^*(y) - \bar{\mu_1} \delta_{k_1} W_{(k_1-1,k_2-1,k_3+1,2)}^*(y) - \mu_1 \bar{\delta}_{k_1} \bar{\delta}_{k_1} + \bar{\mu_1} \delta_{k_1} \bar{\delta}_{k_1} \\
  \text{where} \\
  w_2 = y + \lambda \bar{\delta}_{k_2N} + \beta H(s - k_1) + \mu_1 \delta_{k_1} + \bar{\mu_1} \delta_{k_1} + \mu_1 + \bar{\mu_1} \delta_{k_1} + \mu_1 \bar{\delta}_{k_1} + \bar{\mu_1} \delta_{k_1} \bar{\delta}_{k_1}
\]

Using the expression (3.9) we get $W_1^*(y)$ for a given $y$. This facilitates the applications of the Euler and Post-Widder algorithms in Abate and Whitt [1995] for the numerical inversion of $W_1^*(y)$.

We can exploit the system of equations (3.10) - (3.12) to get a recursive algorithm for computing moments for the conditional and unconditional waiting times.

By differentiating (n+1) times (3.10) - (3.12) the system of equations, and evaluating at $y = 0$, we arrive at

For $k_1 = 0$, $1 \leq k_2 \leq N$, $0 \leq k_3 \leq M$, 
\[
  (\lambda \bar{\delta}_{k_2N} + \beta H(s - k_1) \bar{\delta}_{k_20} - \beta H(s - k_1) \delta_{k_20} - k_3 \bar{\theta} \delta_{k_20}) E \left[ (1) W_{(0,k_2,k_3,0)}^{(n+1)} \right] \\
  - \lambda \bar{\delta}_{k_2N} E \left[ (1) W_{(0,k_2+1,k_3,0)}^{(n+1)} \right] - \beta H(s - k_1) \bar{\delta}_{k_20} E \left[ (1) W_{(Q,k_2,k_3,2)}^{(n+1)} \right] \\
  - \beta H(s - k_1) \delta_{k_20} E \left[ (1) W_{(Q,k_2,k_3,0)}^{(n+1)} \right] - k_3 \bar{\theta} \delta_{k_20} E \left[ (1) W_{(0,k_2,k_3-1,1)}^{(n+1)} \right] \\
  = (n+1) E \left[ (1) W_{(0,k_2,k_3,0)}^{(n)} \right] (3.13)
\]

For $0 \leq k_1 \leq S$, $1 \leq k_2 \leq N$, $0 \leq k_3 \leq M$, 
\[
  w_3 E \left[ (1) W_{(k_1,k_2,k_3,1)}^{(n+1)} \right] - \lambda \bar{\delta}_{k_2N} E \left[ (1) W_{(k_1+1,k_2+1,k_3,1)}^{(n+1)} \right] \\
  - \beta H(s - k_1) E \left[ (1) W_{(k_1,Q,k_2,k_3,1)}^{(n+1)} \right] - q \mu_2 \delta_{k_10} E \left[ (1) W_{(k_1,k_2,k_3,0)}^{(n+1)} \right] \\
  - q \bar{\mu_2} \delta_{k_10} E \left[ (1) W_{(k_1,k_2,k_3+1,0)}^{(n+1)} \right] - q \mu_2 \bar{\delta}_{k_10} E \left[ (1) W_{(k_1,k_2,k_3,2)}^{(n+1)} \right] \\
  - q \bar{\mu_2} \bar{\delta}_{k_10} E \left[ (1) W_{(k_1,k_2,k_3+1,2)}^{(n+1)} \right] = (n+1) E \left[ (1) W_{(k_1,k_2,k_3,1)}^{(n)} \right] (3.14)
\]

where
\[
  w_3 = \lambda \bar{\delta}_{k_2N} + \beta H(s - k_1) + q \mu_2 \delta_{k_10} + q \bar{\mu_2} \delta_{k_10} + q \mu_2 \bar{\delta}_{k_10} + q \bar{\mu_2} \bar{\delta}_{k_10}
\]

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For $1 \leq k_1 \leq S, \quad 1 \leq k_2 \leq N, \quad 0 \leq k_3 \leq M,$

$$w_4 E \left[ (1) W_{(k_1,k_2,k_3,2)}^{(n+1)} \right] - \lambda \delta_{k_2} - N \left[ (1) W_{(k_1,k_2+1,k_3,2)}^{(n+1)} \right] - \beta H (s - k_1) E \left[ (1) W_{(k_1+Q,k_2,k_3,2)}^{(n+1)} \right]$$
$$- p \mu_1 \delta_{k_1} E \left[ (1) W_{(k_1-1,k_2-1,k_3,0)}^{(n+1)} \right] - \bar{p} \mu_1 \delta_{k_1} E \left[ (1) W_{(k_1-1,k_2-1,k_3+1,0)}^{(n+1)} \right]$$
$$- \bar{p} \mu_1 \delta_{k_1} \bar{\delta}_{k_2} E \left[ (1) W_{(k_1-1,k_2-1,k_3+1,2)}^{(n+1)} \right] = (n + 1) E \left[ (1) W_{(k_1,k_2,k_3,2)}^{(n)} \right] (3.15)$$

where

$$w_4 = \lambda \delta_{k_2} - N + \beta H (s - k_1) + p \mu_1 \delta_{k_1} + \bar{p} \mu_1 \delta_{k_1} + \mu_1 + \bar{p} \mu_1 \delta_{k_2} + p \mu_1 \bar{\delta}_{k_1} \bar{\delta}_{k_2} + \bar{p} \mu_1 \bar{\delta}_{k_1} \bar{\delta}_{k_2}$$

Equations (3.13) - (3.15) are used to determine the unknowns $E \left[ (1) W_{(k_1,k_2,k_3,k_4)}^{(n+1)} \right], (k_1, k_2, k_3, k_4) \in E^*$ in terms of the moments of one order less. Noticing that $E \left[ (1) W_{(k_1,k_2,k_3,k_4)}^{(n)} \right] = 1,$ for $n = 0.$ We can obtain the moments up to a desired order in a recursive way.

For determine the moments of $W_1$ we differentiate $W_1^*(y)$ and evaluate at $y = 0,$ we have

$$E[W_1^{(n)}] = \delta_0 + \sum_{k_2=0}^{N-1} \sum_{k_3=0}^{M-1} \sum_{k_4=0}^{1} \pi_{(0,k_2,k_3,k_4)} E \left[ (1) W_{(0,k_2+1,k_3,k_4)}^{(n)} \right]$$
$$+ \sum_{k_1=1}^{S} \sum_{k_2=0}^{N-1} \sum_{k_3=0}^{M} \pi_{(k_1,k_2,k_3,1)} E \left[ (1) W_{(k_1,k_2+1,k_3,1)}^{(n)} \right]$$
$$+ \sum_{k_1=1}^{S} \sum_{k_2=1}^{N-1} \sum_{k_3=0}^{M} \pi_{(k_1,k_2,k_3,2)} E \left[ (1) W_{(k_1,k_2+1,k_3,2)}^{(n)} \right] (1 - \delta_0)$$

which provides the $n^{th}$ moments of the unconditional waiting time in terms of conditional moments of the same order.

### 3.6.2 Waiting time for Feedback customer

In this subsection, we compute the waiting time for feedback customer. Here the tagged feedback customer waiting time, defined as the time between the customer enters into the orbit and joins his service, so we exclude the service time. Let $W_2(y),$ be the probability distribution function for the waiting time in the orbit customer and $W_2^*(y)$ is its Laplace-Stieltjes transforms. The objective is to describe the probability that a customer to wait, the distribution of the waiting time and $n^{th}$ order moments. If orbit size is full then the waiting time of an arrival customer is 0 otherwise the waiting time is positive.
Let us consider the state of this process at an arbitrary time \( t \) and we assume that it is at state \((k_1, k_2, k_3, k_4), k_3 > 0\). We tagged any of those arriving orbiting customer and \((2)^*W \) denotes the waiting time of the customer gets service who is selected randomly from the orbit. Let \( W_2^*(y) = E \left[e^{-yW_2}\right] \) and \((2)^*W \) be corresponding Laplace-Stieltjes transforms for unconditional and conditional waiting time. We have,

\[
W_2^*(y) = \sum_{k_2=0}^{N} \sum_{k_3=0}^{M-1} \pi^{(0,k_2,k_3,1)}(2) W^{*}_{(0,k_2,k_3+1,0)}(y) + \sum_{k_1=1}^{S} \sum_{k_3=0}^{M-1} \pi^{(k_1,0,k_3,1)}(2) W^{*}_{(k_1,0,k_3+1,0)}(y) + \sum_{k_1=1}^{S} \sum_{k_2=1}^{M-1} \sum_{k_3=0}^{N} \pi^{(k_1,k_2,k_3,1)}(2) W^{*}_{(k_1,k_2,k_3+1,2)}(y) + \sum_{k_1=1}^{S} \sum_{k_2=1}^{M-1} \sum_{k_3=0}^{N} \pi^{(1,k_2,k_3,2)}(2) W^{*}_{(0,k_2-1,k_3+1,0)}(y) + \sum_{k_1=2}^{S} \sum_{k_3=0}^{N} \pi^{(k_1,1,k_3,2)}(2) W^{*}_{(k_1-1,0,k_3+1,0)}(y) + \sum_{k_1=2}^{S} \sum_{k_2=2}^{M-1} \sum_{k_3=0}^{N} \pi^{(k_1,k_2,k_3,2)}(2) W^{*}_{(k_1-1,k_2-1,k_3+1,2)}(y) \tag{3.16}
\]

To study \((2)^*W \) we introduce an auxiliary Markov chain on the state space

\[
E^{**} = \{\ast\} \cup \{(0,k_2,k_3,0): k_2 = 0, 1, \ldots, N; k_3 = 1, \ldots, M\} \cup \{(k_1,0,k_3,0): k_1 = 1, 2, \ldots, S; k_3 = 1, 2, \ldots, M\} \cup \{(k_1,k_2,k_3,1): k_1 = 0, 1, \ldots, S; k_2 = 0, 1, 2, \ldots, N; k_3 = 1, \ldots, M\} \cup \{(k_1,k_2,k_3,2): k_1 = 1, 2, \ldots, S; k_2 = 1, 2, \ldots, N; k_3 = 1, \ldots, M\}
\]

where \(\{\ast\}\) represents an absorbing state. The absorption occurs when the feedback customer enters into service.

Being on the state \((k_1, k_2, k_3, k_4)\), we apply the first step argument in this continuous time Markov chain (i.e., we condition on the epoch of the next event and next state of this chain) in order to determine the Laplace-Stieltjes transform \((2)^*W \). Then the functions \((2)^*W \), \((k_1, k_2, k_3, k_4) \in E^{**}\) are the smallest non-negative solution to the system.
For \( k_1 = 0, \ 0 \leq k_2 \leq N, \ 1 \leq k_3 \leq M, \)

\[
(y + \lambda \delta_{k_2,0} + \beta H(s - k_1)\delta_{k_2,0} + \beta H(s - k_1)\delta_{k_2,0} + k\theta)(2)W_{(0,k_2,k_3,0)}^*(y)
- \lambda \delta_{k_2,N}(2)W_{(0,k_2+1,k_3,0)}^*(y) - \beta H(s - k_1)\delta_{k_2,0}(2)W_{(Q,k_2,k_3,2)}^*(y)
- \beta H(s - k_1)\delta_{k_2,0}(2)W_{(Q,k_2,k_3,0)}^*(y) - (k_3 - 1)\theta(2)W_{(0,k_2,k_3-1,1)}^*(y) = \theta \quad (3.17)
\]

For \( 1 \leq k_1 \leq S, \ k_2 = 0, \ 1 \leq k_3 \leq M, \)

\[
(y + \lambda \delta_{k_2,0} + \beta H(s - k_1) + k\theta)(2)W_{(0,k_2,k_3,0)}^*(y) - \beta H(s - k_1)(2)W_{(k_1+Q,0,k_3,0)}^*(y)
- (k_3 - 1)\theta(2)W_{(k_1,0,k_3-1,1)}^*(y) - \lambda \delta_{k_2,N}(2)W_{(k_1,1,k_3,2)}^* = \theta \quad (3.18)
\]

For \( 0 \leq k_1 \leq S, \ 0 \leq k_2 \leq N, \ 1 \leq k_3 \leq M, \)

\[
w_5(2)W_{(k_1,k_2+1,k_3,1)}^*(y) - \lambda \delta_{k_2,N}(2)W_{(k_1,k_2+1,k_3,1)}^*(y) - \beta H(s - k_1)(2)W_{(k_1+Q,k_2,k_3,1)}^*(y)
- q\mu_2\delta_{k_2,1}(2)W_{(k_1,k_2,k_3,0)}^*(y) - q\mu_2\delta_{k_2,0}(2)W_{(k_1,k_2,k_3,0)}^*(y) - \bar{q}\mu_2\delta_{k_2,1}(2)W_{(k_1,k_2,k_3+1,0)}^*(y)
- \bar{q}\mu_2\delta_{k_2,0}(2)W_{(k_1,k_2,k_3+1,0)}^*(y) - q\mu_2\delta_{k_2,1}(2)W_{(k_1,k_2,k_3,2)}^*(y)
- q\mu_2\delta_{k_2,0}(2)W_{(k_1,k_2,k_3,2)}^*(y) = 0 \quad (3.19)
\]

where

\[
w_5 = y + \lambda \delta_{k_2,N} + \beta H(s - k_1) + q\mu_2\delta_{k_2,0} + q\mu_2\delta_{k_2,0} + \bar{q}\mu_2\delta_{k_2,0} + \bar{q}\mu_2\delta_{k_2,0}
\]

For \( 1 \leq k_1 \leq S, \ 1 \leq k_2 \leq N, \ 0 \leq k_3 \leq M, \)

\[
w_6(2)W_{(k_1,k_2,k_3,2)}^*(y) - \lambda \delta_{k_2,N}(2)W_{(k_1,k_2+1,k_3,2)}^*(y) - \beta H(s - k_1)(2)W_{(k_1+Q,k_2,k_3,2)}^*(y)
- p\mu_1\delta_{k_2,1}(2)W_{(k_1,k_2,k_3,0)}^*(y) - p\mu_1\delta_{k_2,1}(2)W_{(k_1,k_2,k_3,0)}^*(y) - \bar{p}\mu_1\delta_{k_2,1}(2)W_{(k_1,k_2,k_3+1,0)}^*(y)
- \bar{p}\mu_1\delta_{k_2,1}(2)W_{(k_1,k_2,k_3+1,0)}^*(y) - p\mu_1\delta_{k_2,1}(2)W_{(k_1,k_2,k_3,2)}^*(y)
- p\mu_1\delta_{k_2,1}(2)W_{(k_1,k_2,k_3,2)}^*(y) = 0 \quad (3.20)
\]

where

\[
w_6 = y + \lambda \delta_{k_2,N} + \beta H(s - k_1) + p\mu_1\delta_{k_2,1} + p\mu_1\delta_{k_2,1} + \bar{p}\mu_1\delta_{k_2,1} + \bar{p}\mu_1\delta_{k_2,1}
\]

We can exploit the system of equations (3.17) - (3.20) to get a recursive algorithm for computing moments for the conditional and unconditional waiting times.
By differentiating \((n+1)\) times \((3.17) - (3.20)\) the system of equations, and evaluating at \(y = 0\), we arrive at

For \(k_1 = 0,\ 0 \leq k_2 \leq N,\ 1 \leq k_3 \leq M,\)

\[
(\lambda \tilde{\delta}_{k_2N} + \beta H(s - k_1)\tilde{\delta}_{k_20} + \beta H(s - k_1)\delta_{k_20} + k_3\theta) E \left[ (2)W^{(n+1)}_{(0,k_2,k_3,0)} \right] \\
- \lambda \tilde{\delta}_{k_2N} E \left[ (2)W^{(n+1)}_{(k_2+1,k_1,3),0} \right] - \beta H(s - k_1)\tilde{\delta}_{k_20} E \left[ (2)W^{(n+1)}_{(Q,k_2,k_3,2)} \right] \\
- \beta H(s - k_1)\delta_{k_20} E \left[ (2)W^{(n+1)}_{(Q,k_2,k_3,0)} \right] - (k_3 - 1)\theta E \left[ (2)W^{(n+1)}_{(0,k_2,k_3-1,1)} \right] \\
= (n + 1) E \left[ (2)W^{(n)}_{(0,k_2,k_3,0)} \right] \tag{3.21}
\]

For \(1 \leq k_1 \leq S,\ k_2 = 0,\ 1 \leq k_3 \leq M,\)

\[
(\lambda \tilde{\delta}_{k_2N} + \beta H(s - k_1) + k_3\theta) E \left[ (2)W^{(n+1)}_{(0,k_2,k_3,0)} \right] - \beta H(s - k_1) E \left[ (2)W^{(n+1)}_{(k_1+Q,k_2,k_3,0)} \right] \\
- (k_3 - 1)\theta E \left[ (2)W^{(n+1)}_{(k_1,0,k_3-1,1)} \right] = (n + 1) E \left[ (2)W^{(n)}_{(0,k_2,k_3,0)} \right] \tag{3.22}
\]

For \(0 \leq k_1 \leq S,\ 0 \leq k_2 \leq N,\ 1 \leq k_3 \leq M,\)

\[
w_7 E \left[ (2)W^{(n+1)}_{(k_1,k_2,k_3,1)} \right] - \lambda \tilde{\delta}_{k_2N} E \left[ (2)W^{(n+1)}_{(k_1,k_2+1,k_3,1)} \right] \\
- \beta H(s - k_1) E \left[ (2)W^{(n+1)}_{(k_1+Q,k_2,k_3,1)} \right] - q\mu_2 \tilde{\delta}_{k_10} E \left[ (2)W^{(n+1)}_{(k_1,k_2,k_3,0)} \right] \\
- q\mu_2 \delta_{k_20} E \left[ (2)W^{(n+1)}_{(k_1,k_2,k_3,0)} \right] - \check{q}\mu_2 \tilde{\delta}_{k_10} \tilde{\delta}_{k_20} E \left[ (2)W^{(n+1)}_{(k_1,k_2,k_3,2)} \right] \\
- \check{q}\mu_2 \tilde{\delta}_{k_10} \tilde{\delta}_{k_20} E \left[ (2)W^{(n+1)}_{(k_1,k_2,k_3+1,2)} \right] = (n + 1) E \left[ (2)W^{(n)}_{(k_1,k_2,k_3,1)} \right] \tag{3.23}
\]

where

\[
w_7 = \lambda \tilde{\delta}_{k_2N} + \beta H(s - k_1) + q\mu_2 \tilde{\delta}_{k_10} + q\mu_2 \delta_{k_20} + \check{q}\mu_2 \tilde{\delta}_{k_10} \tilde{\delta}_{k_20} + \check{q}\mu_2 \tilde{\delta}_{k_10} \tilde{\delta}_{k_20} \]

For \(1 \leq k_1 \leq S,\ 1 \leq k_2 \leq N,\ 0 \leq k_3 \leq M,\)

\[
w_8 E \left[ (2)W^{(n+1)}_{(k_1,k_2,k_3,2)} \right] - \lambda \tilde{\delta}_{k_2N} E \left[ (2)W^{(n+1)}_{(k_1,k_2+1,k_3,2)} \right] - \beta H(s - k_1) E \left[ (2)W^{(n+1)}_{(k_1+Q,k_2,k_3,2)} \right] \\
- p\mu_1 \delta_{k_11} E \left[ (2)W^{(n+1)}_{(k_1-1,k_2-1,k_3,0)} \right] - p\mu_1 \delta_{k_11} E \left[ (2)W^{(n+1)}_{(k_1-1,k_2-1,k_3,0)} \right] \\
- \check{p}\mu_1 \delta_{k_11} E \left[ (2)W^{(n+1)}_{(k_1-1,k_2-1,k_3+1,0)} \right] - \check{p}\mu_1 \delta_{k_11} E \left[ (2)W^{(n+1)}_{(k_1-1,k_2-1,k_3+1,0)} \right] \\
- p\mu_1 \tilde{\delta}_{k_10} \tilde{\delta}_{k_21} E \left[ (2)W^{(n+1)}_{(k_1-1,k_2-1,k_3+2)} \right] - \check{p}\mu_1 \tilde{\delta}_{k_10} \tilde{\delta}_{k_21} E \left[ (2)W^{(n+1)}_{(k_1-1,k_2-1,k_3+2)} \right] \\
= (n + 1) E \left[ (2)W^{(n)}_{(k_1,k_2,k_3,2)} \right] \tag{3.24}
\]
where

\[ w_8 = \lambda \delta_{k_2N} + \beta H(s - k_1) + \mu_1 \delta_{k_1} + \mu_2 \delta_{k_2} + \bar{\mu}_1 \delta_{k_1} + \bar{\mu}_2 \delta_{k_2} \]

Equations (3.21) - (3.24) are used to determine the unknowns \( E \left[ W_{(k_1,k_2,k_3)}^{(n+1)} \right] \), \((k_1, k_2, k_3, k_4) \in E^*\) in terms of the moments of one order less. We can obtain the moments up to a desired order in a recursive way.

For determine the moments of \( W_2 \) we differentiate \( W_2^*(y) \) and evaluate at \( y = 0 \), we have

\[
E[W_2^{(n)}] = \delta_{n0} + \left[ \sum_{k_2=0}^{N} \sum_{k_3=0}^{M-1} \pi_{(0,k_2,k_3+1,1)} \right] E \left[ W_{(0,k_2,k_3+1,0)}^{(n)} \right] \\
+ \sum_{k_1=1}^{S} \sum_{k_2=1}^{M-1} \pi_{(k_1,0,k_3,1)} E \left[ W_{(0,k_2,k_3+1,0)}^{(n)} \right] \\
+ \sum_{k_1=1}^{S} \sum_{k_2=1}^{N} \sum_{k_3=0}^{M-1} \pi_{(k_1,k_2,k_3,1)} E \left[ W_{(k_1,k_2,k_3+1,2)}^{(n)} \right] \\
+ \sum_{k_1=1}^{N} \sum_{k_2=2}^{M-1} \pi_{(1,k_2,k_3,2)} E \left[ W_{(0,k_2-1,k_3+1,0)}^{(n)} \right] \\
+ \sum_{k_1=1}^{S} \sum_{k_2=1}^{M-1} \pi_{(k_1,1,k_3,2)} E \left[ W_{(k_1-1,0,k_3+1,0)}^{(n)} \right] \\
+ \sum_{k_1=2}^{S} \sum_{k_2=2}^{N} \sum_{k_3=0}^{M-1} \pi_{(k_1,k_2,k_3,2)} E \left[ W_{(k_1-1,k_2-1,k_3+1,2)}^{(n)} \right] \left( 1 - \delta_{n0} \right)
\]

which provides the \( n^{th} \) moments of the unconditional waiting time in terms of conditional moments of the same order.

### 3.7 Numerical Analysis

To study the behaviour of the model developed in this work, several examples were performed and a set of representative results is shown here. Although we have not shown the convexity of \( TC(S, s, M, N) \) analytically, our experience with considerable numerical examples indicates the function \( TC(S, s, M, N) \), to be convex. We use simple numerical search procedure to get the optimal values of \( TC, S, s, M \) and \( N \) (say \( TC^*, S^*, s^*, M^* \) and \( N^* \)). Typical three dimensional plots of the expected cost function is given in figure 3.2. The optimal cost value \( TC^* = 0.334002941872642 \) is obtained at \((25, 5, 14, 4)\).
We have studied the effect of varying the system parameters and costs on the optimal values and the results agreed with what one would expect. Some of our results are presented in tables 1 to 4 where the upper entries in each cell give the $S^*$, $s^*$, $M^*$ and $N^*$, respectively, and the lower entry gives the corresponding $TC^*$.

\[
\begin{align*}
\lambda &= 0.4; \theta = 0.5; \mu_1 = 2; \beta = 0.1; \mu_2 = 1; q = 0.6; p = 0.8. \\
c_h &= 0.01; c_s = 3; c_{b1} = 2.2; c_{b2} = 4; c_{qp} = 0.2; c_{qf} = 0.3
\end{align*}
\]

Figure 3.2: A typical three dimensional plot for convexity of total expected cost rate
Example 3.1: In the first example, we study the impact of the arrival rate $\lambda$, the retrial parameter $\theta$, the primary customer’s service parameter $\mu_1$, the feedback customer’s service parameter $\mu_2$ and the replenishment parameter $\beta$, on the optimal values. Towards this end, we first fix the cost values as $c_h = 0.01; c_s = 3; c_b_1 = 2.2; c_b_2 = 4; c_qp = 0.2; c_qf = 0.3$ and $p = 0.6, q = 0.8$. We observe the following from the table 3.1:

1. The total expected cost rate increases when $\theta, \mu_1, \mu_2$ and $\beta$ increase and decrease when $\lambda$ increase.

2. As is to be expected, $S^*$ increases monotonically with $\lambda, \theta$ and $\mu_2$ and monotonically decrease when $\beta$ increase. We cannot predict the behaviour of $S^*$ when the primary customer’s service rate $\mu_1$ increase.

3. The optimal values $s^*$ increases monotonically when the arrival rate $\lambda$, the retrial rate $\theta$ and the feedback customer’s service rate $\mu_2$ increase. We also note that $s^*$ monotonically decreases when $\mu_1$ and $\beta$ increase.

4. As $\lambda, \theta, \mu_1$ and $\beta$ increase, the optimal waiting hall $N^*$ monotonically increases and $N^*$ values cannot predict when $\mu_2$ values increase.

5. The orbit size $M^*$ monotonically decreases with the arrival rate $\lambda$ and the primary customer’s service rate $\mu_1$ and increases monotonically when the retrial rate $\theta$ and the feedback customer’s service rate $\mu_2$ increase. We cannot predict the behaviour of $M^*$ when the reorder rate increase.
### Table 3.1: Effect of parameter on the optimal values

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Example 3.2: In this example, we study the effect of the holding cost $c_h$, the setup cost per order $c_s$, the cost per primary customer lost $c_{b1}$, the cost per feedback customer lost $c_{b2}$, the waiting cost of a primary customer per unit time $c_{qp}$, the waiting cost of a feedback customer per unit time $c_{qf}$ on the optimal values $TC^*$, $S^*$, $s^*$, $N^*$ and $M^*$. For this example, we fix the rate and parameters value as $\lambda = 0.4; \theta = 0.5; \mu_1 = 2; \beta = 0.1; \mu_2 = 1; q = 0.6; p = 0.8$. We observe the following from tables 3.2 to 3.4.

1. The optimal total cost increases when each of $c_h$, $c_s$, $c_{b1}$, $c_{b2}$, $c_{qp}$ and $c_{qf}$ increase.

2. As is to be expected, $S^*$ increases monotonically with $c_s$ and $c_{b1}$. This is because if the setup cost increases then we have to maintain more items to avoid frequent ordering. Similarly, for the balking rate cost of primary customer and the waiting time cost for the primary customers. We also note that, the optimal value of $S$ monotonically decrease when the holding cost $c_h$ and the balking rate cost of feedback customer increase. We cannot predict the behaviour of $S^*$ when the waiting cost of primary customer and the waiting cost of feedback customer increases.

3. As $c_h$, $c_s$, $c_{b2}$ and $c_{qf}$ increase, the optimal reorder level $s^*$ decreases monotonically. We also note that $s^*$ monotonically increases when $c_{b1}$ and $c_{qp}$ increase.

4. The optimal waiting hall $N^*$ monotonically decreases when the holding cost $c_h$, the feedback customer balking rate cost $c_{b2}$ the waiting cost of primary customer $c_{qp}$ and the waiting cost of feedback customer $c_{qf}$ increase. As is to be expected the optimal waiting hall size $N^*$ monotonically increase when the primary customer balking rate cost increases.

5. The orbit size $M^*$ monotonically increases with $c_h$, $c_s$ and $c_{b2}$ and it is decreases monotonically when $c_{b1}$, $c_{qf}$ and $c_{qp}$ increase.
Table 3.2: Influence of costs on the optimal values

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\[ c_h = 0.005 \]
## Table 3.3: Influence of costs on the optimal values

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Table 3.4: Influence of costs on the optimal values
Example 3.3: We now study the impact of the arrival rate $\lambda$, the retrial parameter $\theta$, the primary customer’s service parameter $\mu_1$, the feedback customer’s service parameter $\mu_2$ and the replenishment parameter $\beta$ on the expected waiting time of the primary customer. The graphs for various scenarios are plotted from figure (3.3) - (3.7).

1. As is to be expected $\lambda$ and $E(W_1)$ are directly proportional to each other. This is because, if more customers arrive to the system then their expected waiting time is also increases. We also note that, $E(W_1)$ decreases as $\theta, \mu_1, \mu_2$ and $\beta$ increase.

2. If we increase the size of the waiting hall, the expected waiting time of the primary customer increases. The primary customer’s expected waiting time increases when $M, S$ and $s$ increase.

$$\theta = 1.5, \mu_1 = 3.2, \beta = 1, \mu_2 = 2.5, q = 0.6, p = 0.8$$

Figure 3.3: Effect of $\lambda$ on the primary customer waiting time
\[ \lambda = 2, \mu_1 = 3.2, \beta = 1, \mu_2 = 2.5, q = 0.6, p = 0.8 \]

Figure 3.4: Effect of \( \theta \) on the primary customer waiting time

\[ \lambda = 2, \theta = 1.5, \beta = 1, \mu_2 = 2.5, q = 0.6, p = 0.8 \]

Figure 3.5: Effect of \( \mu_1 \) on the primary customer waiting time
\[ \lambda = 2, \mu_1 = 3.2, \theta = 1.5, \mu_2 = 2.5, q = 0.6, p = 0.8 \]

Figure 3.6: Effect of \( \beta \) on the primary customer waiting time

\[ \lambda = 2, \mu_1 = 3.2, \theta = 1.5, \beta = 1, q = 0.6, p = 0.8 \]

Figure 3.7: Effect of \( \mu_2 \) on the primary customer waiting time
Example 3.4: In this example, we study the influence of the arrival rate $\lambda$, the retrial parameter $\theta$, the primary customer’s service parameter $\mu_1$, the feedback customer’s service parameter $\mu_2$ and the replenishment parameter $\beta$ on the expected waiting time of the orbit customer. The graphs for various scenarios are plotted from figure (3.8) - (3.12).

1. The expected waiting time of the feedback customer $E(W_2)$ increases as $N, M, S$ and $s$ increases.

2. As $\lambda$ increases, $E(W_2)$ increases and $E(W_2)$ decreases as $\theta, \mu_1$ and $\mu_2$ increase. We also note that, as $\beta$ increases, $E(W_2)$ increases.

$$\theta = 1.5, \mu_1 = 3.2, \beta = 1, \mu_2 = 2.5, q = 0.6, p = 0.8$$

Figure 3.8: Effect of $\lambda$ on the feedback customer waiting time
\[ \lambda = 2, \mu_1 = 3.2, \beta = 1, \mu_2 = 2.5, q = 0.6, p = 0.8 \]

Figure 3.9: Effect of \( \theta \) on the feedback customer waiting time

\[ \lambda = 2, \theta = 1.5, \beta = 1, \mu_2 = 2.5, q = 0.6, p = 0.8 \]

Figure 3.10: Effect of \( \mu_1 \) on the feedback customer waiting time
\[ \lambda = 2, \mu_1 = 3.2, \theta = 1.5, \mu_2 = 2.5, q = 0.6, p = 0.8 \]

Figure 3.11: Effect of \( \beta \) on the feedback customer waiting time

\[ \lambda = 2, \mu_1 = 3.2, \theta = 1.5, \beta = 1, q = 0.6, p = 0.8 \]

Figure 3.12: Effect of \( \mu_2 \) on the feedback customer waiting time