CHAPTER VI

SUPERMAGIC CORONATIONS OF GRAPHS
A labeling of graph $G$ is a mapping that carries a set of graph elements into a set of numbers (Usually positive integers) called labels. An edge-magic labeling of a graph with $p$ vertices and $q$ edges will be defined as a one-to-one map taking the vertices and edges onto the integers $1, 2, \ldots, p+q$ with the property that the sum of the labels of an edge and the labels of its end vertices is constant independent of the choice of edge. Moreover, $G$ is said to be supermagic if $f[V(G)] = \{q+1, q+2 \ldots, p+q\}$ and $f[E(G)] = \{1, 2, \ldots, q\}$. In this chapter we have shown that the $n$-crown is supermagic for any positive integer $n$ and thus extended to what was known about the harmoniousness, sequentialness and felicitousness of such graphs. Lastly, we establish two results on $C_m \odot K_n$ for some integer $m$ of certain supermagic graphs to obtain more supermagic graphs.
6.1 Introduction

Let $G$ be a finite simple undirected graph. The set of vertices and edges of a graph $G$ will be denoted by $V(G)$ and $E(G)$ respectively, $p = |V(G)|$ and $q = |E(G)|$. For simplicity, we denote $V(G)$ by $V$ and $E(G)$ by $E$.

A labeling (or valuation) of a graph is a map that carries graph elements to numbers (usually to the positive or non-negative integers). In this chapter, the domain will usually be the set of all vertices and edges. Such a labeling is called total labeling. Some labeling use the vertex set alone, or the edge set alone, and we shall call them vertex labeling and edge labeling respectively. Other domains are possible.

The most complete recent survey of graph labeling is [7].

Kotzig and Rosa [12] defined a magic labeling to be a total labeling in which the labels are the integers from 1 to $p+q$. The sum of labels on an edge and its two end vertices is constant. In 1996 Ringel and Llado [16] redefined this type of labeling (and called the labeling; edge-magic, causing some confusion with papers that have followed the terminology of [13] mentioned below). See also [8, 9]. Recently Enomoto et al. [6] have introduced the same super edge-magic for magic labeling in the sense of Kotzig and Rosa with the added property that the $p$ vertices receive the smaller labels $\{1,2,\ldots,p\}$. In 2010, Akka and Nanda [1] introduced the same type of supermagic for magic labeling with added property that the $p$ vertices receive the bigger labels $\{q+1,q+2,\ldots,q+p\}$ and $q$ edges receive the smaller labels $\{1,2,\ldots,q\}$.

An edge-magic total (EMT) labeling is a one-to-one mapping $f$ from $V \cup E$ onto the integers $1,2,\ldots, p+q$ with the property that for every $(x, y)$ in $E$, $f(x) + f(y) + f(xy) = k$ for some constant $k$. A graph that has an edge-magic total labeling is called an
**edge-magic total graph.** An edge-magic total labeling is called a super edge-magic total (SEMT) labeling [6] if \( f(V) = \{1,2,\ldots,p\} \) and edge-magic total labeling is called a supermagic total labeling (SMT) [1] if \( f(V) = \{q+1,q+2,\ldots,q+p\} \) and a graph that has SEMT (SMT) labeling is called a SEMT (SMT) graph. Research in SEMT (SMT) labeling has been particularly popular during the last decade. For details see the Gallian’s dynamic survey [7].

In 1983, Lih [14] introduced magic labeling of planar graphs where labels extended to faces as well as edges and vertices; an idea which he traced back to 13\(^{th}\) century Chinese roots. M. Bača, (see the example [3,4]) has written extensively on these labelings. A somewhat related sort of magic labeling was defined by Dickson and Rogers in [5].

Lee, Seah and Tan [13] introduced a weaker concept which they called edge-magic in 1992. The edges are labelled and the sums of the vertices are required to be congruent modulo the number of vertices.

Total labeling has also been studied in which the sum of the labels of all edges adjacent to the vertex x plus the label of x itself is constant. A paper on these labelings is in preparation [15].

In order to clarify the terminological confusion mentioned above, we define a labeling to be edge-magic if the sum of all labels associated with an edge equals a constant independent of the choice of edge, and vertex-magic if the same property holds for vertices (This terminology could be extended to other sub structures; face-magic for example). The domain of the labeling is specified by a modifier on the word “labeling”. We shall always require that the labeling is a one-to-one map onto the
appropriate set of consecutive integers starting from 1. For example, Stewart studies vertex-magic edge-labelings and Kotzig and Rosa define edge-magic total labelings. Hypermagic labelings are vertex-magic total labelings.

The following result found in [2] provides a necessary and sufficient condition for graph to be supermagic

**Lemma 6.1.1:** A \((p,q)\)-graph \(G\) is super magic if and only if there exists a bijective function \(f: V(G) \rightarrow \{q + 1, q + 2, \ldots, +q + p\}\) such that the set
\[
S = \{f(x) + f(y) \mid xy \in E(G)\}
\]
consists of \(q\) consecutive integers. In such a case, \(f\) extends to a supermagic labeling of \(G\) with valence \(k^l = q + s\) where \(s = \min\{S\}\) and
\[
S = \{f(x) + f(y) \mid xy \in E(G)\} = \{k^l - 1, k^l - 2, \ldots, k^l - q\}
\]

**Lemma 6.1.2:** [2] If a \((p, q)\) graph \(G\) is Supermagic then \(G\) is felicitous whenever it is a tree or satisfies \(q \geq p\)

Thus, we establish the following relationship between supermagic labelings and felicitous labelings since every **harmonious** labeling is certainly a felicitous labeling.

**Lemma 6.1.3:** [2] If a \((p, q)\)-graph \(G\) is supermagic then \(G\) is felicitous whenever it is tree or satisfies \(q \geq p\).

Throughout this chapter we will utilize the following definitions.

The **Corona** product \(G_1 \circ G_2\) of two graphs \(G_1\) and \(G_2\) is defined as the graph obtained by taking one copy of \(G_1\) (which has order \(p_1\)) and \(p_1\) copies of \(G_2\) and then joining the \(i\)-th vertex of \(G_1\) to every vertex in the \(i\)-th copy of \(G_2\). The graph \(C_m \circ \overline{K}_n\) is called the **n-crown** with cycle length \(m\). We will refer to the 1-crown as the crown with cycle length \(m\). For the sake of brevity, we will refer to the \(C_m \circ \overline{K}_n\) \(n\)-crown with cycle length \(m\) simply as the \(n\)-crown if its cycle length is clear from the context.
A \((p,q)\)-graph \(G\) with \(q \geq p\) is \textbf{harmonious} [11] if there exists an injective function \(f: V(G) \rightarrow \mathbb{Z}_q\) such that each edge \(xy\) of \(G\) is labeled \(f(x) + f(y) \pmod{q}\) and the resulting labels are distinct. Such a function is called \textbf{harmonious labeling}. If \(G\) is a tree (so that \(q = p - 1\)) exactly two vertices are labeled the same, otherwise the definition is the same. For a \((p,q)\) graph \(G\), an injective function \(f: V(G) \rightarrow \{0,1, \ldots, q-1\}\) is a \textbf{sequential labeling} of \(G\) if each edge \(xy\) of \(G\) is labeled \(f(x) + f(y)\) and the resulting edge labels are \(\{m, m+1, \ldots, m+q-1\}\) for some positive integer \(m\). If such a labeling exists, then \(G\) is said to be \textbf{sequential} [10]. In case of a tree, Grace permits the vertex labels to range from 0 to \(q\) with no vertex label used twice. A graph \(G\) of size \(q\) is \textbf{felicitous} if there exists an injective function \(f: V(G) \rightarrow \mathbb{Z}_{q+1}\) such that each edge \(xy\) of \(G\) is labeled \(f(x) + f(y) \pmod{q}\) and the resulting edge labels are distinct. Such a function is called \textbf{felicitous labeling} [17].

In this chapter, we show that a cycle (of length \(m\)) with ‘\(n\)’ pendent edges attached at each vertex, known as an \textbf{n-crown} \(Cm \odot \overline{K}_n\) is supermagic, harmonious, sequential and felicitous. Also, we establish two results that illustrate how attaching pendant edges to the vertices of certain supermagic graphs will result in more supermagic graphs.

\section*{6.2 Coronation of Supermagic graphs}

\textbf{Theorem 6.2.1:} Let \(G\) be a graph of odd order \(p \geq 3\) for which there exists a supermagic labeling \(f\) with the property that,

\[
\max\{f(u) + f(v)/uv \in E(G)\} = p - 1
\]

Then \(G \odot \overline{K}_n\) is supermagic for every positive integer \(n\).
Proof: Let $f$ be a supermagic labeling of $G$ with valence $k'$ and supposing that $f$ has the property that $f(v_i) = q + i$ for every integer $i$ with $1 \leq i \leq p$ where

$$V(G) = \{\frac{vi}{4} \leq i \leq p\}. \text{ And } E(G) = q$$

Further let $V(G \circ \bar{K}_n) = V(G) \cup \{w_i^j / 1 \leq i \leq p \text{ and } 1 \leq j \leq n\}$

and $E(G \circ \bar{K}_n) = E(G) \cup \{v_iw_i^j / 1 \leq i \leq p \text{ and } 1 \leq j \leq n\}.$

Then, the vertex labeling is defined as

$$g: V(G \circ \bar{K}_n) \rightarrow \{q + 1, q + 2, ..., q + p(n + 1)\} \text{ such that } g(v) = f(v) \text{ for every vertex } v \text{ of } G$$

and

$$g(w_i^j) = \begin{cases} 
2pn + q + 1 - i - \frac{p(2j - 1) + 1}{2} & \text{ if } 1 \leq i \leq \frac{p-1}{2}, 1 \leq j \leq n \\
(2n + 1)p + q + 1 - i - \frac{p(2j - 1) + 1}{2} & \text{ if } \frac{p + 1}{2} \leq i \leq p, 1 \leq j \leq n 
\end{cases}$$

To prove that $g$ extends to a supermagic labeling of $G \circ \bar{K}_n$, consider the set

$$S_i^j = \{f(v_i) + f(w_i^j) / 1 \leq i \leq p \text{ and } 1 \leq j \leq n\}$$

and let $m_j = \min\{S_i^j / 1 \leq i \leq p\} = 2(np + p + q) + p + 1 + \frac{p(2j-1)+1}{2}$

$$M_j = \min\{S_i^j / 1 \leq i \leq p\} = 2(np + p + q) + 2p + \frac{p(2j-1)+1}{2}$$

Finally, observe that $m_1 = 2(np + p + q) + \frac{3p+1}{2} + 1, \ M_j + 1 = m_{j+1}$

($i \leq j \leq n - 1$) and $S_i^j$ is a set of consecutive integer $1 \leq i \leq p$ and $1 \leq j \leq n$)

which implies that $g$ extends to a supermagic labeling of $G \circ \bar{K}_n$ with valence $k' + 4np$

By Lemmas 6.1.2 and 6.1.3 we obtain the following corollary:

Corollary 6.2.2: Let $G$ be a graph of odd order $p \geq 3$ for which there exists a supermagic labeling $f$ with the condition that

$$\max \{f(u) + f(v) \in E(G)\} = p + q + 1 - \frac{3p+1}{2}$$

Then $G \circ \bar{K}_n$ is harmonious, sequential and felicitous for every positive integer $n$. 

~ 132 ~
We now provide a similar, though in a sense weaker, result for graphs with even order of at least 4.

**Theorem 6.2.3:** Let $G$ be a graph of even order $p \geq 4$ having a supermagic labeling $f$ with the condition that $\max\{f(u) + f(v) : uv \in E(G)\} = p + q + 1 - \frac{3p}{2}$

Then the graph $H$ got by attaching $n$ edges to each vertex of $G$ except the vertex $v$ with $f(v) = q + p$ is supermagic for every positive integer $n$.

**Proof:** Let $V(G) = \{v_1, v_2, \ldots, v_p\}$ then supermagic labeling of $G$ with valence $k^1$ satisfies the condition that $f(v_i) = q + i$ for $i = 1, 2, \ldots, p$. Now the graph $H$ is defined as follows:

$V(H) = V(G) \cup \{w_i^j / 1 \leq i \leq p - 1 \text{ and } 1 \leq j \leq n\}$

and $E(H) = E(G) \cup \{v_iw_i^j / 1 \leq i \leq p - 1 \text{ and } 1 \leq j \leq n\}$

One can use an argument analogous to the one used in the proof of the previous theorem about vertex labeling $g: V(H) = \{q + 1, q + 2, \ldots, q + p(n + 1) - n\}$ such that $g(v) = f(v)$ for every vertex $v$ of $G$ and

$$g(w_i^j) = \begin{cases} 
2np + q + 1 - i - \frac{p(2j - 1) + 1}{2} & \text{if } 1 \leq \frac{p - 1}{2}, 1 \leq j \leq n \\
(2n + 1)p + q + 1 - i - \frac{p(2j - 1) + 1}{2} & \text{if } \frac{p}{2} \leq i \leq p - 1, 1 \leq j \leq n
\end{cases}$$

extends to a supermagic labeling of $H$ with valence $k^1 + 4n(p-1)$

Again, by Lemmas 6.1.2 and 6.1.3 we have the following corollary

**Corollary 6.2.4:** Let $G$ be a graph of even order $p \geq 4$ having a supermagic labeling $f$ with condition that $\max\{f(u) + f(v) : uv \in E(G)\} = p + q + 1 - \frac{3p}{2}$. Then the
graph $H$ obtained by attaching $n$ pendent edges to each vertex of $G$ except the vertex $v$ with $f(v) = p + q$ is harmonious, sequential and felicitous for every positive integer $n$.

### 6.3 Supermagic Labelings of n-crowns

To proceed to study the supermagicness of $n$-crowns, we start with a result pertaining to 2-regular graphs.

**Theorem 6.3.1:** If $G$ is a supermagic 2-regular, then $G \circ K_n$ is supermagic for every positive integer $n$.

**Proof:** Let $f$ be a supermagic labeling of $G$ with valence $k$. Suppose that $H$ is a component of $G \circ K_n$. Then $H \cong C_r \circ K_n$ for some integer $r \geq 3$.

Let $V(H) = \{v_i/i \in Z_r\} \cup \{u_{i,j}/i \in Z_r \text{ and } 1 \leq j \leq n\}$ and $E(H) = \{v_i v_{i+1}/i \in Z_r\} \cup \{v_i u_{i,j}/i \in Z_r \text{ and } 1 \leq j \leq n\}$

where $Z_r$ denotes the set of integers modulo $r$. Then $f|_H$ extends to a labeling $g$ of $H$ as follows:

\[
g(v_i) = (n + 1)[2r - f(v_i) + 1],
\]
\[
g(v_{i-1}v_i) = (n + 1)[2r - f(v_{i-1}v_i)],
\]
\[
g(u_{i,j}) = (n + 1)[2r - f(v_{i-1}) + 1] - j,
\]
\[
g(v_i u_{i,j}) = (n + 1)[2r - f(v_{i-1}v_i)] + 1 + j
\]

when $i \in Z_r$ and $1 \leq j \leq n$. Therefore, $f$ extends likewise in every component of $G \circ K_n$ and a supermagic labeling of $G \circ K_n$ is obtained with valence

\[
(n + 1)(6r - k + 2) + 1.\]

**Theorem 6.3.2:** [1] The $n$-cycle $C_n$ is supermagic if and only if $n \geq 3$ is odd.
In the following result, we show with considerable more effort that the n-crowns with cycle of length \( m \) are supermagic when \( m \geq 4 \) is even.

**Theorem 6.3.3:** For every two integers \( m \geq 3 \) and \( n \geq 1 \), the n-crown \( G \cong C_m \circ K_n \) is supermagic.

**Proof:** Let \( G \cong C_m \circ K_n \) be the n-crown with

\[
V(G) = \{u_i/1 \leq i \leq m\} \cup \{v_{ij}/1 \leq i \leq m \text{ and } 1 \leq i \leq n\}
\]

and

\[
E(G) = \{u_i u_m\} \cup \{u_i u_{i+1}/1 \leq i \leq m-1\} \cup \{u_i v_{ij}/1 \leq i \leq m \text{ and } 1 \leq i \leq n\}.
\]

Now, notice that if \( m \geq 3 \) is odd then the result follows from Theorem 6.3.1 and 6.3.2. Thus, assume that \( m \geq 4 \) is even for the reminder of the proof and proceed by cases by means of Lemma 6.1.1

**Case I:** For \( m = 4 \), define the vertex labeling

\[
f: V(G) \rightarrow \{4(n + 1) + 1, \ldots, 8(n + 1)\} \text{ such that}
\]

\[
f(u_{2i-1}) = 2m(n + 1) + 1 - i \quad f(u_{2i}) = 2m(n + 1) + 1 - 3i
\]

\[
f(v_{2i-1,1}) = 2m(n + 1) - 2 - 2i \quad f(v_{2i,1}) = 2m(n + 1) - 11 + 4i
\]

When \( i = 1 \) or \( 2 \) and \( f(v_{ij}) = 2m(n + 1) - 4 - 4j + i \quad 1 \leq i \leq 4, 2 \leq j \leq n\)

**Case II:** For \( m = 6 \), define the vertex labeling

\[
f: V(G) \rightarrow \{6(n + 1) + 1, 6(n + 1) + 2, \ldots, +12(n + 1)\} \text{ Such that}
\]

\[
f(u_1) = 12(n + 1) - 8 \quad f(u_2) = 12(n + 1) \quad f(u_3) = 12(n + 1) - 3
\]

\[
f(u_4) = 12(n + 1) - 1 \quad f(u_5) = 12(n + 1) - 4 \quad f(u_6) = 12(n + 1) - 2
\]

\[
f(v_{1,1}) = 12(n + 1) - 5 \quad f(v_{2,1}) = 12(n + 1) - 7 \quad f(v_{3,1}) = 12(n + 1) - 6
\]

\[
f(v_{4,1}) = 12(n + 1) - 12 \quad f(v_{5,1}) = 12(n + 1) - 10 \quad f(v_{6,1}) = 12(n + 1) - 9
\]

and \( f(v_{ij}) = \begin{cases} 
12(n + 1) + 5(i - 1) - 6 & \text{if } 1 \leq i \leq 2 \text{ and } 2 \leq j \leq n \\
12(n + 1) + 2 - i - 6j & \text{if } 3 \leq i \leq 6 \text{ and } 2 \leq j \leq n
\end{cases} \)
**Case III:** For $m = 8$, define the vertex labeling

$$f: V(G) \rightarrow \{8(n + 1) + 1, 8(n + 1) + 2, \ldots, + 16(n + 1)\}$$

Such that

$$f(u_1) = 12(n + 1) \quad f(u_2) = 12(n + 1) - 4 \quad f(u_3) = 12(n + 1) - 1$$

$$f(u_4) = 12(n + 1) - 5 \quad f(u_5) = 12(n + 1) - 2 \quad f(u_6) = 12(n + 1) - 6$$

$$f(u_7) = 12(n + 1) - 3 \quad f(u_8) = 12(n + 1) - 11$$

$$f(v_{1,1}) = 12(n + 1) - 10 \quad f(v_{2,1}) = 12(n + 1) - 12 \quad f(v_{3,1}) = 12(n + 1) - 14$$

$$f(v_{4,1}) = 12(n + 1) - 13 \quad f(v_{5,1}) = 12(n + 1) - 15 \quad f(v_{6,1}) = 12(n + 1) - 7$$

$$f(v_{7,1}) = 12(n + 1) - 9 \quad f(v_{8,1}) = 12(n + 1) - 8$$

and $f(v_{i,j}) = 12(n + 1) - 8(j + 1) + 1$ if $1 \leq i \leq 8$ and $2 \leq j \leq n$.

**Case IV:** Let $m = 8k + 2$ where $k$ is a positive integer and define the vertex labeling

$$f: V(G) \rightarrow \{(8k + 2)(n + 1) + 1, \ldots, + 2(8k + 2)(n + 1)\}$$

such that

$$f(u_l) = \begin{cases} 
2(8k + 2)(n + 1) - 2 - 12k & \text{if } l = 1 \\
2(8k + 2)(n + 1) + 1 - 4k - i & \text{if } l = 2i - 1, \quad 2 \leq i \leq 4k + 1 \\
2(8k + 2)(n + 1) + 1 - i & \text{if } l = 2i, \quad 1 \leq i \leq 4k + 1 
\end{cases}$$

$$f(v_{i,1}) = \begin{cases} 
2(8k + 2)(n + 1) - 8k - i & \text{if } l = 2i - 1, \quad 1 \leq i \leq 2k + 2 \\
2(8k + 2)(n + 1) - 1 - 12k & \text{if } l = 2 \\
2(8k + 2)(n + 1) - 1 - i - 12k & \text{if } l = 2i, 2 \leq i \leq 2k \\
2(8k + 2)(n + 1) - 3 - 2i + 14k & \text{if } l = 4k + 4i - 2, 1 \leq i \leq k \\
2(8k + 2)(n + 1) - 4 + i - 14k & \text{if } l = 4k + i + 3, 1 \leq i \leq 2 \\
2(8k + 2)(n + 1) - 1 - 2i - 10k & \text{if } l = 4k + 4i + 3, 1 \leq i \leq k \\
2(8k + 2)(n + 1) - 2 - 2i - 14k & \text{if } l = 4k + 4i + 4, 1 \leq i \leq k - 1 \\
2(8k + 2)(n + 1) - 2 - 2i - 10k & \text{if } l = 4k + 4i + 5, 1 \leq i \leq k - 1 \\
2(8k + 2)(n + 1) - 2 - 16k & \text{if } l = 8k + 2 
\end{cases}$$

and for $2 \leq j \leq n$ we have that

$$f(v_{2i-1,j}) = \begin{cases} 
2(8k + 2)(n + 1) + 1 - i - 2(4k + 1)j & \text{if } 1 \leq i \leq 2k + 1 \\
2(8k + 2)(n + 1) - i - 2(4k + 1)j & \text{if } 2k + 2 \leq i \leq 4k - 1 
\end{cases}$$

$$f(v_{2i,j}) = \begin{cases} 
2(8k + 2)(n + 1) + 1 - i - (4k + 1)(2j + 1) & \text{if } 2 \leq i \leq 2k \\
2(8k + 2)(n + 1) + 2 - i - (4k + 1)(2j + 1) & \text{if } 2k + 2 \leq i \leq 4k + 1 
\end{cases}$$

$$f(v_{2,j}) = 2(8k + 2)(n + 1) + 1 - 2(4k + 1)(j + 1)$$
\[ f(v_{4k+2,j}) = 2(8k + 2)(n + 1) + 1 - 2(k + 1) - 2(4k + 1)j \]

**Case V:** Let \( m = 8k + 4 \), where \( k \) is a positive integer and define the vertex labeling

\[
f: V(G) \rightarrow \{(8k + 4)(n + 1) + 1, \ldots, 2(8k + 4)(n + 1) \}
\]

\[
f(u_i) = \begin{cases} 
2(8k + 4)(n + 1) + 1 - i & \text{if } l = 2i - 1, \quad 1 \leq i \leq 4k + 2 \\
2(8k + 4)(n + 1) - 1 - i - 4k & \text{if } l = 2i, \quad 1 \leq i \leq 4k + 1 \\
2(8k + 4)(n + 1) - 5 - 12k & \text{if } l = 8k + 4 \\
\end{cases}
\]

\[
f(v_{1,1}) = \begin{cases} 
2(8k + 4)(n + 1) - 4 - 12k & \text{if } l = 1 \\
2(8k + 4)(n + 1) - 7 + 4i - 16k & \text{if } l = 4i - 2, 1 \leq i \leq k \\
2(8k + 4)(n + 1) - 8 + 4i - 16k & \text{if } l = 4i - 1, 1 \leq i \leq k \\
2(8k + 4)(n + 1) - 9 + 4i - 16k & \text{if } l = 4i, \quad 1 \leq i \leq k \\
2(8k + 4)(n + 1) - 6 + 4i - 16k & \text{if } l = 4i + 1, 1 \leq i \leq k \\
2(8k + 4)(n + 1) - 3 - 8k & \text{if } l = 4k + 2 \\
2(8k + 4)(n + 1) - 7 - 16k & \text{if } l = 4k + 3 \\
2(8k + 4)(n + 1) - 6 - 16k & \text{if } l = 8k + 3 \\
2(8k + 4)(n + 1) - 4 - 8k & \text{if } l = 8k + 4 \\
2(8k + 4)(n + 1) - 4 + i - 12k & \text{if } l = 4k + i + 3, 1 \leq i \leq 4k - 1 \\
\end{cases}
\]

and \( (v_{i,j}) = 2(8k + 4)(n + 1) + i - 4(2k + 1)(j + 1) \) if \( 1 \leq i \leq 8k + 4, 2 \leq j \leq n \)

**Case VI:** \( m = 8k + 6 \) where \( k \) is a positive integer and define the vertex labeling

\[
f: V(G) \rightarrow \{(8k + 6)(n + 1) + 1, \ldots, \ldots + 2(8k + 6)(n + 1) \}
\]

\[
f(u_i) = \begin{cases} 
2(8k + 6)(n + 1) - 8 - 12k & \text{if } l = 1 \\
2(8k + 6)(n + 1) - 1 - i - 4k & \text{if } l = 2i - 1, 2 \leq i \leq 4k + 3 \\
2(8k + 6)(n + 1) + 1 - i & \text{if } l = 2i, 1 \leq i \leq 4k + 3 \\
\end{cases}
\]

\[
f(v_{1,1}) = \begin{cases} 
2(8k + 6)(n + 1) - 4 - i - 8k & \text{if } l = 2i - 1, 1 \leq i \leq 2k + 3 \\
2(8k + 6)(n + 1) - 7 - 12k & \text{if } l = 2 \\
2(8k + 6)(n + 1) - 7 - i - 12k & \text{if } l = 2i, 2 \leq i \leq 2k + 1 \\
2(8k + 6)(n + 1) - 13 + 2i - 14k & \text{if } l = 4k + 3 + 1, 1 \leq i \leq 2 \\
2(8k + 6)(n + 1) - 11 - 2i - 14k & \text{if } l = 4k + 4i + 2, 1 \leq i \leq k \\
2(8k + 6)(n + 1) - 8 - 2i - 14k & \text{if } l = 4k + 4i + 4, 1 \leq i \leq k \\
2(8k + 6)(n + 1) - 6 - 2i - 10k & \text{if } l = 4k + 4i + 5, 1 \leq i \leq k \\
2(8k + 6)(n + 1) - 7 - 2i - 10k & \text{if } l = 4k + 4i + 7, 1 \leq i \leq k - 1 \\
2(8k + 6)(n + 1) - 10 - 16k & \text{if } l = 8k + 6 \\
\end{cases}
\]

and for \( 2 \leq j \leq n \), we have that
\[
f(v_{l,j}) = \begin{cases} 
(2(8k + 6)(n + 1) - i - 2(4k + 3)j + 1 & \text{if } l = 2i - 1, 1 \leq i \leq 4k + 3 \\
(2(8k + 6)(n + 1) - i - (4k + 3)(2j + 1) + 1 & \text{if } l = 2i, 1 \leq i \leq 4k + 3 
\end{cases}
\]

**Case VII:** \( m = 16k \), where \( k \) is a positive integer and the vertex labeling is defined as

\[
f: V(G) \to \{16k(n + 1) + 1, \ldots, 32k(n + 1)\} \text{ such that,}
\]

\[
f(u_i) = \begin{cases} 
32k(n + 1) + 1 - i & \text{if } l = 2i - 1 \text{ and } 1 \leq i \leq 8k \\
8k(4n + 1) + 1 - i & \text{if } l = 2i \text{ and } 1 \leq i \leq 8k - 1 \\
8k(4n + 1) + 1 & \text{if } l = 16k
\end{cases}
\]

\[
f(v_{1,1}) = 8k(4n + 1) - 10,
\]

\[
f(v_{2,1}) = 16k(2n + 1) - 10
\]

\[
f(v_{3,1}) = 32kn - 14
\]

\[
f(v_{l,i}) = \begin{cases} 
32kn + 2i & \text{if } l = 2i - 1 \text{ and } 3 \leq i \leq 4k \\
32kn - 1 + 2i & \text{if } l = 2i \text{ and } 2 \leq i \leq 4k \\
32kn - 2 + 3i & \text{if } l = 8k + 2i - 1 \text{ and } 1 \leq i \leq 2 \\
8k(4n + 1) - 4 + 8i & \text{if } l = 8k + 8i - 6 \text{ and } 1 \leq i \leq k \\
8k(4n + 1) - 5 + 8i & \text{if } l = 8k + 8i - 4 \text{ and } 1 \leq i \leq k \\
8k(4n + 1) - 3 + 8i & \text{if } l = 8k + 8i - 3 \text{ and } 1 \leq i \leq k \\
8k(4n + 1) + 1 + 8i & \text{if } l = 8k + 8i - 2 \text{ and } 1 \leq i \leq k \\
8k(4n + 1) - 1 + 8i & \text{if } l = 8k + 8i - 1 \text{ and } 1 \leq i \leq k \\
8k(4n + 1) + 8i & \text{if } l = 8k + 8i \text{ and } 1 \leq i \leq k \\
8k(4n + 1) - 2 + 8i & \text{if } l = 8k + 8i + 1 \text{ and } 1 \leq i \leq k - 1 \\
8k(4n + 1) + 2 + 8i & \text{if } l = 8k + 8i + 3 \text{ and } 1 \leq i \leq k - 1
\end{cases}
\]

and \( f(v_{l,j}) = 16k(2n - j + 1) \text{ if } 1 \leq i \leq 16k \text{ and } 2 \leq j \leq n \)

**Case VIII:** Let \( m = 16k + 8 \), where \( k \) is a positive integer and the vertex labeling is defined as \( f: V(G) = \{8(2k + 1)(n + 1) + 1, \ldots, 16(2k + 1)(n + 1)\} \) such that,

\[
f(u_i) = \begin{cases} 
16(2k + 1)(n + 1) + 1 - i & \text{if } l = 2i - 1, 1 \leq i \leq 8k + 4 \\
16(2k + 1)(n + 1) - 3 - 8k - i & \text{if } l = 2i, 1 \leq i \leq 8k + 3 \\
16(2k + 1)(n + 1) - 11 - 24k & \text{if } l = 16k + 8
\end{cases}
\]

\[
f(v_{1,1}) = 16(2k + 1)(n + 1) - 24k
\]

\[
f(v_{2,1}) = 16(2k + 1)(n + 1) - 16k
\]

\[
f(v_{3,1}) = 16(2k + 1)(n + 1) - 32k
\]
\[
\begin{align*}
\text{if } l = 2i - 1 & \text{ and } 3 \leq i \leq 4k + 2, \\
16(2kn - 1) + 2i & \\
\text{if } l = 2i & \text{ and } 2 \leq i \leq 4k + 2, \\
32kn - 17 + 2i & \\
\text{if } l = 8k + 2i + 3 & \text{ and } 1 \leq i \leq 2, \\
2(16kn - 9) + 3i & \\
\text{if } l = 8k + 8i - 2 & \text{ and } 1 \leq i \leq k + 1, \\
8k(4n + 1) - 15 + 8i & \\
\text{if } l = 8k + i + 7 & \text{ and } 1 \leq i \leq 2, \\
8k(4n + 1) - 10 + i & \\
\text{if } l = 8k + 8i + 2 & \text{ and } 1 \leq i \leq k, \\
8k(4n + 1) - 12 + 8i & \\
\text{if } l = 8k + i + 3 & \text{ and } 1 \leq i \leq k, \\
8k(4n + 1) - 14 + 8i & \\
\text{if } l = 8k + i + 4 & \text{ and } 1 \leq i \leq k, \\
8k(4n + 1) - 13 + 8i & \\
\text{if } l = 8k + 8i + 5 & \text{ and } 1 \leq i \leq k, \\
8k(4n + 1) - 11 + 8i & \\
\text{if } l = 8k + 8i + 7 & \text{ and } 1 \leq i \leq k, \\
8k(4n + 1) - 9 + 8i & \\
\text{if } l = 8k + 8i + 8 & \text{ and } 1 \leq i \leq k, \\
8k(4n + 1) - 8 + 8i & \\
\text{if } l = 8k + 8i + 9 & \text{ and } 1 \leq i \leq k - 1, \\
8k(4n + 1) - 10 + 8i & \\
\end{align*}
\]

and \( f(v_{ij}) = 32k(n + 1) - (16k + 8)(j + 1) + i \) if \( 1 \leq i \leq 16k + 8, 2 \leq j \leq n \).

Therefore, by Lemma 6.1.1, \( f \) extends to a supermagic labeling of \( G \) with valence

\[
48(2k + 1)(n + 1) - \frac{m(4n+5)}{2} + 1.
\]

Using the relationships between supermagic labelings and other labelings mentioned in the introduction, we complete with the following corollary which settles a conjecture by Yegnanarayanan [18].

**Corollary 6.3.4:** For every two integers \( m \geq 3 \) and \( n \geq 1 \), the \( n \)-crown \( G \cong C_m \odot \overline{K}_n \) is harmonious, sequential and felicitous.
References


